Isotropic Entanglement

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Abstract

The method of **"Isotropic Entanglement"** (IE), inspired by Free Probability Theory and Random Matrix Theory predicts the eigenvalue distribution of quantum many-body systems with generic interactions. At the heart is a "Slider", which interpolates between two extrema by matching fourth moments. The first extreme treats the non-commuting terms *classically* and the second treats them *isotropically.* Isotropic means that the eigenvectors are in generic positions. We prove Matching Three Moments and Slider Theorems and further prove that the interpolation is universal, i.e., independent of the choice of local terms. Our examples show that IE provides an accurate picture well beyond what one expects from the first four moments alone.

The term "entanglement" here is meant to be suggestive. We think of the eigenvectors of the two commuting subsets as becoming hopelessly mixed together in nontrivial ways, i.e., entangled, when we consider the eigenvectors of the sum. The term "isotropic" suggests the uniform measure on the eigenvectors enabling the utilization of free probability theory .

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I. ELUSIVE SPECTRA OF HAMILTONIANS

Random matrix techniques have proved to be fruitful in addressing various problems in quantum information theory (QIT) [1–3] and theoretical physics [4, 5]. In condensed matter physics (CMP), quantum spins with random exchange have been extensively studied with surprising phase transition behaviors [6–9].

Moreover, constraint satisfaction problems are at the root of complexity theory [10, 11]. The quantum analogue of the satisfiability problem (QSAT) encodes the constraints by a Hamiltonians acting on quantum spins (or qu*d*its) [12]. Interesting phase transition behaviors have been seen for random QSAT with important implications for the Hardness of typical instances[13–15].

While the application of the spectrum as a whole, not focusing on extremal eigenvalues, may be different in CMP, we point out a few examples where they are studied [16–19]. By comparison, in QIT and CMP the ground state and the first few excited states have been well studied to date [1, 20–25]. To the best of our knowledge the aggregate has not been as well studied perhaps in part because of the complexity [26] and perhaps in part because the *low* energy states have captivated so much interest to date.

Energy *eigenvalue distributions* or the *density of states (DOS)* are needed for calculating the partition function[27, p. 14]. The DOS plays an important role in the theory of solids, where it is used to calculate various physical properties such the internal energy, the density of particles, specific heat capacity, and thermal conductivity. Quantum Many-Body Systems (QMBS) spectra have been elusive for two reasons: 1. The terms that represent the interactions are generally non-commuting. This is pronounced for systems with random interactions (e.g., quantum spin glasses [28, 29]). 2. Standard numerical diagonalization is limited by memory and computer speed. The exact calculation of the spectrum of interacting QMBS has been shown to be difficult [26].

An accurate description of tails of distributions are desirable for CMP. Isotropic Entanglement (IE) provides a direct method for obtaining eigenvalue distributions of quantum spin systems with generic local interactions and does remarkably well in capturing the tails. Indeed interaction is the very source of entanglement generation [30, Section 2.4.1][31] which

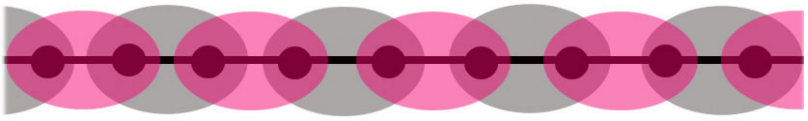


Figure 1: Odd and even summands can separately be locally diagonalized, but not the sum. The overlap of the two subsets at every site generally requires a global diagonalization.

makes QMBS a resource for quantum computation [32] but their study a formidable task on a classical computer.

Suppose we are interested in the eigenvalue distribution of a sum of Hermitian matrices $M = \sum_{i=1}^{N} M_i$. In general, M_i cannot be simultaneously diagonalized, consequently the spectrum of the sum is not the sum of the spectra. Summation mixes the entries in a very complicated manner that depends on eigenvectors. Nevertheless, it seems possible that a one-parameter approximation might suffice.

Though we are not restricted to one dimensional chains, for sake of concreteness, we investigate N interacting d-dimensional quantum spins (qudits) on a line with generic interactions. The Hamiltonian is

$$H = \sum_{l=1}^{N-1} \mathbb{I}_{d^{l-1}} \otimes H_{l,\cdots,l+L-1} \otimes \mathbb{I}_{d^{N-l-(L-1)}}, \tag{1}$$

where the local terms $H_{l,\cdots,l+L-1}$ are finite $d^L \times d^L$ random matrices. We take the case of nearest neighbors interactions, $L = 2$, unless otherwise specified.

The eigenvalue distribution of any commuting subset of H such as the terms with l odd (the "odds") or l even (the "evens") can be obtained using local diagonalization. However, the difficulty in approximating the full spectrum of $H \equiv H_{\text{odd}} + H_{\text{even}}$ is in summing the odds and the evens because of their overlap at every site.

The intuition behind IE is that terms with an overlap, such as $H_{l,l+1}$ and $H_{l+1,l+2}$, introduce randomness and mixing through sharing of a site. Namely, the process of entanglement generation introduces an *isotropicity* between the eigenvectors of evens and odds that can be harnessed to capture the spectrum.

Random Matrix Theory often takes advantage of eigenvectors with Haar measure, the uniform measure on the orthogonal/unitary group. However, the eigenvectors of QMBS have a more special structure (see Eq. 11).

Therefore we created a *hybrid theory*, where we used a finite version of Free Probability

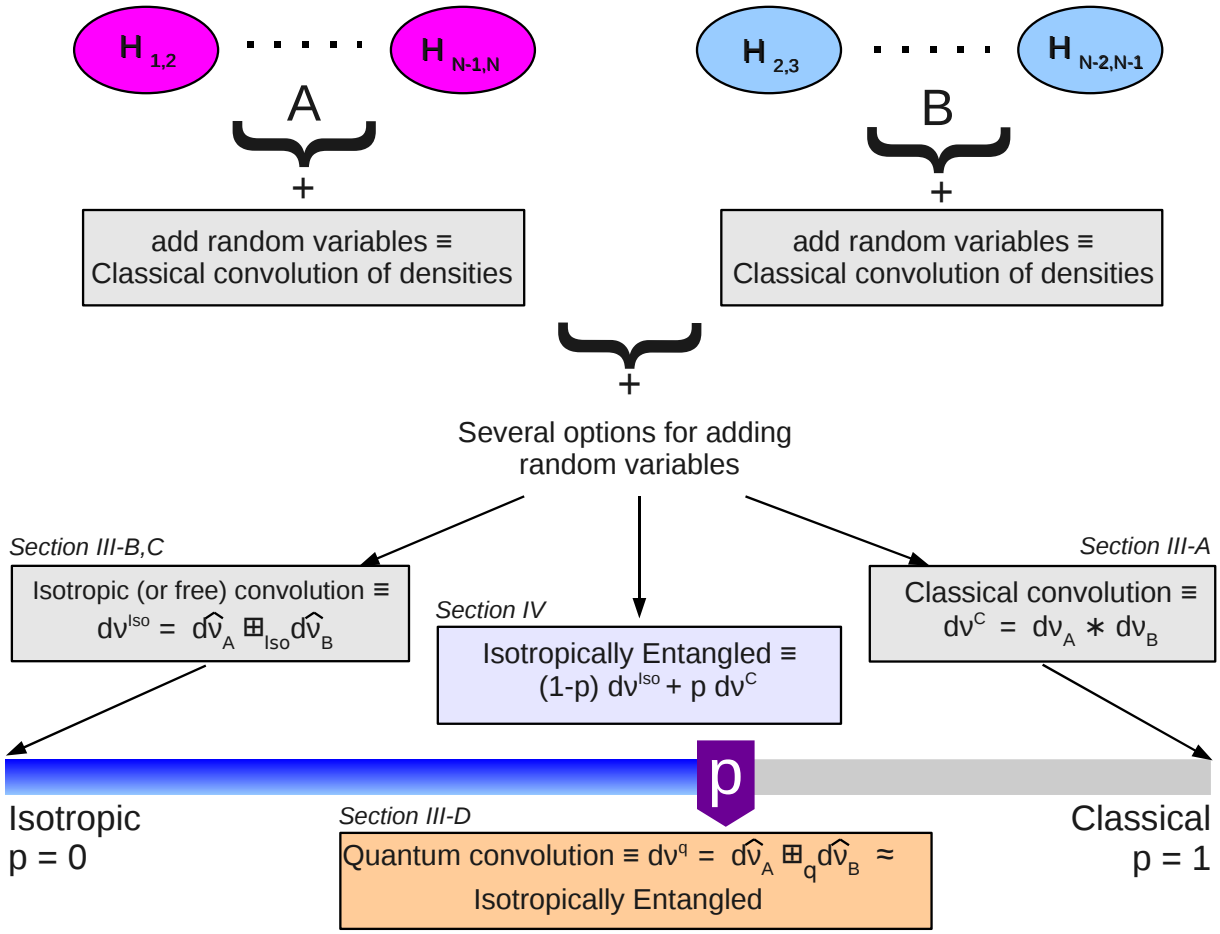


Figure 2: The method of Isotropic Entanglement: Quantum spectra as a convex combination of isotropic and classical distributions. The Slider (bottom) indicates the p that matches the quantum kurtosis as a function of classical ($p = 1$) and isotropic ($p = 0$) kurtoses. To simplify we drop the tensor products (Eq. 8) in the local terms (ellipses on top). Note that isotropic and quantum convolution depend on multivariate densities for the eigenvalues.

Theory and Classical Probability Theory to capture the eigenvalue distribution of Eq. 1. Though such problems can be QMA-complete, our examples show that IE provides an accurate picture well beyond what one expects from the first four moments alone. The *Slider* (bottom of Figure 2) displays the proposed mixture p.

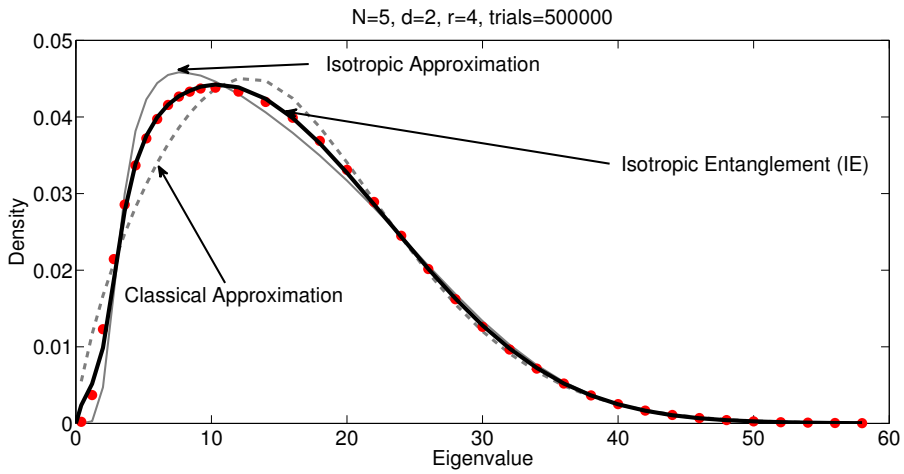


Figure 3: The exact diagonalization in dots and IE compared to the two approximations. The title parameters are explained in the section on numerical results.

II. THE METHOD OF ISOTROPIC ENTANGLEMENT

A. Overview

We propose a method to compute the "density of states" (DOS) or "eigenvalue density" of quantum spin systems with generic local interactions. More generally one wishes to compute the DOS of the sum of non-commuting random matrices from their, individually known, DOS's.

We begin with an example in Figure 3, where we compare exact diagonalization against two approximations:

- Dashed grey curve: *classical* approximation. Notice that it overshoots to the right.
- Solid grey curve: *isotropic* approximation. Notice that it overshoots to the left.
- Solid black curve: *isotropic entanglement (IE).*
- Dots: *exact diagonalization* of the quantum problem given in Eq. 1.

The *classical approximation* ignores eigenvector structure by summing random eigenvalues uniformly from non-commuting matrices. The dashed curve is the convolution of the probability densities of the eigenvalues of each matrix.

The *isotropic approximation* assumes that the eigenvectors are in "general position"; that is, we add the two matrices with correct eigenvalue densities but choose the eigenvectors

from Haar measure. As the matrix size goes to infinity, the resulting distribution is the free convolution of the individual distributions [36].

The exact diagonalization given by red dots, the dashed and solid grey curves have exactly the same first three moments, but differing fourth moments.

Isotropic Entanglement (IE) is a linear combination of the two approximations that is obtained by matching the fourth moments. We show that 1) the fit is better than what might be expected by four moments alone, 2) the combination is always convex for the problems of interest, given by $0 \leq p \leq 1$ and 3) this convex combination is universal depending on the parameter counts of the problem but not the eigenvalue densities of the local terms.

Parameter counts: exponential, polynomial and zero. Because of the *locality* of generic interactions, the complete set of eigenstates has parameter count equal to a polynomial in the number of spins, though the dimensionality is exponential. The classical and isotropic approximations have zero and exponentially many random parameters respectively. This suggests that the problem of interest somehow lies in between the two approximations.

Our work supports a very general principle that one can obtain an accurate representation of inherently exponential problems by approximating them with less complexity. This realization is at the heart of other recent developments in QMBS research such as Matrix Product States [20, 21], and Density Matrix Renormalization Group [22], where the *state* (usually the ground state of $1D$ chains) can be adequately represented by a Matrix Product State (MPS) ansatz whose parameters grow *linearly* with the number of quantum particles. Future work includes explicit treatment of fermionic systems and numerical exploration of higher dimensional systems.

B. Inputs and Outputs of the Theory

In general we consider Hamiltonians $H = H_{\text{odd}} + H_{\text{even}}$, where the local terms that add up to H_{odd} (or H_{even}) form a commuting subset. All the physically relevant quantities such as the lattice structure, N, dimension of the spin d and the rank r are encoded in the eigenvalue densities. The output of the theory is a $0 \leq p \leq 1$ by which the IE distribution is obtained and $d\nu^{IE}$ serves as an approximation to the spectral measure. The inputs can succinctly be expressed in terms of the dimension of the quantum spins, and the nature of the lattice (Figure 4).

$$\boxed{\text{Inputs}}\boxed{H = H_{\text{odd}} + H_{\text{even}}\ ,\ N,\ d,\ \beta} \longrightarrow \boxed{\text{Output}}\boxed{p,\ d\nu^{IE}}$$

Figure 4: Inputs and outputs of the IE theory. See section III for the definition of $d\nu^{IE}$.

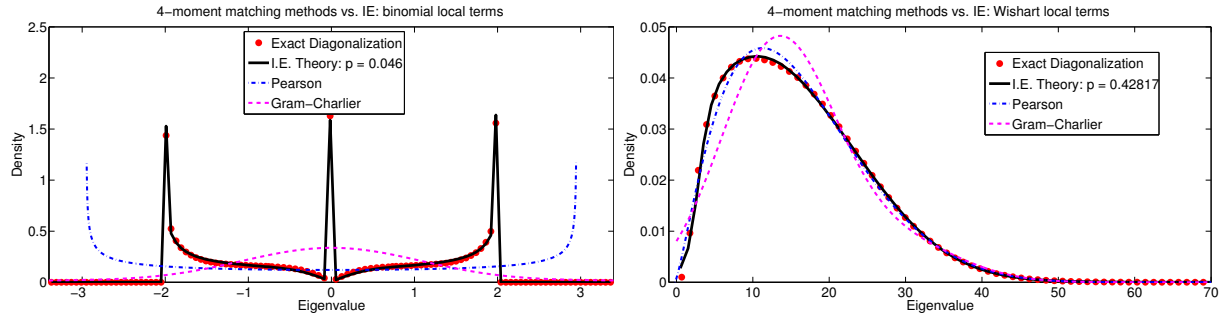


Figure 5: IE vs. Pearson and Gram-Charlier

C. More Than Four Moments of Accuracy?

Alternatives to IE worth considering are 1) Pearson and 2) Gram-Charlier moment fits.

We illustrate in Figure 5 how the IE fit is better than expected when matching four moments. We used the first four moments to approximate the density using the Pearson fit as implemented in MATLAB and also the well-known Gram-Charlier fit [40]. In [39] it was demonstrated that the statistical mechanics methods for obtaining the DOS, when applied to a finite dimensional vector space, lead to a Gaussian distribution in the lowest order. Further, they discovered that successive approximations lead naturally to the Gram-Charlier series [40]. Comparing these against the accuracy of IE leads us to view IE as more than a moment matching methodology.

The departure theorem (Section IV B) shows that in any of the higher moments (> 4) there are many terms in the quantum case that match IE exactly. Further, we conjecture that the effect of the remaining terms are generally less significant.

III. SPECTRA SUMS IN TERMS OF PROBABILITY THEORY

The density of eigenvalues may be thought of as a histogram. Formally for an $m \times m$ matrix M the *eigenvalue distribution* is [33, p. 4][34, p. 101]

$$d\nu_M(x) = \frac{1}{m}\sum_{i=1}^{m} \delta(x - \lambda_i(M)). \tag{2}$$

For a random matrix, there is the expected eigenvalue distribution [35], [36, p. 362]

$$d\nu_M(x) = \frac{1}{m}\mathbb{E}\left[\sum_{i=1}^{m}\delta(x-\lambda_i(M))\right], \tag{3}$$

which is typically a smooth curve [33, p. 101][34, p. 115].

The eigenvalue distributions above are measures on one variable. We will also need the multivariate measure on the m eigenvalues of M:

$$d\hat{\nu}_M(x) = \text{The symmetrized joint distribution of the eigenvalues.}$$

Given the densities for M and M', the question arises: What kind of summation of densities might represent the density for $M + M'$? This question is unanswerable without further information.

One might try to answer this using various assumptions based on probability theory. The first assumption is the familiar "classical" probability theory where the distribution of the sum is obtained by convolution of the density of summands. Another assumption is the modern "free" probability theory; we introduce a finite version to obtain the "isotropic" theory. By finite we mean the matrix models are finite in size as opposed to infinite as required by free probability theory. Our target problem of interest, the "quantum" problem, we will demonstrate, practically falls nicely in between the two. The "Slider" quantifies to what extent the quantum problem falls in between (Figure 2 bottom).

A. Classical

Consider random diagonal matrices A and B of size m, the only randomness is in a uniform choice among the $m!$ possible orders. Then there is no difference between the density of eigenvalue sums and the familiar convolution of densities of random variables,

$$d\nu^c = d\nu_A * d\nu_B. \tag{4}$$

Remark. From this point of view, the diagonal elements of A, say, are identically distributed random variables that need not be independent. Consider Wishart matrices [42], where

there are dependencies among the eigenvalues. To be precise let $\mathbf{a} \in \mathbb{R}^m$ be a symmetric random variable, i.e., $P\mathbf{a}$ has the same distribution as $\mathbf{a}$ for all permutation matrices P. We write, $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_m \end{pmatrix} \equiv \mathrm{diag}(\mathbf{a})$. Similarly for B.

Remark. The classical convolution appears in Figure 2 in two different ways. Firstly, in the definition of A (or B) , the eigenvalues of the odd (or even) terms are added classically. Secondly, A and B are added classically to form one end of the Slider.

B. Free and Isotropic

Free probability [36, is recommended] provides a new natural mathematical "sum" of random variables. This sum is computed by the "free convolution" denoted

$$d\nu^f = d\nu_A \boxplus d\nu_B. \tag{5}$$

Here we assume the random matrices A and B, representing the eigenvalues, have densities $d\nu_A$ and $d\nu_B$. In the large m limit, we can compute the DOS of $A + Q^{-1}BQ$, where Q is a $\beta-$Haar distributed matrix (see Table I).

Comment: In this paper we will not explore the free approach strictly other than observing that it is the infinite limit of the isotropic approach (i.e., $t \to \infty$ in Eq. 6). This infinite limit is independent of the choice of β (see Table I).

	Real $\mathbb{R}$	Complex $\mathbb{C}$	Quaternions $\mathbb{H}$	"Ghosts"
β	1	2	4	general β
Notation	Q	U	S	$\mathcal{Q}_\beta$
Haar matrices	orthogonal	unitary	symplectic	$\beta-$orthogonal

Table I: Various $\beta-$Haar matrices.

We define an isotropic convolution. The isotropic sum depends on a copying parameter t and β (Table I). The new Hamiltonian is the *isotropic Hamiltonian* (*"iso"*):

$$H_{iso} \equiv (A' \otimes \mathbb{I}_t) + Q_\beta^{-1} (\mathbb{I}_t \otimes B') Q_\beta, \tag{6}$$

where Q_β is a $\beta-$Haar distributed matrix, $A = A' \otimes \mathbb{I}_t$ and $B = \mathbb{I}_t \otimes B'$. For the copying parameter $t = d$, H_{iso} has the same dimension as H in Eq. 1; however, $t > d$ allows us to formally treat problems of growing size. We can recover the free convolution by taking the limit: $\lim_{t\to\infty} d\nu^{iso(\beta,t)} = d\nu^f$. The effect of Q_β is to spin the eigenvectors of $\mathbb{I}_t \otimes B$ to point isotropically with respect to the eigenvectors of A. We denote the isotropic eigenvalue distribution by

$$d\nu^{iso(\beta,t)} = d\hat{\nu}_A \boxplus_{iso(\beta,t)} d\hat{\nu}_B \tag{7}$$

omitting t and β when it is clear from the context.

Comment: In Eq. 6, the $\mathbb{I}_t$ and B' in $\mathbb{I}_t \otimes B'$, can appear in any order. We chose this presentation in anticipation of the quantum problem.

Comment: In this paper we primarily consider t to match the dimension of H.

The notion of interpolation between classical and free has been introduced rigorously in [37, 38]. However, here we are on the one hand making use of isotropic (finite case) end point and on the other considering the end points to approximate the unknown quantum convolution

C. Quantum

Let $d\nu^q$ denote the eigenvalue distribution for the Hamiltonian in Eq. 1. This is the distribution that we will approximate by $d\nu^{IE}$. In connection to Figure 1 the Hamiltonian can be written as

$$H \equiv H_{\text{odd}} + H_{\text{even}} = \sum_{l=1,3,5,\cdots} \mathbb{I} \otimes H_{l,l+1} \otimes \mathbb{I} + \sum_{l=2,4,6,\cdots} \mathbb{I} \otimes H_{l,l+1} \otimes \mathbb{I}. \tag{8}$$

We proceed to define a "quantum convolution" on the distributions $d\hat{\nu}_A$ and $d\hat{\nu}_B$, which is β-dependent

$$d\nu^{q(\beta)} = d\hat{\nu}_A \boxplus_q d\hat{\nu}_B. \tag{9}$$

In general, without any connection to a Hamiltonian, let $d\hat{\nu}_A$ and $d\hat{\nu}_B$ be symmetric measures on $\mathbb{R}^{d^N}$. We define $d\nu^{q(\beta)}$ to be the eigenvalue distribution of

$$H = A + Q_q^{-1} B Q_q, \tag{10}$$

where $Q_q = \left(Q_q^{(A)}\right)^{-1} Q_q^{(B)}$ with

$$\begin{aligned} Q_q^{(A)} &= \left[\bigotimes_{i=1}^{(N-1)/2} Q_i^{(O)}\right] \otimes \mathbb{I}_d \text{ and } Q_q^{(B)} = \mathbb{I}_d \otimes \left[\bigotimes_{i=1}^{(N-1)/2} Q_i^{(E)}\right] \qquad N \text{ odd} \\ Q_q^{(A)} &= \left[\bigotimes_{i=1}^{N/2} Q_i^{(O)}\right] \text{ and } Q_q^{(B)} = \mathbb{I}_d \otimes \left[\bigotimes_{i=1}^{N/2-1} Q_i^{(E)}\right] \otimes \mathbb{I}_d \qquad N \text{ even} \end{aligned} \tag{11}$$

and each $Q_i^{(\bullet)}$ is a $\beta-$Haar measure orthogonal matrix of size d^2 and $\mathbb{I}_d$ is an identity matrix of size d and $Q_i^{(\bullet)}$ are all independent.

Comment: A, B and Q_q are $d^N \times d^N$.

Comment: In our examples given in this paper, we assume the local terms are independent and identically distributed (iid) random matrices, each of which has eigenvectors distributed with $\beta-$Haar measure.

The tensor product in (11) succinctly summarizes the departure of the quantum case from a generic matrix as well as from the classical case. First of all the number of parameters in Q_q grows linearly with N whereas in Q it grows exponentially with N. Second, the quantum case possesses isotropicity that makes it different from the classical, whose eigenvectors are a point on the orthogonal group (i.e., the identity matrix).

Comment: General β's can be treated formally [43]. In particular, for quantum mechanical problems β is taken to be 1 or 2 corresponding to real and complex entries in the local terms. $\beta = 4$ corresponds to quaternions.

Definition. The **Hadamard product** of two matrices M_1 and M_2 of the same size, denoted by $M_1 \circ M_2$, is the product of the corresponding elements.

Lemma 1. *The elements of Q_q defined in Eq. 11 are (dependent) random variables with mean zero and variance d^{-N}.*

Proof. Here expectations are taken with respect to the random matrix Q_q which is built from local Haar measure matrices by Eq. 11. The fact that $\mathbb{E}\left(Q_q^{(A)}\right) = \mathbb{E}\left(Q_q^{(B)}\right) = 0_{d^N}$ follows from the Haar distribution of local terms. Thus $\mathbb{E}(Q_q) = 0$ by independence of $Q_q^{(A)}$ and $Q_q^{(B)}$. Further, each element in Q_q involves a dot product between columns of

$Q_q^{(A)}$ and $Q_q^{(B)}$. In every given column of $Q_q^{(A)}$ any nonzero entry is a distinct product of entries of local $Q's$ (see Eq.11). For example the expectation value of the $1,1$ entry is $\mathbb{E}\left(q_{i,1}^{(A)}q_{j,1}^{(A)}q_{i,1}^{(B)}q_{j,1}^{(B)}\right) = \mathbb{E}\left(q_{i,1}^{(A)}q_{j,1}^{(A)}\right)\mathbb{E}\left(q_{i,1}^{(B)}q_{j,1}^{(B)}\right)$. Because of the Haar measure of the local terms, this expectation is zero unless $i = j$. We then have that

$$\begin{gathered}\mathbb{E}\left(Q_q \circ Q_q\right) = \mathbb{E}\left(Q_q^{(A)} \circ Q_q^{(A)}\right)^T \mathbb{E}\left(Q_q^{(B)} \circ Q_q^{(B)}\right) = \\ \begin{cases} \left(\left[\bigotimes_{i=1}^{(N-1)/2} d^{-2} J_{d^2}\right] \otimes \mathbb{I}_d\right)\left(\mathbb{I}_d \otimes \left[\bigotimes_{i=1}^{(N-1)/2} d^{-2} J_{d^2}\right]\right) & N \text{ odd} \\ \left(\bigotimes_{i=1}^{N/2} d^{-2} J_{d^2}\right)\left(\mathbb{I}_d \otimes \left[\bigotimes_{i=1}^{N/2-1} d^{-2} J_{d^2}\right] \otimes \mathbb{I}_d\right) & N \text{ even} \end{cases} \\ = d^{-N} J_{d^N},\end{gathered} \tag{12}$$

where $J_i = i \times i$ matrix of all ones. We use facts such as $(J_i/i)^2 = (J_i/i)$, $(J_i/i) \otimes (J_i/i) = (J_{i^2}/i^2)$ and the variance of the elements of an $i \times i$ $\beta-$Haar matrix is $1/i$. □

IV. THEORY OF ISOTROPIC ENTANGLEMENT

A. Isotropic Entanglement as the Combination of Classical and Isotropic

We create a "Slider" based on the fourth moment. The moment m_k of a random variable defined in terms of its density is $m_k = \int x^k d\nu_M$. For the eigenvalues of an $m \times m$ random matrix, this is $\frac{1}{m}\mathbb{E}\mathrm{Tr}M^k$. In general, the moments of the classical sum and the free sum are different, but the first three moments, $m_1,\ m_2$, and m_3 are theoretically equal [36, p. 191]. Further, to anticipate our target problem, the first three moments of the quantum eigenvalues are also equal to that of the iso and the classical (The Departure and the Three Moments Matching theorems in Section IV B). These moments are usually encoded as the mean, variance, and skewness.

We propose to use the fourth moment (or the excess kurtosis) to choose a correct p from a sliding hybrid sum:

$$d\nu^q \approx d\nu^{IE} = pd\nu^c + (1-p)d\nu^{iso} \tag{13}$$

Therefore, we find p that expresses the kurtosis of the quantum sum (γ_2^q) in terms of the kurtoses of the classical (γ_2^c) and isotropic (γ_2^{iso}) sums:

$$\gamma_2^q = p\gamma_2^c + (1-p)\,\gamma_2^{iso} \Rightarrow \qquad p = \frac{\gamma_2^q - \gamma_2^{iso}}{\gamma_2^c - \gamma_2^{iso}}. \tag{14}$$

Recall that the kurtosis $\gamma_2 \equiv \frac{m_4}{\sigma^4}$, where σ^2 is the variance. Hence kurtosis is the correct statistical quantity that encodes the fourth moments:

$$m_4^c = \frac{1}{d^N}\mathbb{E}\mathrm{Tr}\left(A + \Pi^T B \Pi\right)^4,\ m_4^{iso} = \frac{1}{d^N}\mathbb{E}\mathrm{Tr}\left(A + Q^T B Q\right)^4,\ m_4^q = \frac{1}{d^N}\mathbb{E}\mathrm{Tr}\left(A + Q_q^T B Q_q\right)^4, \tag{15}$$

where Π is a random uniformly distributed permutation matrix, Q is a $\beta-$Haar measure orthogonal matrix of size d^N, and Q_q is given by Eq. 11. *Unless stated otherwise, in the following the expectation values are taken with respect to random eigenvalues A and B and eigenvectors. The expectation values over the eigenvectors are taken with respect to random permutation Π, $\beta-$Haar Q or Q_q matrices for classical, isotropic or quantum cases respectively.*

B. The Departure and The Matching Three Moments Theorems

In general we have the i^{th} moments:

$$\begin{aligned} m_i^{iso} &= \frac{1}{m}\mathbb{E}\mathrm{Tr}\left(A + Q^T B Q\right)^i \\ m_i^q &= \frac{1}{m}\mathbb{E}\mathrm{Tr}\left(A + Q_q^T B Q_q\right)^i, \text{ and} \\ m_i^c &= \frac{1}{m}\mathbb{E}\mathrm{Tr}\left(A + \Pi^T B \Pi\right)^i. \end{aligned}$$

where $m \equiv d^N$. If we expand the moments above we find some terms can be put in the form $\mathbb{E}\mathrm{Tr}\left(A^{m_1} Q_\bullet^T B^{m_2} Q_\bullet\right)$ and the remaining terms can be put in the form $\mathbb{E}\mathrm{Tr}\left\{\ldots Q_\bullet^T B^{\geq 1} Q_\bullet A^{\geq 1} Q_\bullet^T B^{\geq 1} Q_\bullet \ldots\right\}$. The former terms we denote *non-departing*; the remaining terms we denote *departing*.

For example, when $i = 4$,

$$
\begin{aligned}
m_4^{iso} &= \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[A^4+4A^3Q^TBQ+4A^2Q^TB^2Q+4AQ^TB^3Q+\underline{\mathbf{2}\left(\mathbf{AQ^TBQ}\right)^{\mathbf{2}}}+B^4\right]\right\} \quad (16)\\
m_4^{q} &= \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[A^4+4A^3Q_q^TBQ_q+4A^2Q_q^TB^2Q_q+4AQ_q^TB^3Q_q+\underline{\mathbf{2}\left(\mathbf{AQ_q^TBQ_q}\right)^{\mathbf{2}}}+B^4\right]\right\}\\
m_4^{c} &= \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[A^4+4A^3\Pi^TB\Pi+4A^2\Pi^TB^2\Pi+4A\Pi^TB^3\Pi+\underline{\mathbf{2}\left(\mathbf{A\Pi^TB\Pi}\right)^{\mathbf{2}}}+B^4\right]\right\},
\end{aligned}
$$

where the only departing terms and the corresponding classical term are shown as underlined and bold faced.

Theorem. (The Departure Theorem) *The moments of the quantum, isotropic and classical sums differ only in the departing terms:* $\mathbb{E}\,Tr\left\{\ldots Q_\bullet^TB^{\geq 1}Q_\bullet A^{\geq 1}Q_\bullet^TB^{\geq 1}Q_\bullet\ldots\right\}$.

Proof. Below the repeated indices are summed over. If A and B are any diagonal matrices, and $Q_\bullet$ is Q or Q_q or Π of size $m\times m$ then $\mathbb{E}\left(q_{ij}^2\right)=1/m$, by symmetry and by Lemma 1 respectively. Since the $\mathbb{E}\mathrm{Tr}\left(AQ_\bullet^TBQ_\bullet\right)=\mathbb{E}\left(q_{ij}^2a_ib_j\right)$, where expectation is taken over randomly ordered eigenvalues and eigenvectors; the expected value is $m^2\left(\frac{1}{m}\right)\mathbb{E}\left(a_ib_j\right)$ for any i or j. Hence, $\frac{1}{m}\mathbb{E}\mathrm{Tr}\left(AQ_\bullet^TBQ_\bullet\right)=\mathbb{E}\left(a_ib_j\right)=\mathbb{E}\left(a_i\right)\mathbb{E}\left(b_j\right)$, which is equal to the classical value. The first equality is implied by permutation invariance of entries in A and B and the second equality follows from the independence of A and B. □

Therefore, the three cases differ only in the terms $\frac{2}{m}\mathbb{E}\mathrm{Tr}\left(AQ^TBQ\right)^2$, $\frac{2}{m}\mathbb{E}\mathrm{Tr}\left(AQ_q^TBQ_q\right)^2$ and $\frac{2}{m}\mathbb{E}\mathrm{Tr}\left(A\Pi^TB\Pi\right)^2$ in Eq. 16.

Theorem. (The Matching Three Moments Theorem) *The first three moments of the quantum, iso and classical sums are equal.*

Proof. The first three moments are

$$
\begin{aligned}
m_1^{(\bullet)} &= \tfrac{1}{m}\mathbb{E}\mathrm{Tr}\left(A+B\right)\\
m_2^{(\bullet)} &= \tfrac{1}{m}\mathbb{E}\mathrm{Tr}\left(A+Q_\bullet^TBQ_\bullet\right)^2=\tfrac{1}{m}\mathbb{E}\mathrm{Tr}\left(A^2+2AQ_\bullet^TBQ_\bullet+B^2\right) \quad (17)\\
m_3^{(\bullet)} &= \tfrac{1}{m}\mathbb{E}\mathrm{Tr}\left(A+Q_\bullet^TBQ_\bullet\right)^3=\tfrac{1}{m}\mathbb{E}\mathrm{Tr}\left(A^3+3A^2Q_\bullet^TBQ_\bullet+3AQ_\bullet^TB^2Q_\bullet+B^3\right),
\end{aligned}
$$

where $Q_\bullet$ is Q and Q_q for the iso and the quantum sums respectively and we used the familiar trace property $\mathrm{Tr}(M_1M_2)=\mathrm{Tr}(M_2M_1)$. The equality of the first three moments of the iso and quantum with the classical follows from The Departure Theorem. □

number of sites	number of odds or evens (N odd)	number of odds or evens (N even)	size of H	Number of copies
N	$k=\frac{N-1}{2}$	$k_{\text{odd}}=\frac{N}{2},\ k_{\text{even}}=\frac{N-2}{2}$	$m=d^N$	t

dimension of qudits	size of local terms	l^{th} moment	l^{th} cumulant	mean	variance	skewness	kurtosis
d	$n=d^2$	m_l	κ_l	μ	σ^2	γ_1	γ_2

Table II: Notation

Furthermore, in the expansion of any of the moments > 4 all the non-departing terms are exactly captured by IE. These terms are equal to the corresponding terms in the classical and the isotropic and therefore equal to any linear combination of them. The departing terms in higher moments (i.e.,> 4) that are approximated by IE, we conjecture are of little relevance. For example, the fifth moment has only two terms (shown in bold) in its expansion that are departing:

$$m_5=\tfrac{1}{m}\mathbb{E}\mathrm{Tr}\Big(A^5+5A^4Q_\bullet^TBQ_\bullet+5A^3Q_\bullet^TB^2Q_\bullet+5A^2Q_\bullet^TB^3Q_\bullet+\underline{\mathbf{5A\left(AQ_\bullet^TBQ_\bullet\right)^2}}+\underline{\mathbf{5\left(AQ_\bullet^TBQ_\bullet\right)^2Q_\bullet^TBQ_\bullet}}+5AQ_\bullet^TB^4Q_\bullet+B^5\Big) \tag{18}$$

By the Departure Theorem the numerator in Eq. 14 becomes,

$$\gamma_2^q-\gamma_2^{iso}=\frac{\kappa_4^q-\kappa_4^{iso}}{\sigma^4}=\frac{2}{m}\frac{\mathbb{E}\left\{\mathrm{Tr}\left[\left(AQ_q^TBQ_q\right)^2-\left(AQ^TBQ\right)^2\right]\right\}}{\sigma^4} \tag{19}$$

and the denominator in Eq. 14 becomes,

$$\gamma_2^c-\gamma_2^{iso}=\frac{\kappa_4^c-\kappa_4^{iso}}{\sigma^4}=\frac{2}{m}\frac{\mathbb{E}\left\{\mathrm{Tr}\left[\left(A\Pi^TB\Pi\right)^2-\left(AQ^TBQ\right)^2\right]\right\}}{\sigma^4}, \tag{20}$$

where as before, Q is a $\beta-$Haar measure orthogonal matrix of size $m=d^N$, $Q_q=\left(Q_q^{(A)}\right)^T Q_q^{(B)}$ given by Eq. 11 and $\kappa_4^\bullet$ denote the fourth cumulants. Therefore, evaluation of p reduces to the evaluation of the right hand sides of Eqs. 19 and 20.

Below we do not want to restrict ourselves to only chains with odd number of sites and we need to take into account the multiplicity of the eigenvalues as a result of taking the tensor product with identity. It is convenient to denote the size of the matrices involved by $m=d^N=tn^k$, where $n=d^2$ and $k=\frac{N-1}{2}$ and t is the number of copies (Section III B and

Table II).

C. Distribution of A and B

The goal of this section is to express the moments of the entries of A and B (e.g., m_2^A and $m_{1,1}^A$) in terms of the moments of the local terms (e.g for odd local terms $m_2^{\text{odd}}, m_{11}^{\text{odd}}$). Note that A and B are independent. The odd summands that make up A all commute and therefore can be locally diagonalized to give the diagonal matrix A (similarly for B),

$$\begin{aligned} A &= \sum_{i=1,3,\cdots}^{N-2} \mathbb{I} \otimes \Lambda_i \otimes \mathbb{I} \\ B &= \sum_{i=2,4,\cdots}^{N-1} \mathbb{I} \otimes \Lambda_i \otimes \mathbb{I}, \end{aligned} \tag{21}$$

where Λ_i are of size d^2 and are the diagonal matrices of the local eigenvalues.

The diagonal matrices A and B are formed by a direct sum of the local eigenvalues of odds and evens respectively. For open boundary conditions (OBC) each entry has a multiplicity given by Table III.

OBC	N odd	N even
A	d	1
B	d	d^2

Table III: The multiplicity of terms in A and B for OBC. For closed boundary conditions there is no repetition.

Comment: We emphasize that A and B are independent of the eigenvector structures. In particular, A and B are the same among the three cases of isotropic, quantum and classical.

We calculate the moments of A and B. Let us treat the second moment of A (B is done the same way). By the permutation invariance of entries in A

$$
\begin{aligned}
m_2^A \equiv \mathbb{E}\left(a_1^2\right) &= \mathbb{E}\left(\lambda_{i_1}^{(1)} + \cdots + \lambda_{i_k}^{(k)}\right)^2 \\
&= \mathbb{E}\left[k\left(\lambda^2\right) + k\left(k-1\right)\lambda^{(1)}\lambda^{(2)}\right] \\
&= km_2^{\text{odd}} + k\left(k-1\right)m_{1,1}^{\text{odd}}
\end{aligned} \tag{22}
$$

where expectation is taken over randomly chosen local eigenvalues, $m_2^{\text{odd}} \equiv \mathbb{E}\left(\lambda_i^2\right)$ and $m_{1,1}^{\text{odd}} \equiv \mathbb{E}\left(\lambda_i\lambda_j\right)$ for some uniformly chosen i and j with $i \neq j$. The permutation invariance assumption implies $\mathbb{E}\left(a_i^2\right) = \mathbb{E}\left(a_1^2\right)$ for all $i = 1 \cdots m$.

Comment: The key to this argument giving m_2^A is that the indices are not sensitive to the copying that results from the tensor product with $\mathbb{I}_d$ at the boundaries.

Next we calculate the correlation between two diagonal terms, namely $m_{1,1}^A \equiv \mathbb{E}\left(a_i a_j\right)$ for $i \neq j$. We need to incorporate the multiplicity, denoted by t, due to the tensor product with an identity matrix at the end of the chain,

$$
\begin{aligned}
m_{1,1}^A &= \frac{1}{m\left(m-1\right)}\mathbb{E}\left\{\left(\sum_{i_1,\cdots,i_k=1}^{n}\lambda_{i_1}^{(1)} + \cdots + \lambda_{i_k}^{(k)}\right)^2 - \sum_{i_1,\cdots,i_k=1}^{n}\left(\lambda_{i_1}^{(1)} + \cdots + \lambda_{i_k}^{(k)}\right)^2\right\} \\
&= k\left(k-1\right)\mathbb{E}\left(\lambda\right)^2 + k\left\{\text{prob}\left(\lambda^2\right)\mathbb{E}\left(\lambda^2\right) + \text{prob}\left(\lambda_1\lambda_2\right)\mathbb{E}\left(\lambda_1\lambda_2\right)\right\} \\
&= k\left(k-1\right)m_2^{\text{odd}} + \frac{k}{m-1}\left\{\left(tn^{k-1}-1\right)m_2^{\text{odd}} + \left(tn^{k-1}\left(n-1\right)\right)m_{1,1}^{\text{odd}}\right\}
\end{aligned} \tag{23}
$$

where, $\text{prob}\left(\lambda^2\right) = \frac{tn^{k-1}-1}{m-1}$ and $\text{prob}\left(\lambda_1\lambda_2\right) = \frac{tn^{k-1}(n-1)}{m-1}$. Similarly for B.

D. Evaluation and Universality of $p = \frac{\gamma_2^q - \gamma_2^{iso}}{\gamma_2^c - \gamma_2^{iso}}$

Recall the definition of p; from Eqs. 14, 19 and 20 we have,

$$
1 - p = \frac{\mathbb{E}\text{Tr}\left(A\Pi^T B\Pi\right)^2 - \mathbb{E}\text{Tr}\left(AQ_q^T BQ_q\right)^2}{\mathbb{E}\text{Tr}\left(A\Pi^T B\Pi\right)^2 - \mathbb{E}\text{Tr}\left(AQ^T BQ\right)^2}. \tag{24}
$$

The classical case

$$
\frac{1}{m}\mathbb{E}\text{Tr}\left(A\Pi^T B\Pi\right)^2 = \frac{1}{m}\mathbb{E}\sum_{i=1}^{m}a_i^2 b_i^2 = \mathbb{E}\left(a_i^2\right)\mathbb{E}\left(b_i^2\right) = m_2^A m_2^B. \tag{25}
$$

moments	expectation values	count
m_2^2	$\mathbb{E}\left(\left\|q_{i,j}\right\|^4\right) = \frac{\beta+2}{m(m\beta+2)}$	m^2
$m_2 m_{11}$	$\mathbb{E}\left(\left\|q_{1,1}q_{1,2}\right\|^2\right) = \frac{\beta}{m(m\beta+2)}$	$2m^2\left(m-1\right)$
$\left(m_{11}\right)^2$	$\mathbb{E}\left(q_{1,1}\overline{q_{1,2}q_{2,1}}q_{2,2}\right) = -\frac{\beta}{m(m\beta+2)(m-1)}$	$m^2\left(m-1\right)^2$
	$\mathbb{E}\left(q_{13}^2 q_{24}^2\right) = \frac{\beta(n-1)+2}{n(n\beta+2)(n-1)}$	

Table IV: The expectation values and counts of colliding terms in Q when it is $\beta-$Haar distributed. In this section we use the first row; we include the last three rows for the calculations in the appendix.

Comment: Strictly speaking after the first equality we must have used b_{π_i} instead of b_i but we simplified the notation as they are the same in an expectation sense.

The general form for the denominator of Eq. 24 is

$$\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ^T BQ\right)^2\right] = \frac{1}{m}\mathbb{E}\left\{a_l^2 b_l^2 - a_i a_k b_j b_p \left(q_{ji}q_{jk}q_{pk}q_{pi}\right)\right\}. \tag{26}$$

It's worth noting that the arguments leading to Eq. 30 hold even if one fixes A and B and takes expectation values over Π and a permutation invariant Q whose entries have the same expectation value. The right hand side of Eq. 30 is a homogeneous polynomial of order two in the entries of A and B; consequently it necessarily has the form

$$\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ^T BQ\right)^2\right] = c_1\left(B,Q\right)m_2^A + c_2\left(B,Q\right)m_{1,1}^A$$

but Eq. 26 must be zero for $A = I$, for which $m_2^A = m_{1,1}^A = 1$. This implies that $c_1 = -c_2$, allowing us to factor out $\left(m_2^A - m_{1,1}^A\right)$. Similarly, the homogeneity and permutation invariance of B implies,

$$\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ^T BQ\right)^2\right] = \left(m_2^A - m_{1,1}^A\right)\left(D_1\left(Q\right)m_2^B + D_2\left(Q\right)m_{1,1}^B\right).$$

The right hand side should be zero for $B = I$, whereby we can factor out $\left(m_2^B - m_{1,1}^B\right)$

$$\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ^T BQ\right)^2\right] = \left(m_2^A - m_{1,1}^A\right)\left(m_2^B - m_{1,1}^B\right)f\left(Q\right), \tag{27}$$

where $m_2^A = \mathbb{E}\left(a_i^2\right)$, $m_2^B = \mathbb{E}\left(b_j^2\right)$, and $m_{1,1}^A = \mathbb{E}\left(a_i, a_j\right)$, $m_{1,1}^B = \mathbb{E}\left(b_i, b_j\right)$. Moreover $f\left(Q\right)$ is

a homogeneous function of order four in the entries of Q. To evaluate $f(Q)$, it suffices to let A and B be projectors of rank one where A would have only one nonzero entry on the i^{th} position on its diagonal and B only one nonzero entry on the j^{th} position on its diagonal. Further take those nonzero entries to be ones, giving $m_{1,1}^A = m_{1,1}^B = 0$ and $m_2^A = m_2^B = 1/m$,

$$\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ^T BQ\right)^2\right] = \frac{1}{m^2} f(Q) \tag{28}$$

But the left hand side is

$$\begin{aligned}
\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ^T BQ\right)^2\right] &= \frac{1}{m}\mathbb{E}\left[\delta_{ij} - q_{ij}^4\right] \\
&= \frac{1}{m}\left\{\frac{1}{m^2}\sum_{ij}\delta_{ij} - \frac{1}{m^2}\sum_{ij}\mathbb{E}\left(q_{ij}^4\right)\right\} \\
&= \frac{1}{m}\left\{\frac{1}{m} - \mathbb{E}\left(q_{ij}^4\right)\right\},
\end{aligned}$$

where, we used the homogeneity of Q. Consequently, by equating this to $f(Q)/m^2$, we get the desired quantity

$$f(Q) = \left\{1 - m\mathbb{E}\left(q_{ij}^4\right)\right\}$$

Our final result Eq. 27 now reads

$$\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ^T BQ\right)^2\right] = \left(m_2^A - m_{1,1}^A\right)\left(m_2^B - m_{1,1}^B\right)\left\{1 - m\mathbb{E}\left(q_{ij}^4\right)\right\}. \tag{29}$$

The same calculation where each of the terms is obtained separately yields the same result (Appendix). In this paper p is formed by taking Q to have a $\beta-$Haar measure. Expectation values of the entries of Q are listed in the Table IV.

We wish to express everything in terms of the local terms; using Eqs. 22 and 23 as well as $tn^k = m$,

$$
\begin{aligned}
m_2^A - m_{1,1}^A &= \frac{tk\,(n-1)\,n^{k-1}}{m-1}\left(m_2^{\text{odd}} - m_{1,1}^{\text{odd}}\right) \\
m_2^B - m_{1,1}^B &= \frac{tk\,(n-1)\,n^{k-1}}{m-1}\left(m_2^{\text{even}} - m_{1,1}^{\text{even}}\right),
\end{aligned}
$$

giving

$$
\begin{aligned}
\frac{1}{m}\mathbb{E}\left[\text{Tr}\left(A\Pi^T B\Pi\right)^2 - \text{Tr}\left(AQ^T BQ\right)^2\right] &= \left(m_2^{\text{odd}} - m_{1,1}^{\text{odd}}\right)\left(m_2^{\text{even}} - m_{1,1}^{\text{even}}\right) \times \\
&\left(\frac{km\,(n-1)}{n\,(m-1)}\right)^2 \left\{1 - m\mathbb{E}\left(q_{ij}^4\right)\right\}. \qquad (30)
\end{aligned}
$$

We now proceed to the quantum case where we need to evaluate $\frac{1}{m}\mathbb{E}\left[\left(A\Pi^T B\Pi\right)^2 - \text{Tr}\left(AQ_q^T BQ_q\right)^2\right]$. In this case, we cannot directly use the techniques that we used to get Eq. 30 because Q_q is not permutation invariant despite local eigenvectors being so. Before proceeding further we like to prove a useful lemma (Lemma 2). Let us simplify the notation and denote the local terms that are drawn randomly from a known distribution by $H_{l,l+1} \equiv H^{(l)}$ whose eigenvalues are Λ_l as discussed above.

Recall that A represents the *sum* of all the odds and $Q_q^{-1}BQ_q$ the *sum* of all the evens,

$$
H_{\text{odd}} = \sum_{l=1,3,5,\cdots} \mathbb{I} \otimes H^{(l)} \otimes \mathbb{I}, \text{ and } \quad H_{\text{even}} = \sum_{l=2,4,6,\cdots} \mathbb{I} \otimes H^{(l)} \otimes \mathbb{I},
$$

Hence, the expansion of $\frac{1}{m}\mathbb{E}\left[\text{Tr}\left(AQ_q^T BQ_q\right)^2\right]$ amounts to picking an odd term, an even term, then another odd term and another even term, multiplying them together and taking the expectation value of the trace of the product (Figure 6). Therefore, each term in the expansion can have four, three or two different local terms, whose expectation values along with their counts are needed. These expectations are taken with respect to the local terms (dense $d^2 \times d^2$ random matrices).

The expectation values depend on the type of random matrix distribution from which the local terms are drawn. The counting however, depends on the configuration of the lattice

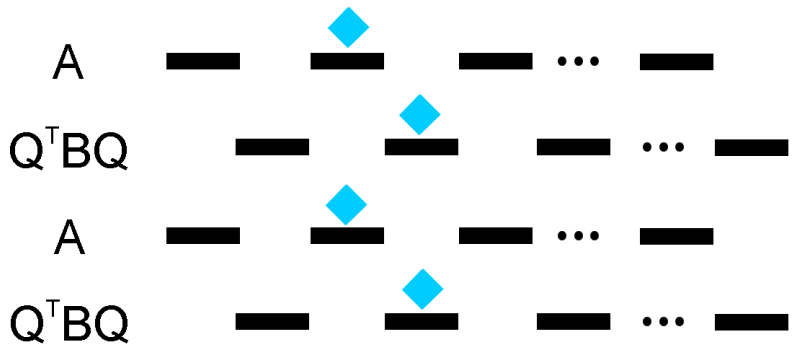


Figure 6: The terms in the expansion of $\frac{1}{m}\mathbb{E}\left[\mathrm{Tr}\left(AQ_q^TBQ_q\right)^2\right]$ can be visualized as picking an element from each row from top to bottom and multiplying. Each row has k of the local terms corresponding to a chain with odd number of terms. Among k^4 terms roughly k^2 of them differ among the classical, isotropic and quantum cases (See Eqs. 31 and 32). An example of such a choice is shown by diamonds.

only. We show the counting of the number of terms, taking care of the boundary terms for an open chain, along with the type of expectation values by which they need to be weighted:

For N odd (k odd terms and k even terms)

$$\begin{gathered}
\text{Four } H^{(\cdot)}\text{'s} : k^2\left(k-1\right)^2 \Rightarrow d^{N-u_1}\mathbb{E}\mathrm{Tr}\left(H^{(l)}\right)^4,\ u_1 \in \{5,\cdots,8\} \\
\text{Three } H^{(\cdot)}\text{'s} : 2k^2\left(k-1\right) \Rightarrow d^{N-u_2}\mathbb{E}\mathrm{Tr}\left(\left[H^{(l)}\right]^2\right)\mathbb{E}\mathrm{Tr}\left(H^{(l)}\right)^2,\ u_2 \in \{4,5,6\} \\
\text{Two } H^{(\cdot)}\text{'s: } \left(k-1\right)^2 \ \text{ Not } \ \text{Entangled} \Rightarrow d^{N-4}\left\{\mathbb{E}\mathrm{Tr}\left(\left[H^{(l)}\right]^2\right)\right\}^2 \\
\text{Two } H^{(\cdot)}\text{'s} : \left(2k-1\right) \ \ \text{Entangled} \Rightarrow d^{N-3}\mathbb{E}\mathrm{Tr}\left[\left(H^{(l)}\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes H^{(l+1)}\right)\left(H^{(l)}\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes H^{(l+1)}\right)\right]
\end{gathered} \tag{31}$$

For N even (k odd terms and $k-1$ even terms)

$$\begin{gathered}
\text{Four } H^{(\cdot)}\text{'s} : k\left(k-1\right)^2\left(k-2\right) \Rightarrow d^{N-u_1}\mathbb{E}\mathrm{Tr}\left(H^{(l)}\right)^4,\ u_1 \in \{5,\cdots,8\} \\
\text{Three } H^{(\cdot)}\text{'s} : k\left(k-1\right)\left(2k-3\right) \Rightarrow d^{N-u_2}\mathbb{E}\mathrm{Tr}\left(\left[H^{(l)}\right]^2\right)\mathbb{E}\mathrm{Tr}\left(H^{(l)}\right)^2,\ u_2 \in \{4,5,6\} \\
\text{Two } H^{(\cdot)}\text{'s} : \left(k-1\right)\left(k-2\right) \ \text{ Not } \ \text{Entangled} \Rightarrow d^{N-4}\left\{\mathbb{E}\mathrm{Tr}\left(\left[H^{(l)}\right]^2\right)\right\}^2 \\
\text{Two } H^{(\cdot)}\text{'s} : 2\left(k-1\right) \ \ \text{Entangled} \Rightarrow d^{N-3}\mathbb{E}\mathrm{Tr}\left[\left(H^{(l)}\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes H^{(l+1)}\right)\left(H^{(l)}\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes H^{(l+1)}\right)\right]
\end{gathered} \tag{32}$$

Here u_1 and u_2 indicate the number of sites that the local terms act on (i.e., occupy). Therefore, $\frac{1}{m}\mathbb{E}\left[\mathrm{Tr}\left(AQ_q^TBQ_q\right)^2\right]$ is obtained by multiplying each type of terms, weighted by the counts and summing. For example for $u_1 = 5$ and $u_2 = 3$, when N is odd,

$$\begin{aligned}\tfrac{1}{m}\mathbb{E}\left[\mathrm{Tr}\left(AQ_q^T BQ_q\right)^2\right] &= \tfrac{1}{m}\Big\{d^{N-5}k^2\left(k-1\right)^2\mathbb{E}\mathrm{Tr}\left(H^{(l)}\right)^4 + \\ 2k^2\left(k-1\right)d^{N-4}\mathbb{E}\mathrm{Tr}\left(\left[H^{(l)}\right]^2\right)\mathbb{E}\mathrm{Tr}\left(H^{(l)}\right)^2 &+ \left(k-1\right)^2 d^{N-4}\left\{\mathbb{E}\mathrm{Tr}\left(\left[H^{(l)}\right]^2\right)\right\}^2 + \\ \left(2k-1\right)d^{N-3}\mathbb{E}\mathrm{Tr}&\left[\left(H^{(l)}\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes H^{(l+1)}\right)\left(H^{(l)}\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes H^{(l+1)}\right)\right]\Big\}\end{aligned} \tag{33}$$

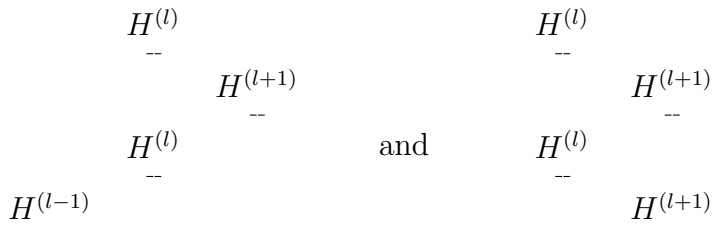
and similarly for N even,

$$\begin{aligned}\tfrac{1}{m}\mathbb{E}\left[\mathrm{Tr}\left(AQ_q^T BQ_q\right)^2\right] &= \tfrac{(k-1)}{m}\Big\{k\left(k-1\right)\left(k-2\right)d^{N-5}\mathbb{E}\mathrm{Tr}\left(H^{(l)}\right)^4 + \\ k\left(2k-3\right)d^{N-4}\mathbb{E}\mathrm{Tr}\left(\left[H^{(l)}\right]^2\right)\mathbb{E}\mathrm{Tr}&\left(H^{(l)}\right)^2 + \left(k-2\right)d^{N-4}\left\{\mathbb{E}\mathrm{Tr}\left(\left[H^{(l)}\right]^2\right)\right\}^2 + \\ 2d^{N-3}\mathbb{E}\mathrm{Tr}&\left[\left(H^{(l)}\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes H^{(l+1)}\right)\left(H^{(l)}\otimes\mathbb{I}\right)\left(\mathbb{I}\otimes H^{(l+1)}\right)\right]\Big\}.\end{aligned} \tag{34}$$

The expectation values depend on the type of random matrix distribution from which the local terms are drawn. We will give explicit examples in the following sections. In the following lemma, we use $\mathbb{E}\left(H^{(l)}\right) = \mu\mathbb{I}_{d^2}$ and $\mathbb{E}\left(H^{(l)}\right)^2 = m_2\mathbb{I}_{d^2}$.

Lemma 2. *In calculating the $\mathbb{E}\,Tr\left(AQ_q^T BQ_q\right)^2$ if at least one of the odds (evens) commutes with one of the evens (odds) then the expectation value is the same as the classical expectation value. Further if the local terms have permutation invariance of eigenvalues then the only quantum expectation value that differs from classical is of Type II (see the proof and the diamonds in figure 6).*

Proof. This can be shown using the trace property $\mathrm{Tr}\left(MP\right) = \mathrm{Tr}\left(PM\right)$. In calculating $\mathbb{E}\mathrm{Tr}\left(H_l^{odd}H_p^{even}H_j^{odd}H_k^{even}\right)$; if any of the odd (even) terms commutes with any of the even (odd) terms to its left or right then they can be swapped. For example one gets $\mathbb{E}\mathrm{Tr}\left(H_l^{odd}H_p^{even}H_k^{even}H_j^{odd}\right) = \mathbb{E}\mathrm{Tr}\left(H_j^{odd}H_l^{odd}H_p^{even}H_k^{even}\right)$ which is just the classical value. Hence the only types of expectations that we need to worry about are



Type I Type II

now we show that with permutation invariance of the local eigenvalues the first type are also classical leaving us with the "diamond terms" alone (Fig. 6). Consider a Type I term, which involves three independent local terms,

$$\begin{aligned}\tfrac{1}{m}\mathbb{E}\mathrm{Tr}\big[\big(\mathbb{I}_{d^2}\otimes H^{(3)}\otimes\mathbb{I}_{d^{N-4}}\big)\big(\mathbb{I}\otimes H^{(2)}\otimes\mathbb{I}_{d^{N-3}}\big)\big(\mathbb{I}_{d^2}\otimes H^{(3)}\otimes\mathbb{I}_{d^{N-4}}\big)\big(\mathbb{I}_{d^3}\otimes H^{(4)}\otimes\mathbb{I}_{d^{N-5}}\big)\big]\\ =\mu^2 m_2.\end{aligned}$$

This follows immediately from the independence of $H^{(4)}$, which allows us to take its expectation value separately giving a μ and leaving us with

$$\frac{\mu}{m}\mathbb{E}\mathrm{Tr}\left[\left(\mathbb{I}_{d^2}\otimes H^{(3)}\otimes\mathbb{I}_{d^{N-4}}\right)^2\left(\mathbb{I}\otimes H^{(2)}\otimes\mathbb{I}_{d^{N-3}}\right)\right]=\mu^2 m_2.$$

Therefore the only relevant terms, shown by diamonds in Fig. 6, are of Type II. As an example of such terms consider (here on repeated indices are summed over)

$$\begin{aligned}&\tfrac{1}{m}\mathbb{E}\mathrm{Tr}\big[\big(H^{(1)}\otimes\mathbb{I}_{d^{N-2}}\big)\big(\mathbb{I}\otimes H^{(2)}\otimes\mathbb{I}_{d^{N-3}}\big)\big(H^{(1)}\otimes\mathbb{I}_{d^{N-2}}\big)\big(\mathbb{I}\otimes H^{(2)}\otimes\mathbb{I}_{d^{N-3}}\big)\big]\\ &=\tfrac{1}{d^3}\left\{\mathbb{E}\left(H^{(1)}_{i_1i_2,j_1j_2}H^{(1)}_{i_1p_2,j_1k_2}\right)\mathbb{E}\left(H^{(2)}_{j_2i_3,k_2k_3}H^{(2)}_{i_2i_3,p_2k_3}\right)\right\},\end{aligned}\tag{35}$$

where the indices with subscript 2 prevent us from treating the two expectation values independently: $H^{(1)}$ and $H^{(2)}$ overlap at the second site. The number of such terms is $2k-1$, where $k=\frac{N-1}{2}$. □

Therefore, we have found a further reduction of the terms from the departure theorem, that distinguishes the quantum problem from the other two. Luckily and interestingly the kurtosis of the quantum case lies in between the classical and the iso. We emphasize that *the only inputs to the theory are the geometry of the lattice (e.g., the number of summands and the inter-connectivity of the local terms) and the moments* that characterizes the type of the local terms.

Comment: The most general treatment would consider Type I terms as well, i.e., there is no assumption of permutation invariance of the eigenvalues of the local terms. This allows one to treat all types of local terms. Here we are confining to random local interactions, where the local eigenvectors are generic or the eigenvalues locally are permutation invariant in the expectation value sense.

The goal is to find p by matching fourth moments

$$1-p=\frac{\mathbb{E}\mathrm{Tr}\left(A\Pi^TB\Pi\right)^2-\mathbb{E}\mathrm{Tr}\left(AQ_q^TBQ_q\right)^2}{\mathbb{E}\mathrm{Tr}\left(A\Pi^TB\Pi\right)^2-\mathbb{E}\mathrm{Tr}\left(AQ^TBQ\right)^2}$$

for which we calculated the denominator resulting in Eq. 30, where $\mathbb{E}\left(|q_{i,j}|^4\right)=\frac{\beta+2}{m(m\beta+2)}$ for $\beta-$Haar Q (Table IV). If the numerator allows a factorization of the moments of the local terms as in Eq. 30, then the value of p will be independent of the covariance matrix (i.e., eigenvalues of the local terms).

Lemma. (Universality) *$p\mapsto p\left(N,d,\beta\right)$, namely, it is independent of the distribution of the local terms.*

Proof. We use a similar techniques as we did in the isotropic case. The general form for the numerator of Eq. 24 is (denoting Lemma 2 by L3)

$$\begin{aligned}\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(A\Pi^TB\Pi\right)^2-\left(AQ_q^TBQ_q\right)^2\right] &\overset{\text{L3}}{=} \frac{(2k-1)}{d^3}\mathbb{E}\mathrm{Tr}\Big\{\left(H^{(l)}\otimes\mathbb{I}_d\right)^2\left(\mathbb{I}_d\otimes H^{(l+1)}\right)^2\\ &\quad-\left[\left(H^{(l)}\otimes\mathbb{I}_d\right)\left(\mathbb{I}_d\otimes H^{(l+1)}\right)\right]^2\Big\}\\ &= \frac{(2k-1)}{d^3}\mathbb{E}\mathrm{Tr}\Big\{\left(Q_l^{-1}\Lambda_lQ_l\otimes\mathbb{I}_d\right)^2\left(\mathbb{I}_d\otimes Q_{l+1}^{-1}\Lambda_{l+1}Q_{l+1}\right)^2\\ &\quad-\left[\left(Q_l^{-1}\Lambda_lQ_l\otimes\mathbb{I}_d\right)\left(\mathbb{I}_d\otimes Q_{l+1}^{-1}\Lambda_{l+1}Q_{l+1}\right)\right]^2\Big\} \qquad (36)\end{aligned}$$

where the expectation on the right hand side is taken with respect to the local terms $H^{(l)}$ and $H^{(l+1)}$. The right hand side is a homogeneous polynomial of order two in the entries of Λ_l, as well as, in the entries of Λ_{l+1}; consequently Eq. 36 necessarily has the form

$$c_1\left(\Lambda^{\text{even}},Q_{\text{odd}},Q_{\text{even}}\right)m_2^{\text{odd}}+c_2\left(H^{\text{even}},Q_{\text{odd}},Q_{\text{even}}\right)m_{1,1}^{\text{odd}}$$

but Eq. 36 must be zero for $\Lambda_l=I$, for which $m_2^{\text{odd}}=m_{1,1}^{\text{odd}}=1$. This implies that $c_1=-c_2$. By permutation invariance of the local terms we can factor out $\left(m_2^{\text{odd}}-m_{1,1}^{\text{odd}}\right)$. Similarly, the homogeneity and permutation invariance of $H^{(l+1)}$ implies,

$$\left(m_2^{\text{odd}}-m_{1,1}^{\text{odd}}\right)\left[D_1\left(Q_{\text{odd}},Q_{\text{even}}\right)m_2^{\text{even}}+D_2\left(Q_{\text{odd}},Q_{\text{even}}\right)m_{1,1}^{\text{even}}\right].$$

The right hand side should be zero for $\Lambda_{l+1}=I$, whereby we can factor out $\left(m_2^{\text{even}}-m_{1,1}^{\text{even}}\right)$; hence the right hand side of Eq. 36 becomes

$$\frac{(2k-1)}{d^3}\left(m_2^{\text{odd}}-m_{1,1}^{\text{odd}}\right)\left(m_2^{\text{even}}-m_{1,1}^{\text{even}}\right)f_q\left(Q_{\text{odd}},Q_{\text{even}}\right) \tag{37}$$

where $f_q\left(Q_{\text{odd}},Q_{\text{even}}\right)$ is a homogeneous function of order four in the entries of Q_{odd} as well as Q_{even}. To evaluate f_q, it suffices to let Λ_l and Λ_{l+1} be projectors of rank one where Λ_l would have only one nonzero entry on the i^{th} position on its diagonal and Λ_{l+1} only one nonzero entry on the j^{th} position on its diagonal. Further take those nonzero entries to be ones, giving $m_{1,1}^A = m_{1,1}^B = 0$ and $m_2^A = m_2^B = 1/n$. Using this choice of local terms the right hand side of Eq. 36 now reads

$$\begin{aligned}\frac{(2k-1)}{d^3}\ \mathbb{E}\text{Tr}\ \Bigg\{&\left(|q_i^{(l)}\rangle\langle q_i^{(l)}|\otimes I_d\right)^2\left(I_d\otimes|q_j^{(l+1)}\rangle\langle q_j^{(l+1)}|\right)^2\\&-\left[\left(|q_i^{(l)}\rangle\langle q_i^{(l)}|\otimes I_d\right)\left(I_d\otimes|q_j^{(l+1)}\rangle\langle q_j^{(l+1)}|\right)\right]^2\Bigg\}\end{aligned} \tag{38}$$

where here the expectation value is taken with respect to random choices of local eigenvectors. Equating this and Eq. 37

$$\begin{aligned}f_q\left(Q_{\text{odd}},Q_{\text{even}}\right)\ =\ n^2\mathbb{E}\text{Tr}\Bigg\{&\left(|q_i^{(l)}\rangle\langle q_i^{(l)}|\otimes I_d\right)^2\left(I_d\otimes|q_j^{(l+1)}\rangle\langle q_j^{(l+1)}|\right)^2\\&-\left[\left(|q_i^{(l)}\rangle\langle q_i^{(l)}|\otimes I_d\right)\left(I_d\otimes|q_j^{(l+1)}\rangle\langle q_j^{(l+1)}|\right)\right]^2\Bigg\}\end{aligned} \tag{39}$$

To simplify notation let us expand these vectors in the computational basis $|q_i^{(l)}\rangle = u_{i_1 i_2}|i_1\rangle|i_2\rangle$ and $|q_j^{(l+1)}\rangle = v_{i_2 i_3}|i_2\rangle|i_3\rangle$. The first term on the right hand side of Eq. 38, the classical term, is obtained by assuming commutativity and using the projector properties,

$$
\begin{aligned}
\mathrm{Tr}\left[\left(|q_i^{(l)}\rangle\langle q_i^{(l)}|\otimes I_d\right)^2\left(I_d\otimes|q_j^{(l+1)}\rangle\langle q_j^{(l+1)}|\right)^2\right] &= \\
\mathrm{Tr}\left[\left(|q_i^{(l)}\rangle\langle q_i^{(l)}|\otimes I_d\right)\left(I_d\otimes|q_j^{(l+1)}\rangle\langle q_j^{(l+1)}|\right)\right] &= \\
\mathrm{Tr}\left[u_{i_1,i_2}\overline{u_{j_1j_2}}v_{j_2i_3}\overline{v_{k_2k_3}}u_{j_1k_2}|i_1i_2i_3\rangle\langle j_1k_2k_3|\right] &= \\
[u_{i_1,i_2}\overline{u_{i_1j_2}}v_{j_2i_3}\overline{v_{i_2i_3}}] = \left(u^\dagger u\right)_{j_2i_2}\left(vv^\dagger\right)_{j_2i_2} &= \\
\mathrm{Tr}\left[\left(u^\dagger u\right)\left(vv^\dagger\right)\right] = \mathrm{Tr}\left[uv\left(uv\right)^\dagger\right] &= \\
\|uv\|_{\mathrm{F}}^2 &= \sum_{i=1}^{d}\sigma_i^2. \qquad (40)
\end{aligned}
$$

where $\|.\|_{\mathrm{F}}$ denotes the Frobenius norm and σ_i are the singular values of uv. The second term, the quantum term, is

$$
\begin{aligned}
\mathrm{Tr}\left[\left(|q_i^{(l)}\rangle\langle q_i^{(l)}|\otimes I_d\right)\left(I_d\otimes|q_j^{(l+1)}\rangle\langle q_j^{(l+1)}|\right)\right]^2 &= \qquad (41)\\
\mathrm{Tr}\left[u_{i_1i_2}\overline{u_{j_1j_2}}v_{j_2i_3}\overline{v_{k_2k_3}}u_{j_1k_2}\overline{u_{m_1m_2}}v_{m_2k_3}\overline{v_{i_2i_3}}|i_1i_2i_3\rangle\langle p_1p_2p_3|\right] &= \\
\left(u^\dagger u\right)_{j_2k_2}\left(vv^\dagger\right)_{m_2k_2}\left(u^\dagger u\right)_{m_2i_2}\left(vv^\dagger\right)_{j_2i_2} &= \\
\left(u^\dagger uvv^\dagger\right)_{j_2m_2}\left(u^\dagger uvv^\dagger\right)_{m_2j_2} = \mathrm{Tr}\left\{\left[uv\left(uv\right)^\dagger\right]^2\right\} &= \\
\left\|uv\left(uv\right)^\dagger\right\|_{\mathrm{F}}^2 &= \sum_{i=1}^{d}\sigma_i^4.
\end{aligned}
$$

where we used the symmetry of $\left(uv\left(uv\right)^\dagger\right)^2 = uv\left(uv\right)^\dagger\left[uv\left(uv\right)^\dagger\right]^\dagger$.

Now we can calculate

$$
f_q\left(Q_{\mathrm{odd}},Q_{\mathrm{even}}\right) = n^2\mathbb{E}\left\{\|uv\|_{\mathrm{F}}^2 - \left\|uv\left(uv\right)^\dagger\right\|_{\mathrm{F}}^2\right\} \qquad (42)
$$

giving us the desired result

$$
\begin{aligned}
\frac{1}{m}\mathbb{E}\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ_q^T BQ_q\right)^2\right] &= d\left(2k-1\right)\left(m_2^{\mathrm{odd}} - m_{1,1}^{\mathrm{odd}}\right)\left(m_2^{\mathrm{even}} - m_{1,1}^{\mathrm{even}}\right) \qquad (43)\\
&\times\ \mathbb{E}\left(\|uv\|_{\mathrm{F}}^2 - \left\|uv\left(uv\right)^\dagger\right\|_{\mathrm{F}}^2\right),
\end{aligned}
$$

from which

$$
\begin{aligned}
1-p \;&=\; \frac{\mathrm{ETr}\left(A\Pi^T B\Pi\right)^2-\mathrm{ETr}\left(AQ_q^{-1}BQ_q\right)^2}{\mathrm{ETr}\left(A\Pi^T B\Pi\right)^2-\mathrm{ETr}\left(AQ^{-1}BQ\right)^2} \\
&=\; \frac{d\left(2k-1\right)\mathbb{E}\left(\left\|uv\right\|_{\mathrm{F}}^2-\left\|uv\left(uv\right)^\dagger\right\|_{\mathrm{F}}^2\right)}{\left(\frac{km(n-1)}{n(m-1)}\right)^2\left\{1-m\mathbb{E}\left(q_{ij}^4\right)\right\}}.
\end{aligned}
\tag{44}
$$

The dependence on the covariance matrix has cancelled- a covariance matrix is one whose element in the i,j position is the covariance between the i^{th} and j^{th} eigenvalue. This shows that p is independent of eigenvalues of the local terms which proves the universality lemma.

Comment: To get the numerator we used permutation invariance of A and B and local terms, to get the denominator we used permutation invariance of Q.

□

Comment: It is interesting that the amount of mixture of the two extremes needed to capture the quantum spectrum is independent of the actual types of local terms. It only depends on the physical parameters of the lattice.

E. The Slider Theorem and a Summary

In this section we make explicit use of $\beta-$Haar properties of Q and local terms. To prove that there exists a $0\leq p\leq 1$ such that the combination in Eq. 14 is convex we need to evaluate the expected Frobenius norms in Eq. 43.

Lemma 3. $\mathbb{E}\left\|uv\right\|_F^2=1/d$ *and* $\mathbb{E}\left\|uv\left(uv\right)^\dagger\right\|_F^2=\frac{\beta^2[3d(d-1)+1]+2\beta(3d-1)+4}{d(\beta d^2+2)^2}$, *when local terms have* $\beta-$*Haar eigenvectors.*

Proof. It is a fact that $G=u\chi_{\beta d^2}$, when u is uniform on a sphere, G is a $d\times d$ $\beta-$Gaussian matrix whose expected Frobenius norm has a $\chi-$distribution denoted here by $\chi_{\beta d^2}$ (similarly for v). Recall that $\mathbb{E}\left(\chi_h^2\right)=h$ and $\mathbb{E}\left(\chi_h^4\right)=h\left(h+2\right)$.

Notation	Type	Count
X	$i \neq k \;\&\; j \neq l$	$d^2 (d-1)^2$
Y	$i = k \;\&\; j \neq l$ or $i \neq k \;\&\; j = l$	$2d^2 (d-1)$
Z	$i = k \;\&\; j = l$	d^2

Table V: Expectation values.

$$\begin{aligned} \mathbb{E}\left\|uv\right\|_{\mathrm{F}}^2 \;\mathbb{E}\left(\chi_{\beta d^2}\right)^2 &= \mathbb{E}\left\|(G_1 G_2)\right\|_{\mathrm{F}}^2 \qquad (45) \\ \Rightarrow \mathbb{E}\left\|uv\right\|_{\mathrm{F}}^2 &= \frac{1}{\left(\beta d^2\right)^2}\mathbb{E}\left\|G_1 G_2\right\|_{\mathrm{F}}^2 = \frac{d^2}{\left(\beta d^2\right)^2}\mathbb{E}\sum_{k=1}^{d}\left(g_{i,k}^{(1)} g_{kj}^{(2)}\right)^2 \\ &= \frac{d^2}{\left(\beta d^2\right)^2} d\left(\beta\right)^2 = \frac{1}{d}. \end{aligned}$$

The quantum case, $u^\dagger u = \frac{G_1^\dagger G_1}{\|G_1\|_{\mathrm{F}}^2} \equiv \frac{W_1}{\|G_1\|_{\mathrm{F}}^2}$, similarly $v^\dagger v = \frac{G_2^\dagger G_2}{\|G_2\|_{\mathrm{F}}^2} \equiv \frac{W_2}{\|G_2\|_{\mathrm{F}}^2}$, where W_1 and W_2 are Wishart matrices.

$$\mathbb{E}\left\|uv\left(uv\right)^\dagger\right\|_{\mathrm{F}}^2 = \frac{\mathbb{E}\mathrm{Tr}\left(W_1 W_2\right)^2}{\mathbb{E}\left(\chi_{d^2\beta}^4\right)\mathbb{E}\left(\chi_{d^2\beta}^4\right)} = \frac{\mathbb{E}\mathrm{Tr}\left(W_1 W_2\right)^2}{\left[d^2\beta\left(d^2\beta+2\right)\right]^2} \qquad (46)$$

hence the complexity of the problem is reduced to finding the expectation of the trace of a product of Wishart matrices.

$$\mathbb{E}\mathrm{Tr}\left(W_1 W_2\right)^2 = \mathbb{E}\mathrm{Tr}\left(W_1 W_2 W_1 W_2\right) = \mathbb{E}\sum_{1\leq ijkl\leq d} x_i x_i^\dagger y_j y_j^\dagger x_k x_k^\dagger y_l y_l^\dagger \equiv \Pi\begin{bmatrix} x_i^\dagger y_j & y_l^\dagger x_i \\ y_j^\dagger x_k & x_k^\dagger y_l \end{bmatrix}, \quad (47)$$

where Π denotes the product of the elements of the matrix. There are three types of expectations summarized in Table V.

In Table V

$$
\begin{aligned}
X &\equiv \mathbb{E}\left[\Pi\left(x_i x_k\right)\left(y_i y_l\right)\right] \\
Y &\equiv \mathbb{E}\left[\left(x_i^\dagger y_j\right)^2\left(x_i^\dagger y_l\right)^2\right] \\
Z &\equiv \mathbb{E}\left[\left(x_i^\dagger y_i\right)^4\right].
\end{aligned}
$$

We now evaluate these expectation values. We have

$$
X = \Pi \begin{pmatrix} \chi_{\beta d} & g_\beta \\ 0 & \chi_{\beta(d-1)} \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}^\dagger \begin{pmatrix} g_\beta & g_\beta \\ g_\beta & g_\beta \\ \mathrm{DC} & \mathrm{DC} \\ \vdots & \vdots \end{pmatrix}
$$

by QR decomposition, where g_β and χ_h denote an element with a $\beta-$Gaussian and χ_h distribution respectively; DC means "Don't Care". Consequently

$$
\begin{aligned}
X &= \Pi \begin{pmatrix} \chi_{\beta d} & g_\beta \\ 0 & \chi_{\beta(d-1)} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \Pi \begin{bmatrix} a\chi_{\beta d} & g_\beta a + \chi_{\beta(d-1)} c \\ b\chi_{\beta d} & g_\beta b + \chi_{\beta(d-1)} d \end{bmatrix} = \Pi \begin{bmatrix} a\chi_{\beta d} & g_\beta a \\ b\chi_{\beta d} & g_\beta b \end{bmatrix} \\
&= \chi_{\beta d}^2 a^2 b^2 g_\beta^2 = \beta^4 d.
\end{aligned}
$$

where we denoted the four independent Gaussian entries by a, b, c, d to not confuse them as one number. From Eq. 47 we have

$$
\begin{aligned}
Y &= \mathbb{E}\left[\left(x_i^\dagger y_j\right)^2\left(x_i^\dagger y_l\right)^2\right] = \mathbb{E}\left(\chi_{d\beta} g_\beta^{(1)}\right)^2\left(\chi_{d\beta} g_\beta^{(2)}\right)^2 = \beta d\left(\beta d + 2\right)\beta^2 \\
Z &= \mathbb{E}\left(x^\dagger y\right)^4 = \mathbb{E}\left(\chi_{\beta d}^4\right)\mathbb{E}\left(\chi_\beta^4\right) = \beta d\left(\beta d + 2\right)\beta\left(\beta + 2\right).
\end{aligned}
$$

Eq. 46 now reads

$$\mathbb{E}\left\|uv\,(uv)^{\dagger}\right\|_F^2 = \frac{\beta^2\left[3d\,(d-1)+1\right]+2\beta\,(3d-1)+4}{d\,(\beta d^2+2)^2}. \tag{48}$$

□

Theorem. (The Slider Theorem) *The quantum kurtosis lies in between the classical and the iso kurtoses,* $\gamma_2^{iso} \leq \gamma_2^q \leq \gamma_2^c$. *Therefore there exists a* $0 \leq p \leq 1$ *such that* $\gamma_2^q = p\gamma_2^c + (1-p)\,\gamma_2^{iso}$. *Further,* $\lim_{N\to\infty} p = 1$.

Proof. We have $\left\{1 - \frac{1}{m}\sum_{ij=1}^{m} q_{ij}^4\right\} \geq 0$, since $\sum_{ij} q_{ij}^4 \leq \sum_{ij} q_{ij}^2 = m$. The last inequality follows from $q_{ij}^2 \leq 1$. Therefore, Eq. 20 is

$$\begin{aligned}\gamma_2^{iso} - \gamma_2^c \;=\; & \frac{2}{\sigma^4}\left(m_2^{\text{odd}} - m_{11}^{\text{odd}}\right)\left(m_2^{\text{even}} - m_{11}^{\text{even}}\right) \times \\ & \left(\frac{km\,(n-1)}{n\,(m-1)}\right)^2 \left\{m\mathbb{E}\left(q_{11}^4\right) - 1\right\} \leq 0.\end{aligned}$$

From Eqs. 41 and 40 and using the fact that the singular values $\sigma_i \leq 1$ we have

$$\left\|uv\,(uv)^{\dagger}\right\|_{\mathrm{F}}^2 = \sum_{i=1}^{d}\sigma_i^4 \leq \sum_{i=1}^{d}\sigma_i^2 = \|uv\|_{\mathrm{F}}^2$$

which proves $\gamma_2^q - \gamma_2^c \leq 0$. In order to establish $\gamma_2^{iso} \leq \gamma_2^q \leq \gamma_2^c$, we need to show that $\gamma_2^c - \gamma_2^q \leq \gamma_2^c - \gamma_2^{iso}$. Eq. 44 after substituting $m\mathbb{E}\left(q_{ij}^4\right) = \frac{\beta+2}{(m\beta+2)}$ from Table IV and Eqs. 45, 48 reads

$$1-p = \left(1-d^{-2k-1}\right)\left[1-\left(\frac{k-1}{k}\right)^2\right]\left\{\left(1-\frac{1-d^{-2k+1}}{1+\beta d^2/2}\right)\left(\frac{d}{d+1}\right)^2\left[\frac{\beta\,(d^3+d^2-2d+1)+4d-2}{(d-1)\,(\beta d^2+2)}\right]\right\} \tag{49}$$

We want to show that $0 \leq 1-p \leq 1$ for any integer $k \geq 1$, $d \geq 2$ and $\beta \geq 1$. All the factors are manifestly ≥ 0, therefore $1-p \geq 0$. The first two factors are clearly ≤ 1 so we need to prove that the term in the braces is too. Further, $k = 1$ provides an upper bound as $\left(1 - \frac{1-d^{-2k+1}}{1+\beta d^2/2}\right) \leq \left(1 - \frac{1-d^{-3}}{1+\beta d^2/2}\right)$. We rewrite the term in the braces

$$\frac{d\,(\beta d^3+2)\left[\beta\,(d^3+d^2-2d+1)+4d-2\right]}{(\beta d^2+2)^2\,(d+1)^2\,(d-1)}, \tag{50}$$

but we can subtract the denominator from the numerator to get

$$(\beta d + 2)\left[\beta\left(d^4 - 2d^3\right) + 2\left(d^3 - d^2 - 1\right)\right] \geq 0 \quad \forall\, d \geq 2.$$

This proves that (50) is less than one. Therefore, the term in the braces is less than one and hence $0 \leq p \leq 1$. Let us note the following limits of interest (recall $N - 1 = 2k$)

$$\begin{aligned} \lim_{d\to\infty} (1-p) &= \frac{2k-1}{k^2} \overset{k=1}{=} 1 \\ \lim_{N\to\infty} (1-p) &\sim \frac{1}{N} \to 0 \end{aligned}$$

the first limit tells us that if we consider having two local terms and take the local dimension to infinity we have essentially free probability theory as expected. The second limit shows that in the thermodynamical limit (i.e., $N \to \infty$) the convex combination slowly approaches the classical end. In the limit where $\beta \to \infty$ the β dependence in $(1-p)$ cancels out. This is a reconfirmation of the fact that in free probability theory, for $\beta \to \infty$, the result should be independent of β. We see that the bounds are tight.

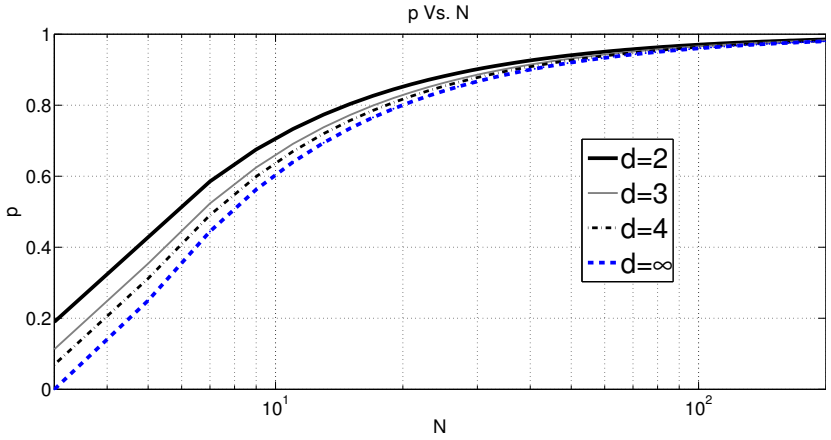


Figure 7: An example: $\beta = 1$: the quantum problem for all d lies in between the iso ($p = 0$) and the classical ($p = 1$).

□

Comment: The non-classical behavior shows itself starting at the fourth moment; further, in the expansion of the fourth moments only the terms that involve *a pair* of local terms *sharing a site* differ. Note that when the QMBS possesses a translational symmetry, there

is an additional complication introduced by the dependence of the local terms. Though, in this case, the non-iid nature of the local terms complicates the matter theoretically, we have not seen a practical limitation of IE in our numerical experiments.

Comment: One could from the beginning use free approximation instead of isotropic ($m \to \infty$), in which case the proofs are simplified.

We now *summarize* the main thesis of this work. We are interested in the eigenvalue distribution of

$$H \equiv H_{\text{odd}} + H_{\text{even}} = \sum_{l=1,3,5,\cdots} \mathbb{I} \otimes H_{l,l+1} \otimes \mathbb{I} + \sum_{l=2,4,6,\cdots} \mathbb{I} \otimes H_{l,l+1} \otimes \mathbb{I},$$

which in a basis in that H_{odd} is diagonal reads $H = A + Q_q^{-1} B Q_q$. Since this problem has little hope in being solved exactly we consider along with it two known approximations:

$$\begin{aligned} H_c &= A + \Pi^{-1} B \Pi \\ H &= A + Q_q^{-1} B Q_q \\ H_{iso} &= A + Q^{-1} B Q. \end{aligned}$$

We proved that the first three moments of the three foregoing equations are equal. We then calculated their fourth moments as encoded by their kurtoses (γ_2's) analytically and proved that there exists a $0 \leq p \leq 1$ such that

$$\gamma_2^q = p\gamma_2^c + (1-p)\,\gamma_2^{iso}.$$

It turned out that the only terms in the expansion of the fourth moments that were relevant were

$$1 - p = \frac{\mathbb{E}\text{Tr}\left\{ \left(A\Pi^{-1}B\Pi\right)^2 - \left(AQ_q^{-1}BQ_q\right)^2 \right\}}{\mathbb{E}\text{Tr}\left\{ \left(A\Pi^{-1}B\Pi\right)^2 - \left(AQ^{-1}BQ\right)^2 \right\}}. \tag{51}$$

Through direct calculation we found that the numerator $\mathbb{E}\text{Tr}\left\{ \left(A\Pi^{-1}B\Pi\right)^2 - \left(AQ_q^{-1}BQ_q\right)^2 \right\}$ evaluates to be

$$d\left(2k-1\right)\left(m_2^{\text{odd}}-m_{1,1}^{\text{odd}}\right)\left(m_2^{\text{even}}-m_{1,1}^{\text{even}}\right)\mathbb{E}\left(\|uv\|_{\text{F}}^2-\left\|uv\left(uv\right)^{\dagger}\right\|_{\text{F}}^2\right),$$

and the denominator $\mathbb{E}\text{Tr}\left\{\left(A\Pi^{-1}B\Pi\right)^2-\left(AQ^{-1}BQ\right)^2\right\}$

$$\left(m_2^{\text{odd}}-m_{1,1}^{\text{odd}}\right)\left(m_2^{\text{even}}-m_{1,1}^{\text{even}}\right)\left(\frac{km\left(n-1\right)}{n\left(m-1\right)}\right)^2\left\{1-m\mathbb{E}\left(q_{ij}^4\right)\right\}.$$

Therefore $1-p$ does not depend on the local distribution and can generally be expressed as

$$1-p=\frac{d\left(2k-1\right)\mathbb{E}\left(\|uv\|_{\text{F}}^2-\left\|uv\left(uv\right)^{\dagger}\right\|_{\text{F}}^2\right)}{\left(\frac{km(n-1)}{n(m-1)}\right)^2\left\{1-m\mathbb{E}\left(q_{ij}^4\right)\right\}}.$$

If we further assume that the local eigenvectors are $\beta-$Haar distributed we get

$$\begin{aligned}1-p\;=\;&\left(1-d^{-2k-1}\right)\left[1-\left(\frac{k-1}{k}\right)^2\right]\left(1-\frac{1-d^{-2k+1}}{1+\beta d^2/2}\right)\left(\frac{d}{d+1}\right)^2\\&\times\;\left[\frac{\beta\left(d^3+d^2-2d+1\right)+4d-2}{\left(d-1\right)\left(\beta d^2+2\right)}\right].\end{aligned}$$

Next we asserted that this p can be used to approximate the distribution

$$d\nu^q\approx d\nu^{IE}=pd\nu^c+\left(1-p\right)d\nu^{iso}.$$

We argued that the spectra obtained using Isotropic Entanglement (IE) are accurate well beyond four moments.

For illustration, we apply IE theory in full detail to a chain with Wishart matrices as local terms. Other types of local terms (e.g. GOE, random ±1 eigenvalues) can be treated similarly; therefore in Section VI we show the plots comparing IE with exact diagonalization for these cases.

V. A DETAILED EXAMPLE: WISHART MATRICES AS LOCAL TERMS

As an example take a chain with odd number of sites and for the local terms in Eq. 1 pick $H^{(l)} = W^T W$, where W is a rank r matrix whose elements are picked randomly from a Gaussian distribution ($\beta = 1$); these matrices $W^T W$ are known as *Wishart matrices.* Clearly the maximum possible rank is $r = d^2$ for each of the local terms.

Any cumulant is equal to the corresponding cumulant of one local term, denoted by κ, times the number of summands in Eq. 1. In particular, the fourth cumulant of H is $\kappa_4^{(N-1)} = (N-1)\,\kappa_4$. Below we drop the superscripts when the quantity pertains to the whole chain. Next we recall the definitions in terms of cumulants of the mean (μ), the variance (σ^2), the skewness (γ_1), and the kurtosis (γ_2)

$$\mu \equiv \kappa_1 \qquad \sigma^2 \equiv \kappa_2 \qquad \gamma_1 \equiv \tfrac{\kappa_3}{\sigma^3} \qquad \gamma_2 \equiv \tfrac{\kappa_4}{\sigma^4} = \tfrac{m_4}{\sigma^4} - 3 \ . \tag{52}$$

A. Evaluation of $p = \frac{\gamma_2^q - \gamma_2^{iso}}{\gamma_2^c - \gamma_2^{iso}}$

The moments of the local terms are obtained from MOPS [41];

$$\begin{gathered}
m_1 = \beta r \\
m_2 = \beta r \left[\beta \left(r + n - 1\right) + 2\right] \\
m_3 = \beta r \left\{\beta^2 \left[n^2 + (r-1)\left(3n + r - 2\right)\right] + 6\beta \left(n + r - 1\right) + 8\right\} \\
m_4 = \beta r \left\{48 + \beta^3 \left[n^3 + 6n^2(r-1) + n\left(6r - 11\right)(r-1) - 6\left(r^2+1\right) + r^3 + 11r\right]\right. \\
\left. +2\beta^2 \left[6\left(n^2 + r^2\right) + 17\left(n(r-1) - r\right) + 11\right] + 44\beta\left(n + r - 1\right)\right\} \\
m_{1,1} = \beta^2 r \left(r - 1\right)
\end{gathered} \tag{53}$$

which for real matrices $\beta = 1$ yields

$$\begin{gathered}
m_1 = r \\
m_2 = r\left(r + n + 1\right) \\
m_3 = r\left(n^2 + 3n + 3rn + 3r + r^2 + 4\right) \\
m_4 = r\left(6n^2 + 21n + 6rn^2 + 17rn + 21r + 6nr^2 + 6r^2 + n^3 + r^3 + 20\right) \\
m_{1,1} = r\left(r - 1\right).
\end{gathered} \tag{54}$$

The mean, variance, skewness, and kurtosis are obtained from the foregoing relations, through the cumulants Eq. 52. We drop the superscripts when the quantity pertains to the whole chain. Therefore, using Eq. 52, we have

$$\begin{array}{cc} \mu \equiv (N-1)\, r & \sigma^2 \equiv r\,(N-1)\,(n+1) \\ \gamma_1 \equiv \frac{n^2+3n+4}{(n+1)^{3/2}\sqrt{r(N-1)}} & \gamma_2^{(c)} \equiv \frac{n^2(n+6)-rn(n+1)+21n+2r+20}{r(N-1)(n+1)^2}. \end{array} \tag{55}$$

From Eq. 22 we readily obtain

$$\frac{1}{m}\mathbb{E}\mathrm{Tr}\left(A\Pi^T B\Pi\right)^2 = r^2k^2\left(rk+n+1\right)^2. \tag{56}$$

By The Matching Three Moments theorem we immediately have the mean, the variance and the skewness for the isotropic case

$$\begin{array}{cc} \mu = (N-1)\, r & \sigma^2 = r\,(N-1)\,(n+1) \\ \gamma_1 = \frac{n^2+3n+4}{(n+1)^{3/2}\sqrt{r(N-1)}}. & \end{array}$$

Note that the denominator in Eq. 14 becomes,

$$\gamma_2^c - \gamma_2^{iso} = \frac{\kappa_4^{(c)} - \kappa_4^{(iso)}}{\sigma^4} = \frac{2}{m}\frac{\mathbb{E}\left\{\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ^T BQ\right)^2\right]\right\}}{r^2\left(N-1\right)^2\left(n+1\right)^2}. \tag{57}$$

In the case of Wishart matrices, $m_1^{\mathrm{odd}} = m_1^{\mathrm{even}} = r$, and $m_2^{\mathrm{odd}} = m_2^{\mathrm{even}} = r\,(r+n+1)$, $m_{11}^{\mathrm{odd}} = m_{11}^{\mathrm{even}} = r\,(r-1)$ given by Eqs. 22 and 23 respectively. Therefore we can substitute these into Eq. 30

$$\begin{aligned} \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\left(A\Pi^T B\Pi\right)^2 - \left(AQ^T BQ\right)^2\right]\right\} &= \left(m_2^{(A)} - m_{1,1}^{(A)}\right)\left(m_2^{(B)} - m_{1,1}^{(B)}\right)\left\{1 - m\mathbb{E}\left(q_{ij}^4\right)\right\} \\ &= \frac{\beta\,(m-1)}{(m\beta+2)}\left(\frac{km\,(n-1)}{n\,(m-1)}\right)^2\left(m_2 - m_{1,1}\right)^2 \\ &= \frac{\beta k^2 m^2\,(n-1)^2}{(m\beta+2)\,(m-1)\,n^2}\left(m_2 - m_{1,1}\right)^2 \end{aligned} \tag{58}$$

One can also calculate each of the terms separately and obtain the same results (see Appendix for the alternative).

From Eq. 43 we have

$$\begin{aligned}\frac{1}{m}\mathbb{E}\left[\mathrm{Tr}\left(A\Pi^T B\Pi\right)^2 - \mathrm{Tr}\left(AQ_q^T BQ_q\right)^2\right] &= d\left(2k-1\right)\left(m_2 - m_{1,1}\right)^2 \\ &\times\ \mathbb{E}\left(\|uv\|_{\mathrm{F}}^2 - \left\|uv\left(uv\right)^\dagger\right\|_{\mathrm{F}}^2\right) \\ &= \left(2k-1\right)\left(m_2 - m_{1,1}\right)^2 \\ &\times\ \left\{\frac{1+d\left(d^3+d-3\right)}{\left(d^2+2\right)^2}\right\}. \end{aligned} \tag{59}$$

We can divide Eq. 59 by Eq. 58 to evaluate the parameter p

$$p = \frac{k^2}{2k-1}\frac{\beta m^2}{\left(m\beta+2\right)\left(m-1\right)n^2}\frac{\left(n-1\right)^2\left(n+2\right)^2}{\left[1+\left(n^2+n-3d\right)\right]};$$

note that the dependence on the covariance matrix has cancelled out as expected by the universality lemma.

B. Summary of Our Findings and Further Numerical Results

We summarize the results along with numerical experiments in Tables VI and VIII to show the equivalence of the first three moments and the departure of the three cases in their fourth moment. As said above,

$$Q_c = \mathbb{I}_{d^N} \qquad Q_{iso} \equiv Q \text{ Haar } d^N \times d^N \qquad Q_q = \left(Q_q^{(A)}\right)^T Q_q^{(B)} \tag{60}$$

where, $\left(Q_q^{(A)}\right)^T Q_q^{(B)}$ is given by Eq. 11. In addition, from Eq. 55 we can define Δ to be the part of the kurtosis that is equal among the three cases

$$\Delta = \gamma_2^c - \frac{2m_2^A m_2^B}{\sigma^4} = \gamma_2^c - \frac{1}{2}\frac{\left(rk+n+1\right)^2}{\left(n+1\right)^2}.$$

Using Δ we can obtain the full kurtosis for the iso and quantum case, and therefore (see Table VI for a theoretical summary):

$$p = \frac{\gamma_2^q - \gamma_2^{iso}}{\gamma_2^c - \gamma_2^{iso}}. \tag{61}$$

$\beta = 1$ Wishart	Iso	Quantum	Classical
Mean μ	$r\,(N-1)$		
Variance σ^2	$r\,(N-1)\,(d^2+1)$		
Skewness γ_1	$\frac{d^4+3d^2+4}{\sqrt{r(N-1)(d^2+1)^3}}$		
$\frac{1}{m}\mathbb{E}\left[\mathrm{Tr}\left(AQ_\bullet^T BQ_\bullet\right)^2\right]$	$m_2^A m_2^B -$ Eq. 58	$m_2^A m_2^B -$ Eq. 59	$r^2k^2\,(rk+n+1)^2$
Kurtosis $\gamma_2^{(\bullet)}$	$\frac{2}{m\sigma^4}\mathbb{E}\left[\mathrm{Tr}\left(AQ^T BQ\right)^2\right]+\Delta$	$\frac{2}{m\sigma^4}\mathbb{E}\left[\mathrm{Tr}\left(AQ_q^T BQ_q\right)^2\right]+\Delta$	Eq. 55

Table VI: Summary of the results when the local terms are Wishart matrices. The fourth moment is where the three cases differ.

The numerical convergence of the kurtoses to the theoretical values were rather slow. To make sure the results are consistent we did a large run with 500 million trials for $N = 5$, $d = 2$, $r = 3$ and $\beta = 1$ and obtained four digits of accuracy

$$\gamma_2^c - \gamma_2^{iso} = 0.39340 \qquad \text{Numerical experiment}$$
$$\gamma_2^c - \gamma_2^{iso} = 0.39347 \qquad \text{Theoretical value.}$$

Convergence is faster if one calculates p based on the departing terms alone (Eq. 51). In this case, for full rank Wishart matrices with $N = 5$ and $d = 2$

$\beta = 1$, trials: 5 Million	$1-p$
Numerical Experiment	0.57189
Theoretical Value	0.57183

$\beta = 2$, trials: 10 Million	$1-p$
Numerical Experiment	0.63912
Theoretical Value	0.63938

Below we compare our theory against exact diagonalization for various number of sites N, local ranks r, and site dimensionality d (Figures 8-12).

Experiments based on 500000 trials						
$N=3$	Theoretical value			Numerical Experiment		
	Iso	Quantum	Classical	Iso	Quantum	Classical
Mean μ	8			8.007	8.007	7.999
Variance σ^2	40			40.041	40.031	39.976
Skewness γ_1	$\frac{8}{25}\sqrt{10}=1.01192$			1.009	1.009	1.011
Kurtosis γ_2	$\frac{516}{875}=0.590$	$\frac{33}{50}=0.660$	$\frac{24}{25}=0.960$	0.575	0.645	0.953
Experiments based on 500000 trials						
$N=5$	Theoretical value			Numerical Experiment		
	Iso	Quantum	Classical	Iso	Quantum	Classical
Mean μ	16			15.999	15.999	16.004
Variance σ^2	80			79.993	80.005	80.066
Skewness γ_1	$\frac{8}{25}\sqrt{5}=0.716$			0.715	0.715	0.717
Kurtosis γ_2	$\frac{228}{2635}=0.087$	$\frac{51}{200}=0.255$	$\frac{12}{25}=0.48$	0.085	0.255	0.485
Experiments based on 300000 trials						
$N=7$	Theoretical value			Numerical Experiment		
	Iso	Quantum	Classical	Iso	Quantum	Classical
Mean μ	24			23.000	23.000	24.095
Variance σ^2	120			120.008	120.015	120.573
Skewness γ_1	$\frac{8}{75}\sqrt{30}=0.584$			0.585	0.585	0.588
Kurtosis γ_2	$-\frac{16904}{206375}=-0.082$	$\frac{23}{150}=0.153$	$\frac{8}{25}=0.320$	-0.079	0.156	0.331
Experiments based on 40000 trials						
$N=9$	Theoretical value			Numerical Experiment		
	Iso	Quantum	Classical	Iso	Quantum	Classical
Mean μ	32			32.027	32.027	31.777
Variance σ^2	160			160.074	160.049	157.480
Skewness γ_1	$\frac{4}{25}\sqrt{10}=0.506$			0.505	0.506	0.500
Kurtosis γ_2	$-\frac{539142}{3283175}=-0.164$	$\frac{87}{800}=0.109$	$\frac{6}{25}=0.240$	-0.165	0.109	0.213
Experiments based on 2000 trials						
$N=11$	Theoretical value			Numerical Experiment		
	Iso	Quantum	Classical	Iso	Quantum	Classical
Mean μ	40			39.973	39.973	39.974
Variance σ^2	200			200.822	200.876	197.350
Skewness γ_1	$\frac{8}{25}\sqrt{2}=0.452548$			0.4618	0.4538	0.407
Kurtosis γ_2	$-\frac{11162424}{52454375}=-0.213$	$\frac{21}{250}=0.084$	$\frac{24}{125}=0.192$	-0.189	0.093	0.102

Table VIII: The mean, variance and skewness of classical, iso and quantum results match. However, the fourth moments (kurtoses) differ. Here we are showing results for $d=2,\ r=4$ with an accuracy of three decimal points.

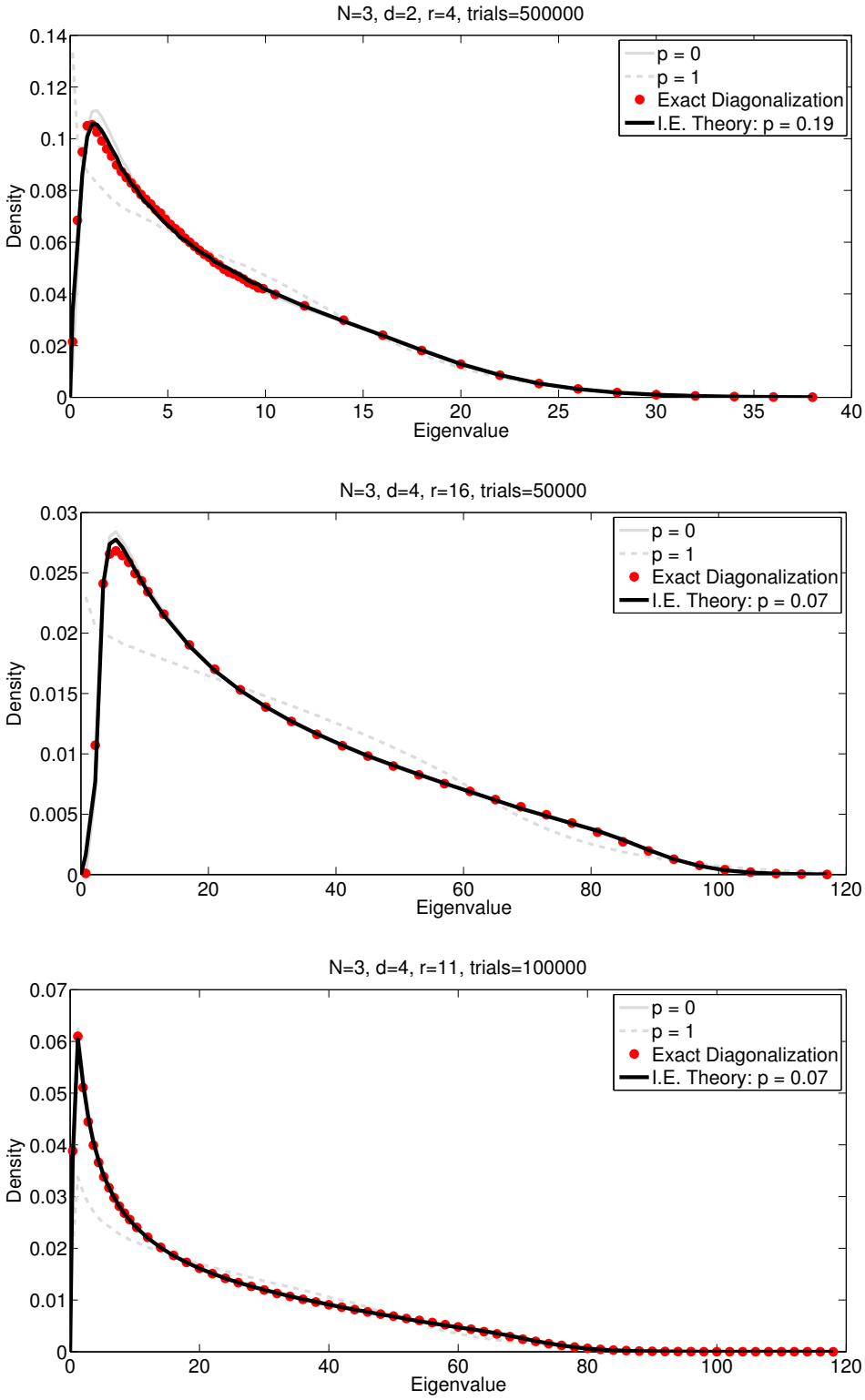


Figure 8: $N = 3$ examples. Note that the last two plots have the same p despite having different ranks r. This is a consequence of the Universality Lemma since they have the same N and d.

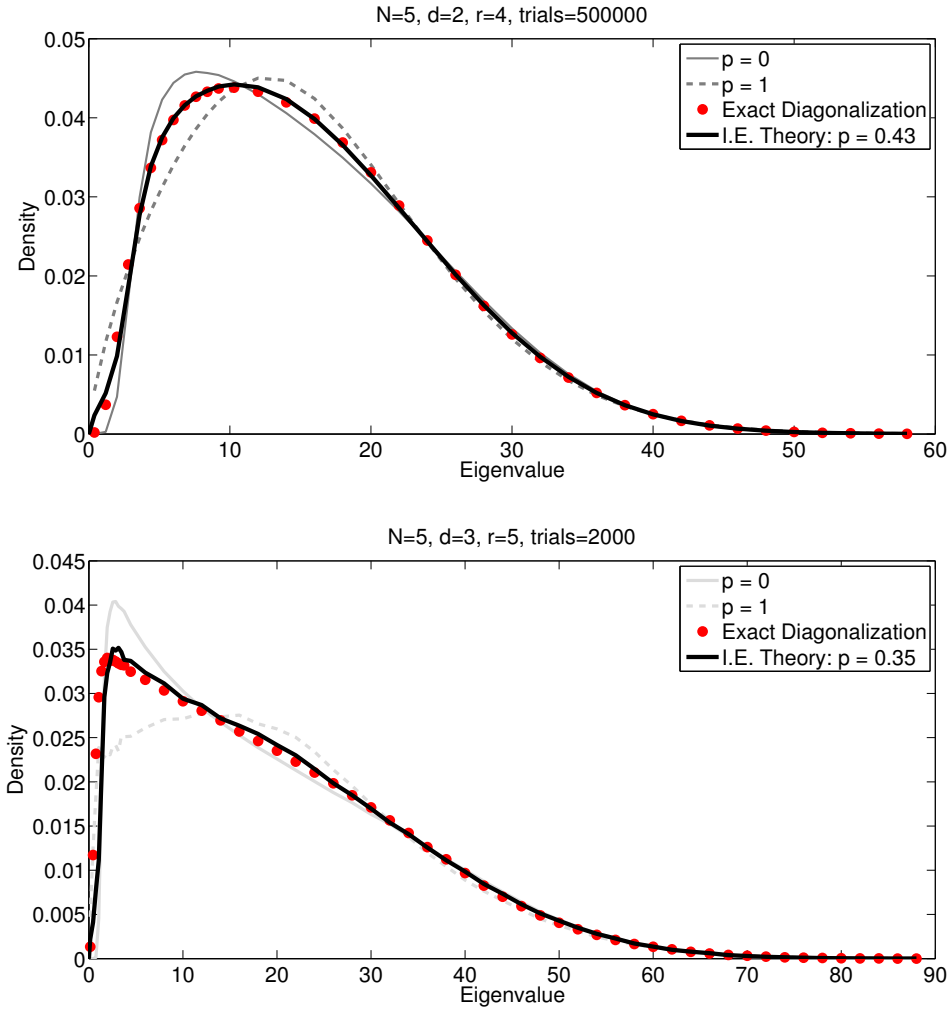


Figure 9: $N = 5$

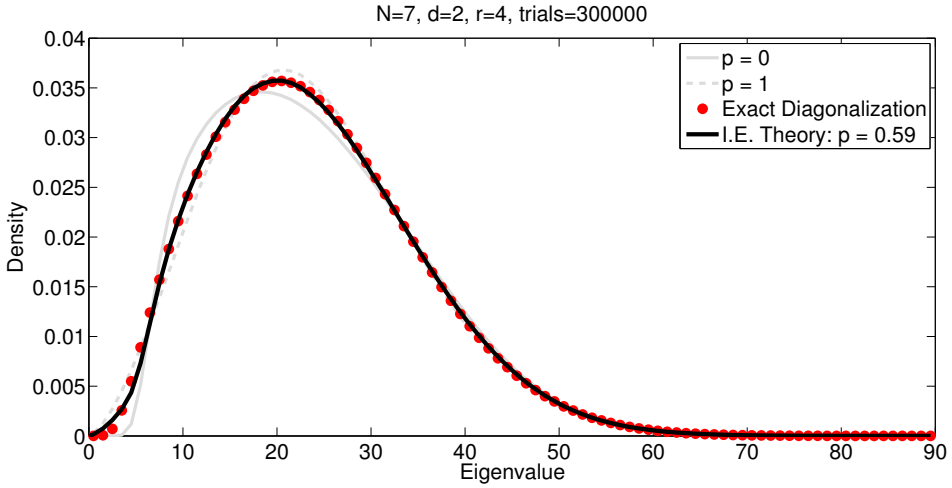


Figure 10: $N = 7$

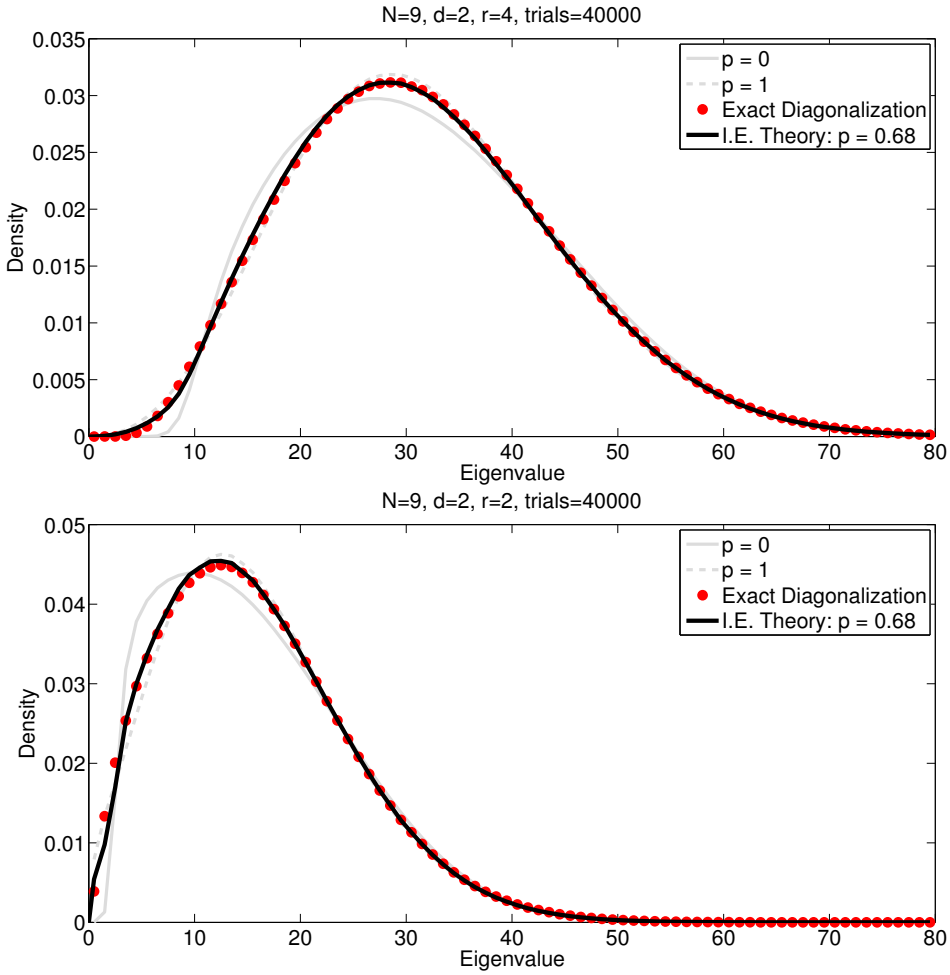


Figure 11: $N = 9$. Note that the two plots have the same p despite having different local ranks.

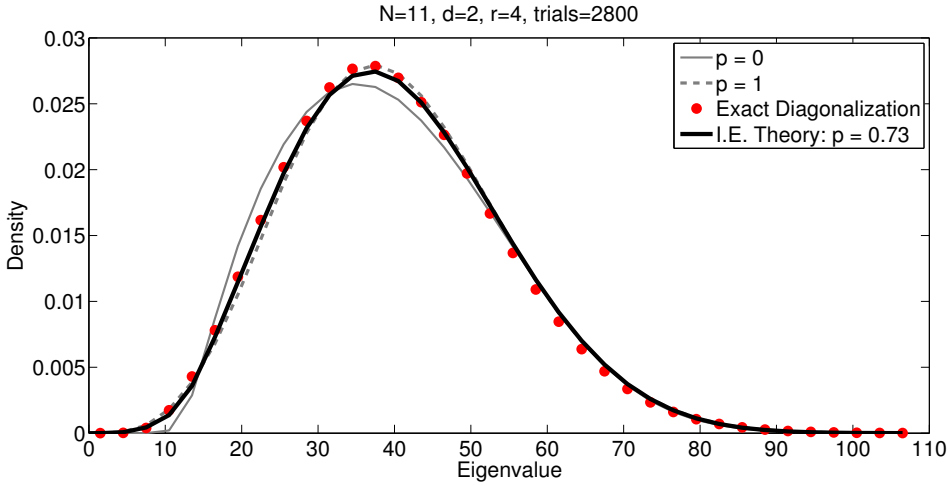


Figure 12: $N = 11$

VI. OTHER EXAMPLES OF LOCAL TERMS

Because of the Universality lemma, p is independent of the type of local distribution. Furthermore, as discussed above, the application of the theory for other types of local terms is entirely similar to the Wishart case. Therefore, we only show the results in this section. As a second example consider GOE's as local terms, i.e., $H_{l,l+1} = \frac{G^T+G}{2}$, where G is a full rank matrix whose elements are real Gaussian random numbers.

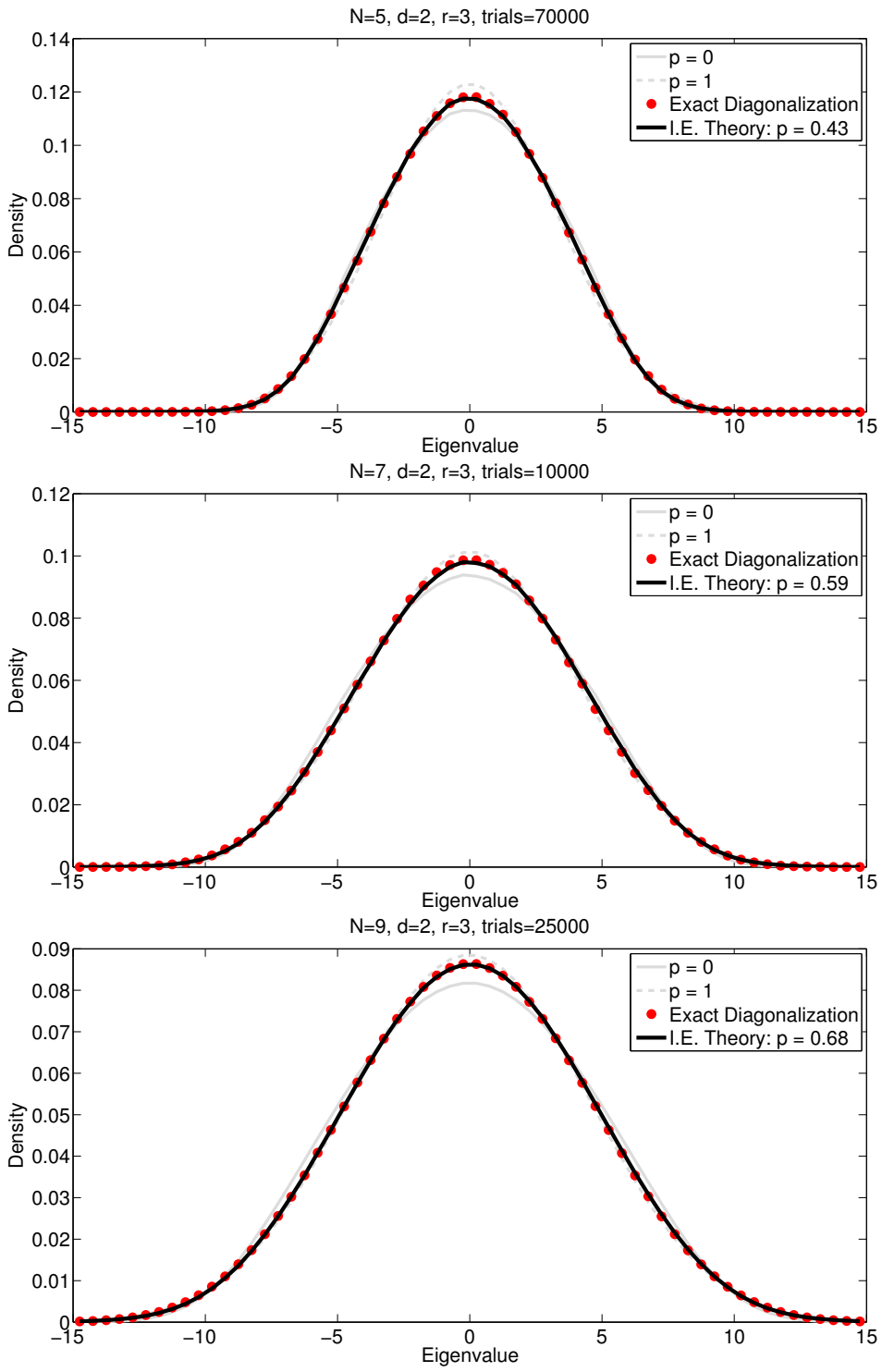


Figure 13: GOE's as local terms

Lastly take the local terms to have Haar eigenvectors but with random eigenvalues ± 1, i.e., $H_{l,l+1} = Q_l^T \Lambda_l Q_l$, where Λ_l is a diagonal matrix whose elements are binary random variables ± 1 (Figure 14).

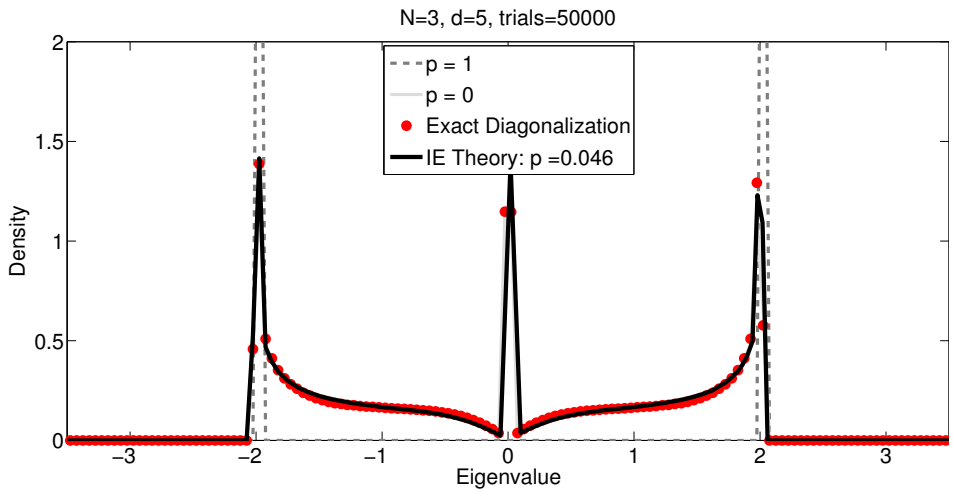


Figure 14: Local terms have a random binomial distribution.

In this case the classical treatment of the local terms leads to a binomial distribution. As expected $p = 1$ in Figure 14 has three atoms at $-2, 0, 2$ corresponding to the randomized sum of the eigenvalues from the two local terms. The exact diagonalization, however, shows that the quantum chain has a much richer structure closer to iso; i.e, $p = 0$. This is captured quite well by IE with $p = 0.046$.

VII. BEYOND NEAREST NEIGHBORS INTERACTION: $L > 2$

If one fixes all the parameters in the problem and compares $L > 2$ with nearest neighbor interactions, then one expects the former to act more isotropic as the number of random parameters in Eq. 1 are more. When the number of random parameters introduced by the local terms, i.e., $(N - L + 1)\, d^L$ and d^N are comparable, we find that we can approximate the spectrum with a high accuracy by taking the summands to be all isotropic[45] (See Figures 15-17).

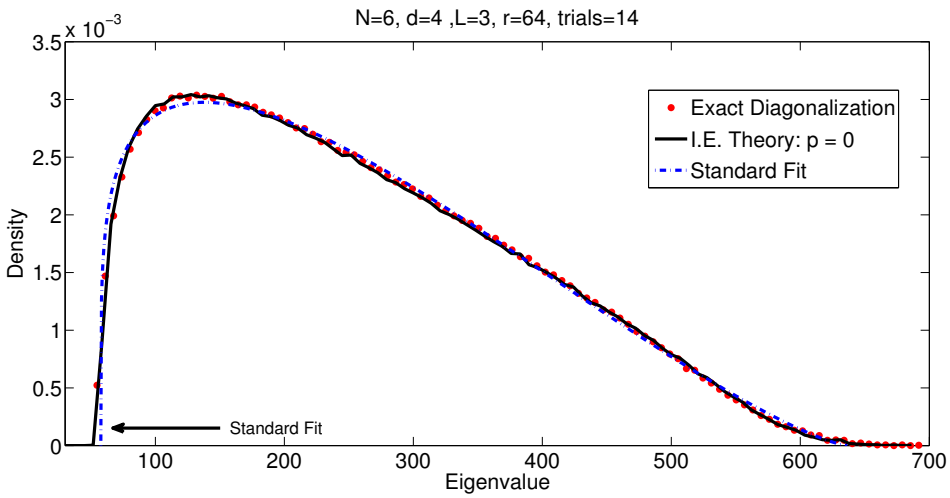


Figure 15: IE method approximates the quantum spectrum by $H^{IE} = \sum_{l=1}^{4} Q_l^T H_{l,\cdots,l+2} Q_l$

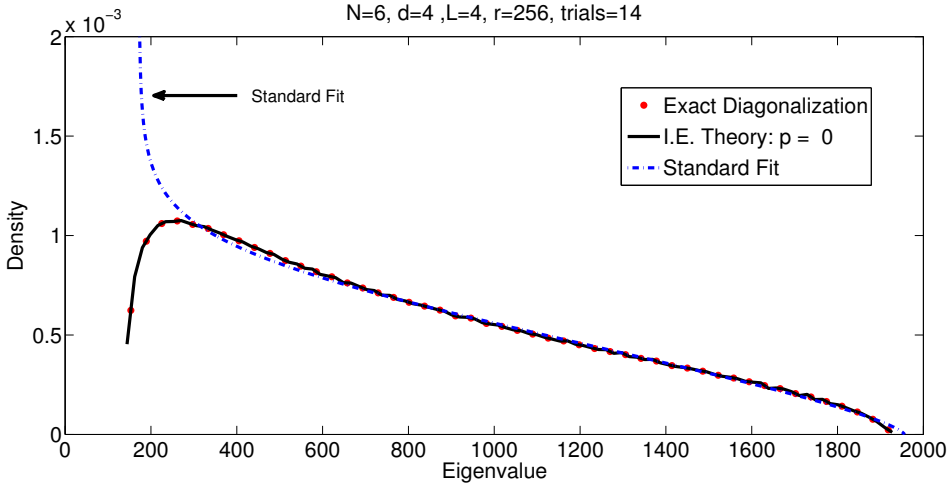


Figure 16: IE method approximates the quantum spectrum by $H^{IE} = \sum_{l=1}^{3} Q_l^T H_{l,\cdots,l+3} Q_l$.

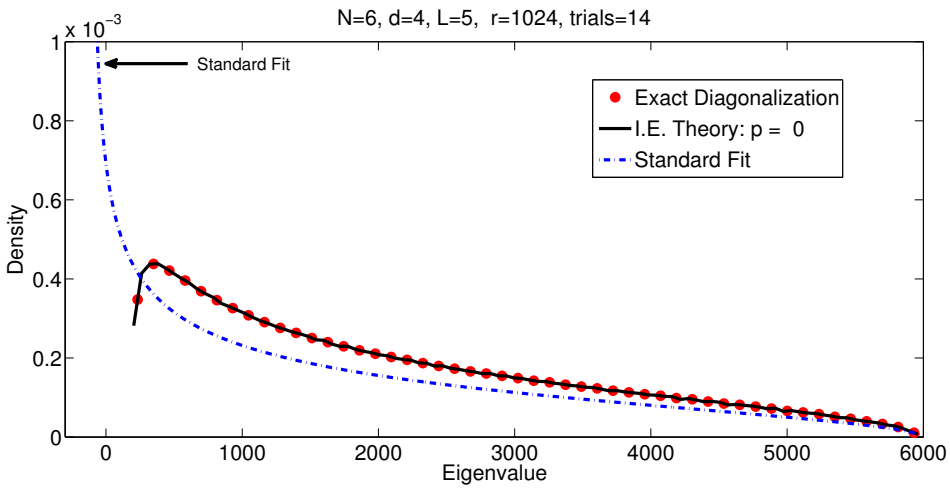


Figure 17: IE method approximates the quantum spectrum by $H^{IE} = \sum_{l=1}^{2} Q_l^T H_{l,\cdots,l+4} Q_l$

Most distributions built solely from the first four moments, would give smooth curves. Roughly speaking, the mean indicates the center of the distribution, variance its width, skewness its bending away from the center and kurtosis how tall and skinny versus how short and fat the distribution is. Therefore, it is hard to imagine that the kinks, cusps and local extrema of the quantum problem (as seen in some of our examples and in particular Figure in 14) could be captured by fitting only the first four moments of the QMBS Hamiltonian to a known distribution. It is remarkable that a one parameter (i.e., p) interpolation between the isotropic and classical suffices in capturing the richness of the spectra of QMBS.

VIII. CONJECTURES AND OPEN PROBLEMS

In this paper we have offered a method that successfully captures the density of states of QMBS with generic local interaction with an accuracy higher than one expects solely from the first four moments. We would like to direct the reader's attention to open problems that we believe are within reach.

1. We conjecture that the higher moments may be analyzed for their significance. For example, one can show that the fraction of departing terms in the expansion of the higher moments (e.g. analogous to bold faced and underlined terms in Eqs. 16,18 but for higher moments) is asymptotically upper bounded by $1/N^3$. In SectionII C we conjectured that their expectation values would not change the moments significantly. It would be of interest to know if

$$\begin{aligned}\mathbb{E}\mathrm{Tr}\left\{\ldots Q^{-1}B^{\geq 1}QA^{\geq 1}Q^{-1}B^{\geq 1}Q\ldots\right\} &\leq \\ \mathbb{E}\mathrm{Tr}\left\{\ldots Q_q^{-1}B^{\geq 1}Q_qA^{\geq 1}Q_q^{-1}B^{\geq 1}Q_q\ldots\right\} &\leq \\ \mathbb{E}\mathrm{Tr}\left\{\ldots \Pi^{-1}B^{\geq 1}\Pi A^{\geq 1}\Pi^{-1}B^{\geq 1}\Pi\ldots\right\} &\quad .\end{aligned}$$

For example, one wonders if

$$\mathbb{E}\mathrm{Tr}\left\{\left(AQ^{-1}BQ\right)^k\right\} \leq \mathbb{E}\mathrm{Tr}\left\{\left(AQ_q^{-1}BQ_q\right)^k\right\} \leq \mathbb{E}\mathrm{Tr}\left\{\left(A\Pi^{-1}B\Pi\right)^k\right\}$$

for $k > 2$; we have proved that the inequality becomes an equality for $k = 1$ (Departure Theorem) and holds for $k = 2$ (Slider Theorem).

2. Though we focus on a decomposition for spin chains, we believe that the main theorems may generalize to higher dimensional graphs. Further rigorous and numerical work in higher dimensions would be of interest.

3. At the end of Section IV D we proposed that more general local terms might be treated by explicitly including the extra terms (Type I terms).

4. Application of this method to slightly disordered systems would be of interest in CMP. In this case, the assumption of fully random local terms needs to be relaxed.

5. In our numerical work, we see that the method gives accurate answers in the presence of an external field. It would be nice to formally extend the results and calculate thermodynamical quantities.

6. We derived our results for general β but numerically tested $\beta = 1, 2$. We acknowledge that general β remains an abstraction.

7. Readers may wonder whether it is better to consider "iso" or the infinite limit which is "free". We have not fully investigated these choices, and it is indeed possible that one or the other is better suited for various purposes.

8. A grander goal would be to apply the ideas of this paper to very general sums of matrices.

A Comparison with the Literature

We are aware of two other works ([38] and [37]) that formulate some form of interpolation between a "free" object and a "classical" object: In [38], a random unitary matrix is explicitly constructed through a Brownian motion process starting at time $t = 0$, and going to time $t = \infty$. "Classical" corresponds to $t = 0$, and "free" corresponds to $t = \infty$. The random unitary matrix starts non-random and is randomized continuously until it fully reaches Haar measure. In [37], through detailed combinatorial constructions and investigation into Fock space representations of Fermions and Bosons, unique measures are constructed that interpolate between the limit of the classical central limit theorem, the gaussian, and the free central limit theorem, the semicircle. The curve also continues on to $t = -1$, which corresponds to two non-random atoms.

An unknown question is whether the unitary construction in [38] leads to the same convolution interpolate as this paper where we take a convex combination. Another unknown question is whether IE and [38] lead to an analog of a limit of a central limit theorem which would match that of [37].

We outline in our matrix below features found in each paper. The empty boxes indicate opportunities for research.

	Application	Unitary Matrix Construction	Interpolate Convolution	Iterate Convolution to a CLT
IE	✔		✔	
[38]		✔	✔	
[37]				✔

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IX. APPENDIX

To help the reader with the random quantities that appear in this paper, we provide explanations of the exact nature of the random variables that are being averaged. A common assumption is that we either assume a uniformly randomly chosen eigenvalue from a random

matrix or we assume a collection of eigenvalues that may be randomly ordered, Random ordering can be imposed or a direct result of the eigenvector matrix having the right property. Calculating each of the terms separately and then subtracting gives the same results.

$$\frac{1}{m}\mathbb{E}\left[\mathrm{Tr}\left(AQ^TBQ\right)^2\right] = \frac{1}{m}\mathbb{E}\left\{\sum_{1\le i_1,i_2,j_1,j_2\le m} a_{i_1}a_{i_2}b_{j_1}b_{j_2}\left(q_{i_1j_1}q_{i_1j_2}q_{i_2j_1}q_{i_2j_2}\right)\right\}, \tag{62}$$

where a_i and b_j are elements of matrices A and$B$ respectively. The right hand side of Eq. 62 can have terms with two collisions (i.e., $i_1 = i_2$ and $j_1 = j_2$), one collision (i.e. $i_1 \neq i_2$ exclusive-or $j_1 \neq j_2$), or no collisions (i.e., $i_1 \neq i_2$ and $j_1 \neq j_2$). Our goal now is to group terms based on the number of collisions. The pre-factors for two, one and no collisions along with the counts are summarized in Table IV. Using the latter we can sum the three types of contributions, to get the expectation

$$\begin{array}{c}\frac{1}{m}\mathbb{E}\left[\mathrm{Tr}\left(AQ^TBQ\right)^2\right] = \frac{(\beta+2)}{(m\beta+2)}\mathbb{E}\left(a^2\right)\mathbb{E}\left(b^2\right) + \\ \frac{\beta(m-1)}{(m\beta+2)}\left[\mathbb{E}\left(b^2\right)\mathbb{E}\left(a_1a_2\right) + \mathbb{E}\left(a^2\right)\mathbb{E}\left(b_1b_2\right)\right] - \frac{\beta(m-1)}{(m\beta+2)}\mathbb{E}\left(a_1a_2\right)\mathbb{E}\left(b_1b_2\right).\end{array} \tag{63}$$

If we take the local terms to be from the same distribution we can further express the foregoing equation

$$\frac{1}{m}\mathbb{E}\left[\mathrm{Tr}\left(AQ^TBQ\right)^2\right] = \frac{1}{(m\beta+2)}\left[\left(\beta+2\right)m_2^2 + \beta\left(m-1\right)\mathbb{E}\left(a_1a_2\right)\left\{2m_2 - \mathbb{E}\left(a_1a_2\right)\right\}\right]. \tag{64}$$

The quantity of interest is the difference of the classical and the iso (see Eq. 20),

$$\begin{array}{c}\frac{1}{m}\mathbb{E}\mathrm{Tr}\left(A\Pi^TB\Pi\right)^2 - \frac{1}{m}\mathbb{E}\mathrm{Tr}\left(AQ^TBQ\right)^2 = \\ \frac{\beta(m-1)}{(m\beta+2)}\left\{\mathbb{E}\left(a^2\right)\mathbb{E}\left(b^2\right) - \mathbb{E}\left(b^2\right)\mathbb{E}\left(a_1a_2\right) - \mathbb{E}\left(a^2\right)\mathbb{E}\left(b_1b_2\right) + \mathbb{E}\left(a_1a_2\right)\mathbb{E}\left(b_1b_2\right)\right\} = \\ \frac{\beta(m-1)}{(m\beta+2)}\left\{m_2^{(A)}m_2^{(B)} - m_2^{(B)}m_{1,1}^{(A)} - m_2^{(A)}m_{1,1}^{(B)} + m_{1,1}^{(A)}m_{1,1}^{(B)}\right\} = \\ \frac{\beta(m-1)}{(m\beta+2)}\left(m_2^{(A)} - m_{1,1}^{(A)}\right)\left(m_2^{(B)} - m_{1,1}^{(B)}\right) =\end{array} \tag{65}$$

If we assume that the local terms have the same distribution: $m_2 \equiv m_2^{(A)} = m_2^{(B)}$, $m_{11} \equiv m_{1,1}^{(A)} = m_{1,1}^{(B)}$ as in Eq. 64, the foregoing equation simplifies to

$$\frac{1}{m}\mathbb{E}\mathrm{Tr}\left(A\Pi^TB\Pi\right)^2 - \frac{1}{m}\mathbb{E}\mathrm{Tr}\left(AQ^TBQ\right)^2 = \frac{\beta\left(m-1\right)}{(m\beta+2)}\left(m_2 - m_{1,1}\right)^2.$$

In the example of Wishart matrices as local terms we have

$$\begin{array}{c} m_2 \equiv \mathbb{E}\left(a^2\right) = rk\left(rk+n+1\right) \\ m_{1,1} \equiv \mathbb{E}\left(a_1a_2\right) = k\left(k-1\right)r^2 + \frac{kr}{m-1}\left\{\left(tn^{k-1}-1\right)\left(n+r+1\right) + tn^{k-1}\left(n-1\right)\left(r-1\right)\right\} \\ = k\left(k-1\right)r^2 + \frac{kr}{m-1}\left\{tn^{k-1}(nr+2) - n - r - 1\right\}.\end{array} \tag{66}$$