Beyond Universality in Random Matrix Theory

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July 11, 2014

Abstract

In order to have a better understanding of finite random matrices with non-Gaussian entries, we study the $1/N$ expansion of local eigenvalue statistics in both the bulk and at the hard edge of the spectrum of random matrices. This gives valuable information about the smallest singular value not seen in universality laws. In particular, we show the dependence on the fourth moment (or the kurtosis) of the entries. This work makes use of the so-called complex deformed GUE and Laguerre ensembles.

1 Beyond Universality

The desire to assess the applicability of universality results in random matrix theory has pressed the need to go beyond universality, in particular the need to understand the influence of finite n and what happens if the matrix deviates from Gaussian normality. In this note, we provide exact asymptotic correction formulas for the smallest singular value of complex matrices and bulk statistics for complex Wigner matrices.

"Universality," a term found in statistical mechanics, is widely found in the field of random matrix theory. The universality principle loosely states that eigenvalue statistics of interest will behave asymptotically as if the matrix elements were Gaussian. The spirit of the term is that the eigenvalue statistics will not care about the details of the matrix elements.

It is important to extend our knowledge of random matrices beyond universality. In particular, we should understand the role played by

- finite n and
- non Gaussian random variables.

From an applications viewpoint, it is very valuable to have an estimate for the departure from universality. Real problems require that n be finite, not infinite, and it has long been observed computationally that ∞ comes very fast in random matrix theory. The applications beg to know how fast. From a theoretical viewpoint, there is much to be gained in searching for proofs that closely follow the underlying mechanisms of the mathematics. We might distinguish "mechanism oblivious" proofs whose bounds require n to be well outside imaginably useful ranges, with "mechanism aware" proofs that hold close to the underlying workings of random matrices. We encourage such "mechanism aware" proofs.

In this article, we study the influence of the fourth cumulant on the local statistics of the eigenvalues of random matrices of Wigner and Wishart type.

On one hand, we study the asymptotic expansion of the smallest eigenvalue density of large random sample covariance matrices. The behavior of smallest eigenvalues of sample covariance matrices when p/n is close to one (and more generally) is somewhat well understood now. We refer the reader to [11], [29], [13], [4], [5]. The impact of the fourth cumulant of the entries is of interest here; we show its contribution to the distribution function of the smallest eigenvalue density of large random sample covariance matrices as an additional error term of order of the inverse of the dimension.

On the other hand, we consider the influence of the fourth moment in the local fluctuations in the bulk. Here, we consider Wigner matrices and prove a conjecture of Tao and Vu [27] that the fourth moment brings a correction to the fluctuation of the expectation of the eigenvalues in the bulk of order of the inverse of the dimension.

In both cases, we consider the simplest random matrix ensembles that are called Gaussian divisible, that is whose entries can be describe as the convolution of a distribution by the Gaussian law. To be more precise, we consider the so-called Gaussian-divisible ensembles, also known as Johansson-Laguerre and Johansson-Wigner ensembles. This ensemble, defined hereafter, has been first considered in [19] and has the remarkable property that the induced joint eigenvalue density can be computed. It is given in terms of the Itzykson-Zuber-Harich-Chandra integral. From such a formula, saddle point analysis allows to study the local statistics of the eigenvalues. It turns out that in both cases under study, the contribution of the fourth moment to the local statistics can be inferred from a central limit theorem for the linear statistics of Wigner and Wishart random matrices. The covariance of the latter is well known to depend on the fourth moments, from which our results follow.

2 Discussion and Simulations

2.1 Preliminaries: Real Kurtosis

Definition 1. *The* **kurtosis** *of a distribution is*

$$\gamma = \frac{\kappa_4^{\Re}}{\sigma_{\Re}^4} = \frac{\mu_4}{\sigma_{\Re}^4} - 3,$$

where $\kappa_4^{\Re}$ is the fourth cumulant of the real part, $\sigma_{\Re}^2$ is the variance of the real part, and μ_4 is the fourth moment about the mean.

note: From a software viewpoint, commands such as `randn` make it natural to take the real and the imaginary parts to separately have mean 0, variance 1, and also to consider the real kurtosis.

Example of Kurtoses γ for distributions with mean 0, and $\sigma^2 = 1$:

Distribution	γ	Univariate Code
normal	0	`randn`
Uniform $[-\sqrt{3}, \sqrt{3}]$	-1.2	`(rand -.5)*sqrt(12)`
Bernoulli	-2	`sign(randn)`
Gamma	6	`rand(Gamma()) - 1`

For the matrices themselves, we compute the smallest eigenvalues of the Gram matrix constructed from $(n+\nu)\times n$ complex random matrices with Julia [6] code provided for the reader's convenience:

RM	Complex Matrix Code
normal	`randn(n+ν,n)+im*randn(n+ν,n)`
Uniform	`((rand(n+ν,n)-.5)+im*rand(n+ν,n) -.5))*sqrt(12)`
Bernoulli	`sign(randn(n+ν, n))+im*sign(randn(n+ν,n))`
Gamma	`(rand(Gamma(),n+ν,n)-1) + im*(rand(Gamma(),n+ν,n)-1)`

2.2 Smallest Singular Value Experiments

Let A be a random $n+\nu$ by n complex matrix with iid real and complex entries all with mean 0, variance 1 and kurtosis γ. In the next several subsections we display special cases of our results, with experiment vs. theory curves for $\nu = 0, 1$, and 2.

We consider the distribution

$$F(x) = \mathbb{P}\left(x \le n\lambda_{\min}(A^T A)\right) = \mathbb{P}\left(x \le n\left(\sigma_{\min}(A)\right)^2\right),$$

where $\sigma_{\min}(A)$ is the smallest singular value of A. We also consider the density

$$f(x) = \frac{d}{dx}F(x).$$

In the plots to follow we took a number of cases when $n = 20, 40$ and sometimes $n = 80$. We computed $2,000,000$ random samples on each of 60 processors using Julia [6], for a total of $120,000,000$ samples of each experiment. The runs used 75% of the processors on a machine equipped with 8 Intel Xeon E7-8850-2.0 GHz-24M-10 Core Xeon MP Processors. This scale experiment, which is made easy by the Julia system, allows us to obtain visibility on the higher order terms that would be hard to see otherwise. Typical runs took about an hour for $n = 20$, three hours for $n = 40$, and twelve hours for $n = 80$.

We remark that we are only aware of two or three instances where parallel computing has been used in random matrix experiments. Working with Julia is pioneering in showing just how easy this can be, giving the random matrix experimenter a new tool for honing in on phenomena that would have been nearly impossible to detect using conventional methods.

2.3 Example: Square Complex Matrices ($\nu = 0$)

Consider taking, a 20 by 20 random matrix with independent real and imaginary entries that are uniformly distributed on $[-\sqrt{3}, \sqrt{3}]$.

```
((rand(20,20)-.5) + im*(randn(20,20)-.5))*sqrt(12).
```

This matrix has real and complex entries that have mean 0, variance 1, and kurtosis $\gamma = -1.2$.

An experimenter wants to understand how the smallest singular value compares with that of the complex Gaussian matrix

```
randn(20,20) + im*randn(20,20).
```

The law for complex matrices [10, 11] in this case valid for all finite sized matrices, is that $n\lambda_{\text{min}} = n\sigma^2_{\text{min}}$ is exactly exponentially distributed: $f(x) = \frac{1}{2}e^{-x/2}$. Universality theorems say that the uniform curve will match the Gaussian in the limit as matrix sizes go to ∞. The experimenter obtains the curves in Figure 1 (taking both $n = 20$ and $n = 40$).

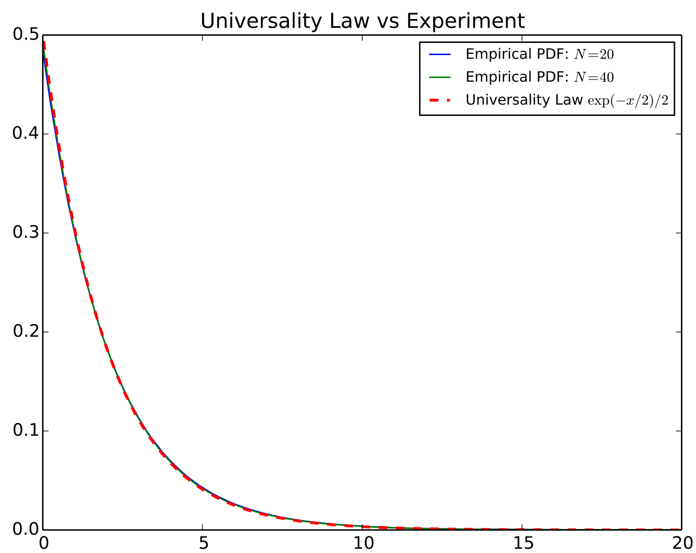

Figure 1: Universality Law vs Experiment: $n = 20$ and $n = 40$ already resemble $n = \infty$

Impressed that $n = 20$ and $n = 40$ are so close, he or she might look at the proof of the universality theorem only to find that no useful bounds are available at $n = 20, 40$.

The results in this paper gives the following correction in terms of the kurtosis (when $\nu = 0$):

$$f(x) = e^{-x/2} \left(\frac{1}{2} + \frac{\gamma}{n} (\frac{1}{4} - \frac{x}{8}) \right) + O\left(\frac{1}{n^2} \right).$$

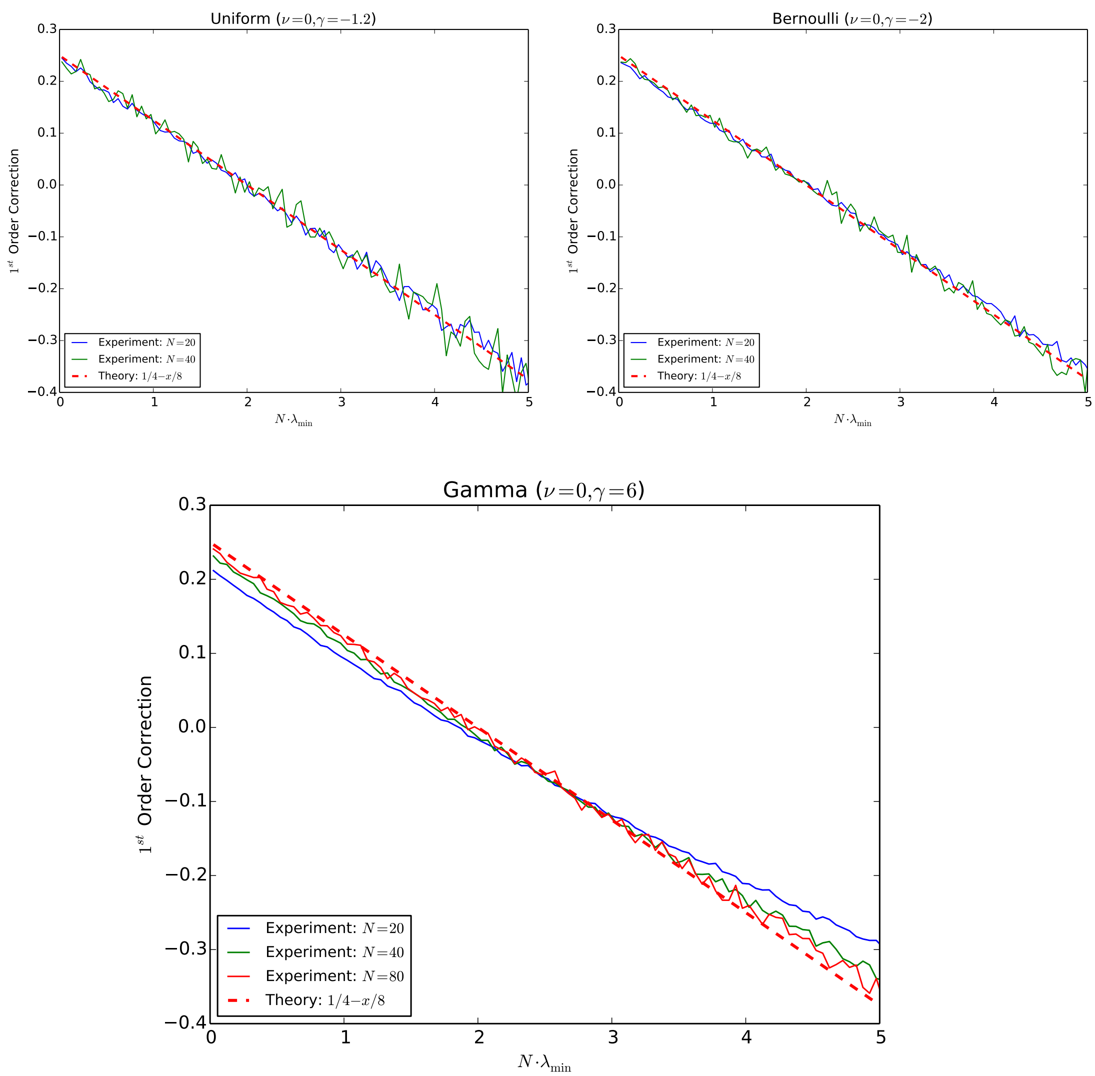

Figure 2: Correction for square matrices Uniform, Bernoulli, ($\nu = 0$). Monte carlo simulations are histogrammed, 0th order term subtracted, and result multiplied by $ne^{x/2}/\gamma$. Bottom curve shows convergence for $n = 20, 40, 80$ for a distribution with positive kurtosis.

On the bottom of Figure 1, with the benefit of 60 computational processors, we can magnify the departure from universality with Monte Carlo experiments, showing that the departure truly fits $\frac{\gamma}{n}(\frac{1}{4} - \frac{x}{8})e^{-x/2}$. This experiment can be run and rerun many times, with many distributions, kurtoses that are positive and negative, small values of n, and the correction term works very well.

2.4 Example: $n+1$ by n complex matrices ($\nu = 1$)

The correction to the density can be written as

$$f(x) = e^{-x/2}\left(\frac{1}{2}I_2(s) + \frac{1+\gamma}{8n}\left(sI_1(s) - xI_2(s)\right)\right) + O(\frac{1}{n^2}),$$

where $I_1(x)$ and $I_2(x)$ are Bessel functions and $s = \sqrt{2x}$.

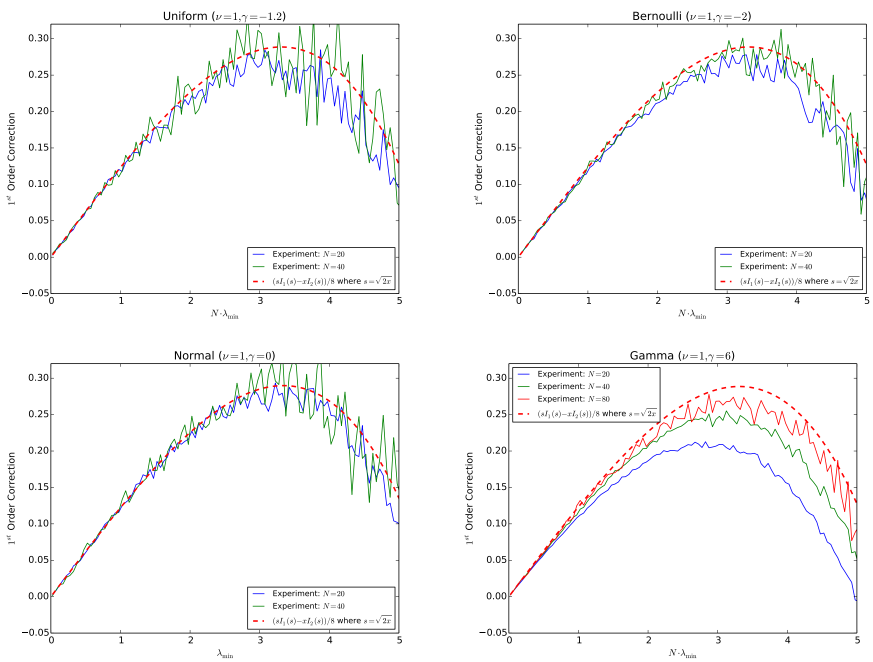

Figure 3: Correction for $\nu = 1$. Uniform, Bernoulli, normal, and Gamma; Monte carlo simulations are histogrammed, 0th order term subtracted, and result multiplied by $ne^{x/2}/(1+\gamma)$. Bottom right curve shows convergence for $n = 20, 40, 80$ for a distribution with positive kurtosis.

2.5 Example: $n+2$ by n complex matrices ($\nu = 2$)

The correction to the density for $\nu = 2$ can be written

$$f(x) = \frac{1}{2}e^{-x/2}\left([I_2^2(s) - I_1(s)I_3(s)] + \frac{2+\gamma}{2n}\left[(x+4)I_1^2(s) - 2sI_0(s)I_1(s) - (x-2)I_2^2(s)\right]\right),$$

where I_0, I_1, I_2, and I_3 are Bessel functions, and $s = \sqrt{2x}$.

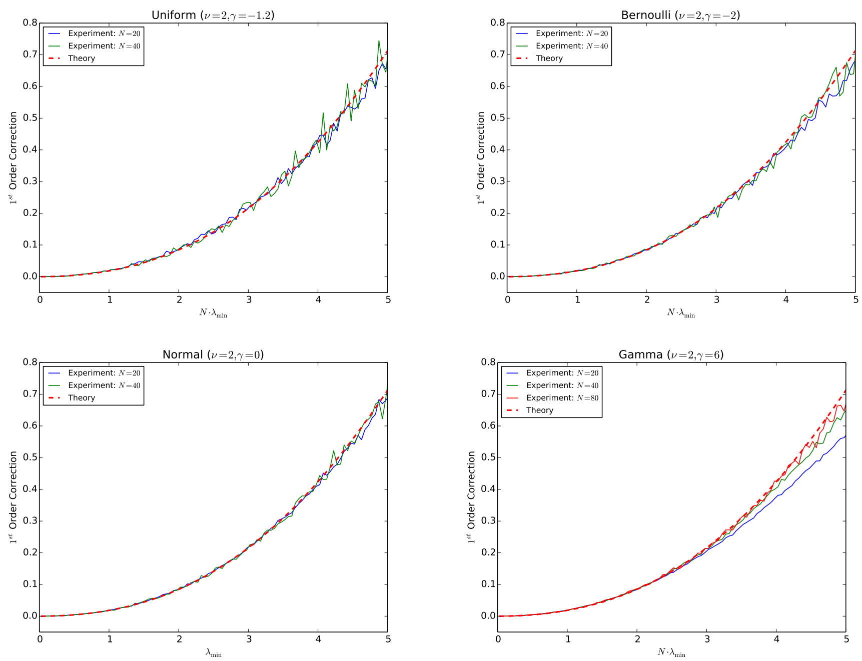

Figure 4: Correction for $\nu = 2$. Uniform, Bernoulli, normal, and Gamma; Monte carlo simulations are histogrammed, 0th order term subtracted, and result multiplied by $ne^{x/2}/(2+\gamma)$. Bottom right curve shows convergence for $n = 20, 40, 80$ for a distribution with positive kurtosis.

3 Models and Results

In this section, we define the models we will study and state the results. Let some real parameter $a > 0$ be given. Consider a matrix M of size $p \times n$:

$$M = W + aV$$

where

- $V = (V_{ij})_{1\le i\le p; 1\le j\le n}$ has i.i.d. entries with complex $\mathcal{N}_{\mathbb{C}}(0,1)$ distribution, which means that both $\Re V_{ij}$ and $\Im V_{ij}$ are real i.i.d. $\mathcal{N}(0,1/2)$ random variables,

- $W = (W_{ij})_{1 \le i \le p; 1 \le j \le n}$ is a random matrix with entries being mutually independent random variables with distribution $P_{ij}, 1 \le j \le n$ independent of n and p, with uniformly bounded fourth moment,
- W is independent of V,
- $\nu := p - n \ge 0$ is a fixed integer independent of n.

We then form the Johansson-Laguerre matrix:

$$\frac{1}{n} M^* M = \left(\frac{1}{\sqrt{n}} (W + aV) \right)^* \left(\frac{1}{\sqrt{n}} (W + aV) \right). \tag{1}$$

When W is fixed, the above ensemble is known as the *Deformed Laguerre Ensemble.*

We assume that the probability distributions $P_{j,k}$ satisfies

$$\int z dP_{j,k}(z) = 0, \int |zz^*| dP_{j,k}(z) = \sigma_1^2 = \frac{1}{4}. \tag{2}$$

Hypothesis (2) ensures the convergence of the spectral measure of H^*H to the Marchenko-Pastur distribution with density

$$\rho(x) = \frac{2}{\pi} \frac{\sqrt{1-x}}{\sqrt{x}}. \tag{3}$$

Condition (2) implies also that the limiting spectral measure of $\frac{1}{n} M^* M$ is then given by Marchenko-Pastur's law with parameter $1/4 + a^2$; we denote $\rho = \rho_a$ the density of this probability measure.

For technical reasons, we assume that the entries of W have sub-exponential tails: There exist $C, c, \theta > 0$ so that for all $i, j \in \mathbb{N}^2$, all $t \ge 0$

$$P_{j,l}(|z| \ge t) \le C e^{-ct^\theta}. \tag{4}$$

This hypothesis could be weakened to requiring enough finite moments.

Finally we assume that the fourth moments do not depend on j, k and let κ_4 be the difference between the fourth moment of $P_{j,k}$ and the Gaussian case, namely in the case where $\beta = 2$

$$\kappa_4 = \int |zz^*|^2 dP_{j,k} - 8^{-1}$$

(Thus, with the notation of Definition 1, $\kappa_4 = 2\gamma\sigma_{\Re}^4 = 2\kappa_4^{\Re}$.)

Then our main result is the following. Let $\sigma := \sqrt{4^{-1} + a^2}$.

Theorem 3.1. *Let g_n be the density of the hard edge in the Gaussian case with entries of constant complex variance $\sigma^2 = 2\sigma_{\Re}^2$:*

$$g_n(s) = \mathbb{P}\left(\lambda_{min} \ge \frac{s}{n} \right)$$

Then, for all $s > 0$, if our distribution has complex fourth cumulant $\kappa_4 = 2\kappa_4^{\Re}$, then

$$\mathbb{P}\left(\lambda_{min} \geq \frac{s}{n}\right) = g_n(s) + \frac{sg_n'(s)}{\sigma^4 n}\kappa_4 + o(\frac{1}{n}).$$

We note that this formula is scale invariant.

As a consequence, we obtain:

Corollary 3.2. *For the ν for which Conjectures 1 and 2 are true (see section 4.2),*

$$\mathbb{P}\left(\lambda_{min} \geq \frac{s}{n}\right) = g_\infty(s) + (\nu + \frac{\kappa_4}{\sigma^4})\frac{sg_\infty'(s)}{n} + o(\frac{1}{n}).$$

Note: Conjectures 1 and 2 were verified for $\nu = 0, \ldots, 25$ thanks to mathematica and maple.

Note: The g_n formulation involves Laguerre polynomials and exponentials. The g_∞ formulation involves Bessel functions and exponentials.

For the Wigner ensemble we consider the matrix

$$M_n = \frac{1}{\sqrt{n}}(W + aV)$$

where W a Wigner matrix with complex independent entries above the diagonal with law μ which has sub exponential moments: there exists $C, c > 0$, and $\alpha > 0$ such that for all $t \geq 0$

$$\mu(|x| \geq t) \leq C \exp\{-ct^\alpha\},$$

and satisfies

$$\int x d\mu(x) = 0, \int |x|^2 d\mu(x) = 1/4, \int x^3 d\mu(x) = 0.$$

The same assumptions are also assumed to hold true for μ'. V is a GUE random matrix with i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$ entries. We denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the ordered eigenvalues of M_n. By Wigner's theorem, it is known that the spectral measure of M_n

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}$$

converges weakly to the semi-circle distribution with density

$$\sigma_{sc}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\mathbb{1}_{|x| \leq 2\sigma}; \sigma^2 = 1/4 + a^2. \tag{5}$$

This is the Gaussian-divisible ensemble studied by Johansson [19]. We study the dependency of the one point correlation function ρ_n of this ensemble, given as the probability measure on $\mathbb{R}$ so that for any bounded measurable function f

$$\mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} f(\lambda_i)] = \int f(x)\rho_n(x)dx$$

as well as the localization of the quantiles of ρ_n with respect to the quantiles of the limiting semi-circle distribution. In particular, we study the $1/n$ expansion of this localization, showing that it depends on the fourth moment of μ. Define $N_n(x) := \frac{1}{n}\sharp\{i, \lambda_i \leq x\}$, with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $N_{sc}(x) = \int_{-\infty}^{x} d\sigma_{sc}(u)$, with σ_{sc} defined in 27. Let us define the quantiles $\hat{\gamma}_i$ (resp. γ_i) by

$$\hat{\gamma}_i := \inf\left\{y, \mathbb{E}N_n(y) = \frac{i}{n}\right\} \text{ resp. } \sigma_{sc}((-\infty, \gamma_i]) = i/n.$$

We shall prove that

Theorem 3.3. *Let $\varepsilon > 0$. There exists functions C, D on $[-2+\varepsilon, 2-\varepsilon]$, independent of the distributions μ, μ', such that for all $x \in [-2+\varepsilon, 2-\varepsilon]$*

$$\rho_n(x) = \sigma_{sc}(x) + \frac{1}{n}C(x) + \frac{1}{n}\kappa_4 D(x) + o(\frac{1}{n})\,.$$

For all $i \in [n\varepsilon, n(1-\varepsilon)]$ for some $\varepsilon > 0$, there exists a constant C_i independent of κ_4 so that

$$\hat{\gamma}_i - \gamma_i \quad = \quad \frac{C_i}{n} + \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + o(\frac{1}{n}). \tag{6}$$

This is a version of the rescaled Tao-Vu conjecture 1.7 in [27] (using the fact that the variance of the entries of W is $1/4$ instead of 1) where $\mathbb{E}[\lambda_i]$ is replaced by $\hat{\gamma}_i$. A similar result could be derived for Johansson-Laguerre ensembles. We do not present the detail of the computation here, which would ressemble the Wigner case.

4 Smallest Singular Values of $n+\nu$ by n complex Gaussian matrices

Theorem 3.1 depends on the partition function for Gaussian matrices, which itself depends on ν and n. In this section, we investigate these dependencies.

4.1 Known exact results

It is worthwhile to review what exact representations are known for the smallest singular values of complex Gaussians.

We consider the finite n density $f_n^\nu(x)$, the finite n distribution $F_n^\nu(x)$ (which was denoted g_n in the previous section when the variance could vary), and their asymptotic values $f_\infty^\nu(x)$ and $F_\infty^\nu(x)$. We have found the first form in the list below useful for symbolic and numerical computation. In the formulas to follow, we assume $\sigma_{\Re}^2 = 1$ so that a command such as `randn()` can be used without modification for the real and imaginary parts. All formulas concern $n\lambda_{\min} = n\sigma_{\min}^2$ and its asymptotics. We present in the array below eight different formulations of the exact distribution F_n^ν.

1. Determinant: ν by ν	[14, 15]
2. Painléve III	[14, Eq. (8.93)]
3. Determinant: n by n	[8]
4. Fredholm Determinant	[7, 28]
5. Multivariate Integral Recurrence	[11, 15]
6. Finite sum of Schur Polynomials (evaluated at $\mathbb{I}$)	[9]
7. Hypergeometric Function of Matrix Argument	[9]
8. Confluent Hypergeometric Function of Matrix Argument	[24]

Table 1: Exact Results for smallest singular values of complex Gaussians (smallest eigenvalues of complex Wishart or Laguerre Ensembles)

Some of these formulations allow one or both of ν or n to extend beyond integers to real positive values. Assuming ν and n are integers [11, Theorem 5.4], the probability density $f_n^\nu(x)$ takes the form $x^\nu e^{-x/2}$ times a polynomial of degree $(n-1)\nu$ and $1 - F_n^\nu(x)$ is $e^{-x/2}$ times a polynomial of degree $n\nu$.

Remark: A helpful trick to compare normalizations used by different authors is to inspect the exponential term. The 2 in $e^{-x/2}$ denotes total complex variance 2 (twice the real variance of 1). In general the total complex variance $\sigma^2 = 2\sigma_{\Re}^2$ will appear in the denominator.

In the next paragraphs, we discuss the eight formulations introduced above.

4.1.1 Determinant: ν by ν determinant

The quantities of primary use are the beautiful ν by ν determinant formulas for the distributions by Forrester and Hughes [15] in terms of Bessel functions and Laguerre polynomials. The infinite formulas also appear in [14, Equation (8.98)].

$$F_\infty^\nu(x) = 1 - e^{-x/2} \det[I_{i-j}(\sqrt{2x})]_{i,j=1,\ldots,\nu}.$$
$$f_\infty^\nu(x) = \tfrac{1}{2} e^{-x/2} \det[I_{2+i-j}(\sqrt{2x})]_{i,j=1,\ldots,\nu}.$$
$$F_n^\nu(x) = 1 - e^{-x/2} \det\left[L_{n+i-j}^{(j-i)}(-x/2n)\right]_{i,j=1,\ldots,\nu}.$$
$$f_n^\nu(x) = \left(\tfrac{x}{2n}\right)^\nu \tfrac{(n-1)!}{2(n+\nu-1)!} e^{-x/2} \det\left[L_{n-1+i-j}^{(j-i+2)}(-x/2n)\right]_{i,j=1,\ldots,\nu}.$$

Recall that $I_j(x) = I_{-j}(x)$. To facilitate reading of the relevant ν by ν

determinants we provide expanded views:

$$\det[I_{i-j}(\sqrt{2x})]_{i,j=1,\ldots,\nu} = \left|\begin{matrix} I_0 & I_1 & I_2 & \cdots & I_{\nu-1} \\ I_1 & I_0 & I_1 & \cdots & I_{\nu-2} \\ I_2 & I_1 & I_0 & \cdots & I_{\nu-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{\nu-1} & I_{\nu-2} & I_{\nu-3} & \cdots & I_0 \end{matrix}\right|_{\text{Bessel functions evaluated at } \sqrt{2x}}$$

$$\det[I_{2+i-j}(\sqrt{2x})]_{i,j=1,\ldots,\nu} = \left|\begin{matrix} I_2 & I_1 & I_0 & \cdots & I_{\nu-3} \\ I_3 & I_2 & I_1 & \cdots & I_{\nu-4} \\ I_4 & I_1 & I_2 & \cdots & I_{\nu-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{\nu+1} & I_{\nu} & I_{\nu-1} & \cdots & I_2 \end{matrix}\right|_{\text{Bessel functions evaluated at } \sqrt{2x}}$$

$$\det\left[L_{n+i-j}^{(j-i)}(-\frac{x}{2n})\right]_{i,j=1,\ldots,\nu} = \left|\begin{matrix} L_n & L_{n-1}^{(1)} & L_{n-2}^{(2)} & \cdots & L_{n-\nu+1}^{(\nu-1)} \\ L_{n+1}^{(-1)} & L_n & L_{n-1}^{(1)} & \cdots & L_{n-\nu+2}^{(\nu-2)} \\ L_{n+2}^{(-2)} & L_{n+1}^{(-1)} & L_n & \cdots & L_{n-\nu+3}^{(\nu-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n+\nu-1}^{(1-\nu)} & L_{n+\nu-2}^{(2-\nu)} & L_{n+\nu-3}^{(3-\nu)} & \cdots & L_n \end{matrix}\right|_{\text{evaluated at } -x/2n}$$

$$\det\left[L_{n-1+i-j}^{(j-i+2)}(-\frac{x}{2n})\right]_{i,j=1,\ldots,\nu} = \left|\begin{matrix} L_{n-1}^{(2)} & L_{n-2}^{(3)} & L_{n-3}^{(4)} & \cdots & L_{n-\nu}^{(\nu+1)} \\ L_{n}^{(1)} & L_{n-1}^{(2)} & L_{n-2}^{(3)} & \cdots & L_{n-\nu+1}^{(\nu)} \\ L_{n+1} & L_{n}^{(1)} & L_{n-1}^{(2)} & \cdots & L_{n-\nu+2}^{(\nu-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n+\nu-2}^{(3-\nu)} & L_{n+\nu-3}^{(4-\nu)} & L_{n+\nu-4}^{(5-\nu)} & \cdots & L_{n-1}^{(2)} \end{matrix}\right|_{\text{evaluated at } -x/2n}$$

The following Mathematica code symbolically computes these distributions

```
M[x_, v_]     := Table[ BesselI[Abs[i - j], x],     {i,v}, {j,v}];
m[x_, v_]     := Table[ BesselI[Abs[2 + i - j], x], {i,v}, {j,v}]
M[x_, n_, v_] := Table[ LaguerreL[n+i-j, j - i,  -x/(2*n)], {i,v}, {j,v}];
m[x_, n_, v_] := Table[ LaguerreL[n-1+i-j,j-i+2, -x/(2*n)], {i,v}, {j,v}];
F[x_, v_  ]   :=  1 - Exp[-x/2]*Det [M[Sqrt[2 x], v]]
f[x_, v_]     := (1/2)*Exp[-x/2]*Det[m[Sqrt[2 x], v]]
F[x_, n_, v_] :=  1 - Exp[-x/2]*Det [M[x,n,v]]
f[x_, n_, v_] := (x/(2 n))^v*((n - 1)!/(2 (n+v-1)!))*Exp[-x/2]*Det[m[x,n,v]]
```

4.1.2 Painléve III

According to [14, Eq. (8.93)], [7, p. 814-815], [28,29] we have the formula valid for all $\nu > 0$

$$F_\infty^\nu(x) = \exp\left(-\int_0^{2t} \sigma(s)\frac{ds}{s}\right),$$

where $\sigma(s)$ is the solution to a Painléve III differential equation. Please consult the references taking care to match the normalization.

4.1.3 n by n determinant:

Following standard techniques to set up the multivariate integral and applying a continuous version of the Cauchy-Binet theorem (Gram's Formula) [22, e.g., Appendix A.12] or [30, e.g. Eqs. (1.3) and (5.2)] one can work out an $n \times n$ determinant valid for any ν, so long as n is an integer [8].

$$F_n^\nu(x) = \frac{\det(M(m,\nu,x/2))}{\det(M(m,\nu,0))}.$$

where

$$M(m,\nu,x) = \begin{pmatrix} \Gamma(\nu+1,x) & \Gamma(\nu+2,x) & \Gamma(\nu+3,x) & \cdots & \Gamma(\nu+m,x) \\ \Gamma(\nu+2,x) & \Gamma(\nu+3,x) & \Gamma(\nu+4,x) & \cdots & \Gamma(\nu+m+1,x) \\ \Gamma(\nu+3,x) & \Gamma(\nu+4,x) & \Gamma(\nu+5,x) & \cdots & \Gamma(\nu+m+2,x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Gamma(\nu+m,x) & \Gamma(\nu+m+1,x) & \Gamma(\nu+m+2,x) & \cdots & \Gamma(\nu+2m-1,x) \end{pmatrix}.$$

4.1.4 Remaining Formulas in Table 1

The Fredholm determinant is a standard procedure. The multivariate integral recurrence was computed in the real case in [11] and in the complex case in [15]. Various hypergeometric representations may be found in [9], but to date we are not aware of the complex representation of the confluent representation in [24] which probably is worth pursuing.

4.2 Asymptotics of Smallest Singular Value Densities of Complex Gaussians

A very useful expansion extends a result from [15, (3.29)]

Lemma 4.1. *As $n \to \infty$, we have the first two terms in the asymptotic expansion of scaled Laguerre polynomials whose degree and constant parameter sum to n:*

$$L_{n-k}^{(k)}(-x/n) \sim n^k \left\{\frac{I_k(2\sqrt{x})}{x^{k/2}} - \frac{1}{2n}\left(\frac{I_{k-2}(2\sqrt{x})}{x^{(k-2)/2}}\right) + O\left(\frac{1}{n^2}\right)\right\}$$

Proof. We omit the tedious details but this (and indeed generalizations of this result) may be computed either through direct expansion of the Laguerre polynomial or through the differential equation it satisfies.

We can use the lemma above to obtain asymptotics of the distribution $F_n^{(\nu)}(x)$. As a result, we have ample evidence to believe the following conjecture:

Conjecture 1. *(Verified correct for $\nu = 0, 1, 2, \ldots, 25$) Let $F_n^{(\nu)}(x)$ be the distribution of $n\sigma_{\min}^2$ of an $n+\nu$ by n complex Gaussian. We propose that*

$$F_n^{(\nu)}(x) = F_\infty^{(\nu)}(x) + \frac{\nu}{2n} x f_\infty^{(\nu)}(x) + O(\frac{1}{n^2})$$

note: The above is readily checked to be scale invariant, so it is not necessary to state the particular variances in the matrix as long as they are equal.

In light of Lemma 4.1, our conjecture may be deduced from

Conjecture 2. *Consider the Bessel function (evaluated at x) determinant*

$$\begin{vmatrix} I_0 & I_1 & I_2 & \cdots & I_{\nu-1} \\ I_1 & I_0 & I_1 & \cdots & I_{\nu-2} \\ I_2 & I_1 & I_0 & \cdots & I_{\nu-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{\nu-1} & I_{\nu-2} & I_{\nu-3} & \cdots & I_0 \end{vmatrix}.$$

We propose that the following determinant equation is an equality for $\nu \geq 2$, where the first/second determinant below on the left side of the equal sign is identical to the above except for the first/second column respectively.

$$\begin{vmatrix} I_2 & I_1 & I_2 \cdots I_{\nu-1} \\ I_3 & I_0 & I_1 \cdots I_{\nu-2} \\ I_4 & I_1 & I_0 \cdots I_{\nu-3} \\ \vdots & \vdots & \vdots \ddots \vdots \\ I_{\nu+1} & I_{\nu-2} & I_{\nu-3} \cdots I_0 \end{vmatrix} + \begin{vmatrix} I_0 & I_1 & I_2 \cdots I_{\nu-1} \\ I_1 & I_2 & I_1 \cdots I_{\nu-2} \\ I_2 & I_3 & I_0 \cdots I_{\nu-3} \\ \vdots & \vdots & \vdots \ddots \vdots \\ I_{\nu-1} & I_\nu & I_{\nu-3} \cdots I_0 \end{vmatrix} = \nu \begin{vmatrix} I_2 & I_1 & I_0 \cdots I_{\nu-3} \\ I_3 & I_2 & I_1 \cdots I_{\nu-4} \\ I_4 & I_1 & I_2 \cdots I_{\nu-5} \\ \vdots & \vdots & \vdots \ddots \vdots \\ I_{\nu+1} & I_\nu & I_{\nu-1} \cdots I_2 \end{vmatrix}.$$

Proof. This may be obtained by comparing the asymptotics of $F_n^\nu(x)$ using Lemma 4.1, and taking the derivative of the determinant for $F_\infty^\nu(x)$, using the derivative of $\frac{d}{dx} I_j(x) = \frac{1}{2}(I_{j+1}(x) + I_{j-1}(x))$ and the usual multilinear properties of determinants.

Remark: This conjecture has been verified symbolically for $\nu = 2, \ldots, 25$ symbolically in Mathematica and Maple, and numerically for larger values.

Our main interest in this conjecture is that once granted it would give the following corollary of Theorem 3.1. (Verified at this time for $\nu \leq 25$.)

Conjecture 3. *Suppose we have a non-Gaussian $n+\nu$ by n random matrix with real kurtosis γ. Then with $\lambda_{\min}$ as the square of the smallest singular value,*

$$P(n\lambda_{\min} \geq x) = F_\infty^\nu(x) + \frac{\nu+\gamma}{2n} f_\infty^\nu(x) + O(1/n^2).$$

5 The smallest eigenvalue in Johansson-Laguerre ensemble

5.1 Reminder on Johansson-Laguerre ensemble

We here recall some important facts about the Johansson-Laguerre ensemble, that we use in the following.

Notations: We call $\mu_{n,p}$ the law of the sample covariance matrix $\frac{1}{n}M^*M$ defined in (1). We denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the ordered eigenvalues of the random sample covariance matrix $\frac{1}{n}M^*M$. We also set

$$H = \frac{W}{\sqrt{n}},$$

and denote the distribution of the random matrix H by P_n. The ordered eigenvalues of HH^* are denoted by $y_1(H) \leq y_2(H) \leq \cdots \leq y_n(H)$.

We can now state the known results about the joint eigenvalue density (j.e.d.) induced by the Johansson-Laguerre ensemble. By construction, this is obtained as the integral w.r.t. P_n of the j.e.d. of the Deformed Laguerre Ensemble . The latter has been first computed by [16] and [18].

We now set

$$s = \frac{a^2}{n}.$$

Proposition 5.1. *The symmetrized eigenvalue measure on $\mathbb{R}_+^n$ induced by $\mu_{n,p}$ has a density w.r.t. Lebesgue measure given by*

$$g(x_1, \ldots, x_n) = \int dP_n(H) \frac{\Delta(x)}{\Delta(y(H))} det\Big(\frac{e^{-\frac{y_i(H)+x_j}{2t}}}{2t} I_\nu(\frac{\sqrt{y_i(H)x_j}}{t})\Big(\frac{x_j}{y_i(H)}\Big)^{\frac{\nu}{2}}\Big)_{i,j=1}^n, \tag{7}$$

where $t = \frac{a^2}{2n} = \frac{s}{2}$, and $\Delta(x) = \prod_{i<j}(x_i - x_j)$.

From the above computation, all eigenvalue statistics can in principle be computed. In particular, the m-point correlation functions of $\mu_{n,p}$ defined by $R_m(u_1, \ldots, u_m) = \frac{n!}{(n-m)!} \int_{\mathbb{R}_+^{n-m}} g(u_1, \ldots, u_n) \prod_{i=m+1}^n du_i$ are given by the integral w.r.t. to $dP_n(H)$ of those of the Deformed Laguerre Ensemble, i.e. the covariance matrix $n^{-1}M_nM_n^*$ when H is given. Let $R_m(u, v; y(H))$ be the m-point correlation function of the Deformed Laguerre Ensemble (defined by the fixed matrix H). Then

Proposition 5.2.

$$R_m(u_1, \ldots, u_m) = \int_{M_{p,n}(\mathbb{C})} dP_n(H) R_m(u_1, \ldots, u_m; y(H)).$$

The second remarkable fact is that the Deformed Laguerre Ensemble induces a determinantal random point field, that is all the m-point correlation functions are given by the determinant of a $m \times m$ matrix involving the same *correlation kernel*.

Proposition 5.3. *Let m be a given integer. Then one has that*

$$R_m(u_1, \ldots, u_m; y(H)) = \det \left(K_n(u_i, u_j; y(H))\right)_{i,j=1}^m,$$

where the correlation kernel K_n is defined in Theorem 5.4 below.

There are two important facts about this determinantal structure. The fundamental characteristic of the correlation kernel is that it depends only on the spectrum of HH^* and more precisely on its spectral measure. Since we are interested in the determinant of matrices with entries $K_n(x_i, x_j; y)$, we can consider the correlation kernel up to a conjugation: $K_n(x_i, x_j)\frac{f(x_i)}{f(x_j)}$. This has no impact on correlation functions and we may use this fact later.

Theorem 5.4. *The correlation kernel of the Deformed Laguerre Ensemble (H is fixed) is also given by*

$$\begin{aligned} K_n(u, v; y(H)) &= \frac{1}{i\pi s^3} e^{i\nu\pi} \int_\Gamma \int_\gamma dw dz wz K_B\Big(\frac{2zu^{1/2}}{s}, \frac{2wv^{1/2}}{s}\Big) \Big(\frac{w}{z}\Big)^\nu \\ &\times \prod_{i=1}^n \frac{w^2 - y_i(H)}{z^2 - y_i(H)} \exp\{\frac{w^2 - z^2}{s}\}\Big(1 - s\sum_{i=1}^n \frac{y_i(H)}{(w^2 - y_i(H))(z^2 - y_i(H))}\Big). \end{aligned} \tag{8}$$

where the contour Γ is symmetric around 0 and encircles the $\pm\sqrt{y_i(H)}$, γ is the imaginary axis oriented positively $0 \longrightarrow +\infty$, $0 \longrightarrow -\infty$, and K_B is the kernel defined by

$$K_B(x, y) = \frac{xI_\nu'(x)I_\nu(y) - yI_\nu'(y)I_\nu(x)}{x^2 - y^2}. \tag{9}$$

For ease of exposition, we drop from now on the dependency of the correlation kernel K_n on the spectrum of H and write $K_n(u, v)$ for $K_n(u, v; y(H))$. The goal of this section is to deduce Theorem 3.1 by a careful asymptotic analysis of the above formulas.

5.2 Asymptotic expansion of the partition function at the hard edge

The main result of this section is to prove the following expansion for the partition function at the hard edge: Set $\alpha = \sigma^2/4$ with $\sigma = \sqrt{1/4 + a^2}$.

Theorem 1. *There exists a non-negative function g_n^0, depending on n, so that*

$$\mathbb{P}\left(\lambda_{min} \geq \frac{\alpha s}{n^2}\right) = g_n^0(s) + \frac{1}{n}\partial_\beta g_n^0(\beta s)|_{\beta=1} \int dP_n(H)[\Delta_n(H)] + o(\frac{1}{n})$$

where

$$\Delta_n(H) = \frac{-1}{v_c^{\pm} m_0'(v_c^{\pm})} X_n(v_c^{\pm})$$

with $X_n(z) = \sum_{i=1}^n \frac{1}{y_i(H)-z} - n m_0(z)$, $m_0(z) = \int (x-z)^{-1}\rho(dx)$ *is the Stieltjes transform of the Marchenko-Pastur distribution* ρ, $(y_i)_{1\leq i\leq n}$ *are the eigenvalues of* H, *and* $v_c^{\pm} = (w_c^{\pm})^2$ *where*

$$w_c^{\pm} = \pm i(R - 1/R)/2, \quad R := \sqrt{1+4a^2}.$$

We will estimate the term $\int dP_n(H)[\Delta_n(H)]$ in terms of the kurtosis in the next section. We prove Theorem 1 in the next subsections.

5.2.1 Expansion of the correlation kernel

Let $z_c^{\pm}$ be the critical points of

$$F_n(w) := w^2/a^2 + \frac{1}{n}\sum_{i=1}^n \ln(w^2 - y_i), \tag{10}$$

where the y_i are the eigenvalues of H^*H. Then we have the following Lemma. Let K_n be the kernel defined in Theorem 5.4.

Lemma 5.5. *There exists a smooth function* A *such that for all* x, y

$$\frac{\alpha}{n^2} K_n(u\alpha n^{-2}, v\alpha n^{-2}; y(H)) =$$
$$\widetilde{K_B}(u,v) + \frac{A(u,v)}{n} + ((z_c/w_c)^2 - 1)\frac{\partial}{\partial\beta}\Big|_{\beta=1} \beta\widetilde{K_B}(\beta u, \beta v) + o(\frac{1}{n}).$$

where $\widetilde{K_B}$ *is the usual Bessel kernel*

$$\widetilde{K_B}(u,v) := e^{i\nu\pi} K_B(i\sqrt{u}, i\sqrt{v})$$

with K_B *defined in* (9).

Proof

To focus on local eigenvalue statistics at the hard edge, we consider

$$u = \left(\frac{a^2}{2nr_0}\right)^2 x; \quad v = \left(\frac{a^2}{2nr_0}\right)^2 y, \text{ where } r_0 \text{ will be fixed later.}$$

As $\nu = p - n$ is a fixed integer independent of n, this readily implies that the Bessel kernel shall not play a role in the large exponential term of the correlation kernel. In other words, the large exponential term to be considered is F_n defined in (10). The correlation kernel can then be re-written as

$$K_n(u,v) = \frac{1}{i\pi s^3} e^{i\nu\pi} \int_\Gamma \int_\gamma dwdz\, w\, z\, K_B\Big(\frac{zx^{1/2}}{r_0}, \frac{wy^{1/2}}{r_0}\Big)\left(\frac{w}{z}\right)^\nu$$

$$\times \exp\{nF_n(w) - nF_n(z)\}\tilde{g}(w,z), \tag{11}$$

where

$$\tilde{g}(w,z) := a^2 g(w,z) = 1 - s\sum_{i=1}^{n} \frac{y_i}{(w^2-y_i)(z^2-y_i)} = \frac{a^2}{2}\frac{wF_n'(w) - zF_n'(z)}{w^2-z^2}.$$

We note that $F_n(w) = H_n(w^2)$ where $H_n(w) = w/a^2 + \frac{1}{n}\sum_{i=1}^{n}\ln(w - y_i)$.

We may compare the exponential term F_n to its "limit", using the convergence of the spectral measure of H^*H to the Marchenko-Pastur distribution ρ. Set

$$F(w) := w^2/a^2 + \int \ln(w^2 - y)d\rho(y).$$

It was proved in [5] that this asymptotic exponential term has two conjugated critical points satisfying $F'(w) = 0$ and which are given by

$$w_c^{\pm} = \pm i(R - 1/R)/2, \quad R := \sqrt{1 + 4a^2}.$$

Let us also denote by $z_c^{\pm}$ the true non real critical points (which can be seen to exist and be conjugate [5]) associated to F_n. These critical points do depend on n but for ease of notation we do not stress this dependence. These critical points satisfy

$$F_n'(z^{\pm}) = 0, z_c^+ = -z_c^-$$

and it is not difficult to see that they are also on the imaginary axis.

We now refer to the results established in [5] to claim the following facts:

- there exist constants C and $\xi > 0$ such that

$$|z_c^{\pm} - w_c^{\pm}| \leq Cn^{-\xi}.$$

  This comes from concentration results for the spectral measure of H established in [17] and [2].

- Fix $\theta > 0$. By the saddle point analysis performed in [5], the contribution of the parts of the contours γ and Γ within $\{|w - z_c^{\pm}| \geq n^{\theta}n^{-1/2}\}$ is $O(e^{-cn^{\theta}})$ for some $c > 0$. This contribution "far from the critical points " is thus exponentially negligible. In the sequel we will choose $\theta = 1/11$. The choice of $1/11$ is arbitrary.

- We can thus restrict both the w and z integrals to neighborhoods of width $n^{1/11}n^{-1/2}$ of the critical points $z_c^{\pm}$.

Also , we can assume that the parts of the contours Γ and γ that will contribute to the asymptotics are symmetric w.r.t. $z_c^{\pm}$. This comes from the fact that the initials contours exhibit this symmetry and the location of the critical points. A plot of the oriented contours close to critical points is given in Figure 5.2.1.

Let us now make the change of variables

$$w = z_c^1 + sn^{-1/2}; \quad z = z_c^2 + tn^{-1/2};$$

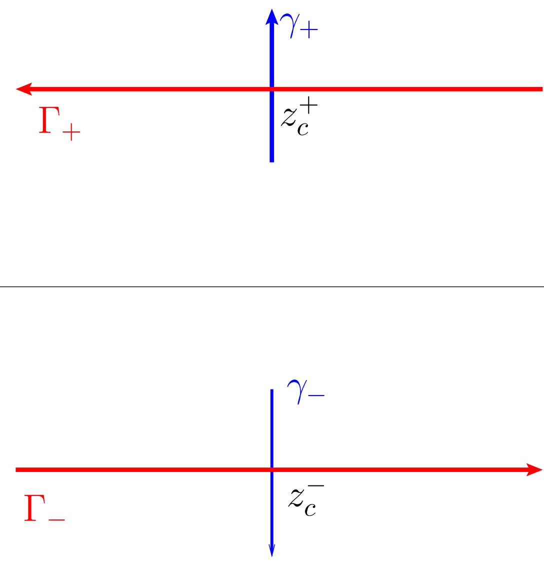

Figure 5: Contours close to the critical points

where $z_c^1, z_c^2 = z_c^{\pm}$ and the $\pm$ depends on the part of the contours γ and Γ under consideration and s, t satisfy $|s|, |t| \leq n^{1/11}$. Then we perform the Taylor expansion of each of the terms arising in both z and w integrands. Then one has that

$$
\begin{aligned}
& e^{nF(z_c^{\pm}+sn^{-1/2})-nF(z_c^{\pm})} \\
&= e^{F''(z_c^{\pm})\frac{s^2}{2}+\sum_{i=3}^{5} F^{(i)}(z_c^{\pm})\frac{s^i}{i!n^{i/2-1}}}(1+O(n^{-23/22})) \\
&= e^{F''(z_c^{\pm})\frac{s^2}{2}} + \frac{1}{n^{1/2}}\underbrace{e^{F''(z_c^{\pm})\frac{s^2}{2}}\frac{F^3(z_c^{\pm})}{6}s}_{e_1(s)} \\
&+\frac{1}{n}\underbrace{e^{F''(z_c^{\pm})\frac{s^2}{2}}\Big(\frac{F^{(4)}(z_c^{\pm})s^4}{4!}+\Big(\frac{F^3(z_c^{\pm})}{6}\Big)^2\frac{s^6}{2}\Big)}_{e_2(s)}+o(\frac{1}{n})e^{F''(z_c^{\pm})\frac{s^2}{2}}.
\end{aligned}
\tag{12}
$$

as $|s| \leq n^{1/11}$. For each term in the integrand, one has to consider the contribution of equal or opposite critical points. In the following, we denote by z_c, z_c^1, z_c^2 any of the two critical points (allowing z_c to take different values with a slight abuse of notation). We then perform the Taylor expansion of each of the functions arising in the integrands.

$$
wz = z_c^1 z_c^2 + n^{-1/2}\underbrace{(sz_c^2+tz_c^1)}_{v_1(s,t)}+\frac{1}{n}\underbrace{st}_{v_2(s,t)},
$$

$$g\left(z_c^1+\frac{s}{n^{1/2}},z_c^2+\frac{t}{n^{1/2}}\right)=\frac{F_n''(z_c)}{2}\mathbb{1}_{z_c^1=z_c^2}+\frac{1}{\sqrt{n}}\underbrace{\left(s\frac{\partial}{\partial x_1}+t\frac{\partial}{\partial x_2}\right)g(x_1,x_2)\Big|_{z_c^1,z_c^2}}_{g_1(s,t)}$$

$$+\frac{1}{n}\underbrace{\left((\frac{s^2}{2}\frac{\partial^2}{\partial x_1^2}g(x_1,x_2)+\frac{t^2}{2}\frac{\partial^2}{\partial x_2^2}+st\frac{\partial^2}{\partial x_2\partial x_1})g(x_1,x_2)\Big|_{z_c^1,z_c^2}\right)}_{g_2(s,t)}+o(\frac{1}{n}).$$

$$\left(\frac{w}{z}\right)^\nu=(z_c^1/z_c^2)^\nu+n^{-1/2}\underbrace{(z_c^1/z_c^2)^\nu(\frac{\nu s}{z_c^1}-\frac{\nu t}{z_c^2})}_{r_1(s,t)}$$

$$+\frac{1}{n}\underbrace{(z_c^1/z_c^2)^\nu\left(\frac{\nu(\nu-1)s^2}{(z_c^1)^2}+\frac{\nu(\nu+1)t^2}{(z_c^2)^2}-\frac{\nu^2st}{z_c^1z_c^2}\right)}_{r_2(s,t)}+o(\frac{1}{n}).$$

$$K_B\left(\frac{zx^{1/2}}{r_0},\frac{wy^{1/2}}{r_0}\right)=K_B\left(\frac{z_cx^{1/2}}{r_0},\frac{z_cy^{1/2}}{r_0}\right)$$

$$+\frac{1}{\sqrt{n}}\underbrace{\left(s\frac{\partial}{\partial x_1}+t\frac{\partial}{\partial x_2}\right)\Big|_{z_c,z_c}K_B\left(\frac{x_1x^{1/2}}{r_0},\frac{x_2y^{1/2}}{r_0}\right)}_{h_1(s,t)}$$

$$+\frac{1}{n}\underbrace{\left(\frac{s^2}{2}\frac{\partial^2}{\partial x_1^2}+\frac{t^2}{2}\frac{\partial^2}{\partial x_2^2}+st\frac{\partial^2}{\partial x_1\partial x_2}\right)\Big|_{z_c,z_c}K_B\left(\frac{x_1x^{1/2}}{r_0},\frac{x_2y^{1/2}}{r_0}\right)}_{h_2(s,t)}+o(\frac{1}{n}). \tag{13}$$

In all the lines above, $z_c^1/z_c^2=\pm 1$ depending on equal or opposite critical points. Also one can note that the o are uniform as long as $|s|,|t|<n^{1/11}$.

We now choose

$$r_0=|w_c^+|.$$

Combining the whole contribution of neighborhoods of a pair of equal critical points e.g., denoted by $K_n(u,v)_{equal}$, we find that it has an expansion of the form

$$\begin{aligned}
&\frac{a^4}{4n^2r_0^2}K_n(u,v)_{equal}=\sum_{z_c=z_c^\pm}\\
&\frac{\pm}{4i\pi}e^{i\nu\pi}\int_{\mathbb{R}}\int_{i\mathbb{R}}dsdt\frac{|z_c|^2}{r_0^2}\left(K_B(\frac{z_cx^{1/2}}{|w_c^+|},\frac{z_cy^{1/2}}{|w_c^+|})+\sum_{i=1}^2\frac{h_i(s,t)}{n^{i/2}}+o(\frac{1}{n})\right)\\
&\times\left(\frac{F''(z_c)}{2}+\sum_{i=1}^2\frac{g_i(s,t)}{n^{i/2}}+o(\frac{1}{n})\right)(1+\sum_{i=1}^2\frac{r_i(s,t)}{n^{i/2}}+o(\frac{1}{n}))\\
&\times(1+\sum_{i=1}^2 v_i(s,t)n^{-i/2}z_c^{-2})\Big(\exp\{F''(z_c)(s^2-t^2)/2\}(1+o(\frac{1}{n}))\\
&+n^{-1/2}(e_1(s)-e_1(t))+\frac{1}{n}(-e_1(s)e_1(t)+e_2(s)-e_2(t))\Big),
\end{aligned} \tag{14}$$

where h_i, e_i, r_i, v_i and g_i defined above have no singularity.

It is not difficult also to see that h_1, g_1, r_1, e_1 are odd functions in s as well as in t: because of the symmetry of the contour, their contribution will thus vanish. The first non zero lower order term in the asymptotic expansion will thus come from the combined contributions $h_1g_1, g_1r_1, r_1h_1, h_1e_1, g_1e_1, r_1e_1, r_1v_1$... and those from h_2, g_2, r_2, e_2, v_2.Therefore one can check that one gets the expansion

$$\frac{\alpha}{n^2}K_n(\frac{\alpha x}{n^2},\frac{\alpha y}{n^2})_{equal} = \frac{e^{i\nu\pi}}{2}(\frac{|z_c^{\pm}|}{|w_c^{\pm}|})^2 K_B(\frac{z_c^{\pm}x^{1/2}}{|w_c^+|},\frac{z_c^{\pm}y^{1/2}}{|w_c^+|}) + \frac{a_1(z_c^{\pm};x,y)}{n} + o(\frac{1}{n}), \tag{15}$$

where a_1 is a function of $z_c^{\pm}, x, y$ only. a_1 is a smooth and non vanishing function a priori.

We can write the first term above as $(\frac{z_c^{\pm}}{w_c^{\pm}})^2\widetilde{K_B}((\frac{z_c^{\pm}}{w_c^{\pm}})^2x^{1/2},(\frac{z_c^{\pm}}{w_c^{\pm}})^2y^{1/2})$ so that we deduce that

$$\begin{aligned}&e^{i\nu\pi}(\frac{z_c^{\pm}}{|w_c^{\pm}|})^2K_B(\frac{z_c^{\pm}x}{|w_c^+|},\frac{z_c^{\pm}y}{|w_c^+|})\\&=\widetilde{K_B}(x,y)+\left((\frac{z_c^{\pm}}{w_c^{\pm}})^2-1\right)\partial_\beta(\beta\widetilde{K_B}(\beta x,\beta y))|_{\beta=1}+o(z_c^{\pm}-w_c^{\pm})\,.\end{aligned}$$

One can do the same thing for the combined contribution of opposite critical points and get a similar result. We refer to [5] for more detail about this fact.

5.2.2 Asymptotic expansion of the density

The distribution of the smallest eigenvalue of M_n is defined by

$$\mathbb{P}\left(\lambda_{min}\geq\frac{\alpha s}{n^2}\right)=\int dP_n(H)\det(I-\tilde{K}_n)_{L^2(0,s)},$$

where $\tilde{K}_n$ is the rescaled correlation kernel $\frac{\alpha}{n^2}K_n(x\alpha n^{-2}, y\alpha n^{-2})$. In the above we choose $\alpha = (a^2/2r_0)^2$. The limiting correlation kernel is then, at the first order, the Bessel kernel:

$$\widetilde{K_B}(x,y):=e^{i\nu\pi}K_B(i\sqrt{x},i\sqrt{y}).$$

The error terms are ordered according to their order of magnitude: the first order error term, in the order of $O(n^{-1})$, can thus come from two terms in (15), namely
-the deterministic part that is $a_1(z_c^{\pm};x,y)$. These terms yield a contribution in the order of $\frac{1}{n}$. However it is clear that as a_1 is smooth

$$a_1(z_c^{\pm};x,y)=a_1(w_c^{\pm};x,y)+o(1)\,.$$

As a consequence there is no fourth moment contribution in these $\frac{1}{n}$ terms. We denote the contribution of the deterministic error from all the combined (equal

or not) critical points by $A(x,y)/n$.
-the kernel (arising 4 times due to the combination of critical points)

$$e^{i\nu\pi}(\frac{z_c^+}{|w_c^+|})^2 K_B(\frac{z_c^+}{|w_c^+|}(\sqrt{x},\sqrt{y})) = \widetilde{K_B}(x,y) + \int_1^{|z_c^+/w_c^+|^2} \frac{\partial}{\partial\beta}\beta\widetilde{K_B}(\beta x,\beta y)d\beta.$$

Combining all the arguments above, one then gets the following:

$$\begin{aligned}&\frac{\alpha}{n^2}K_n(x\alpha n^{-2}, y\alpha n^{-2})\\ &= \widetilde{K_B}(x,y) + \frac{A(x,y)}{n} + ((z_c^+/w_c^+)^2-1)\frac{\partial}{\partial\beta}\Big|_{\beta=1}\beta\widetilde{K_B}(\beta x,\beta y) + o(\frac{1}{n}).\end{aligned}$$

The Fredholm determinant can be developed to obtain that

$$\begin{aligned}&\det(I-\tilde{K}_n)_{L^2(0,s)}\\ &= \sum_k \frac{(-1)^k}{k!}\int_{[0,s]^k} \det\left(\tilde{K}_n(x_i,x_j)\right)_{i,j=1}^k\\ &= \sum_k \frac{(-1)^k}{k!}\int_{[0,s]^k} \det\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k \det\left(I+G(x_i,x_j)\right)_{i,j=1}^k,\end{aligned} \tag{16}$$

where we have set

$$G(x_i,x_j) = \left(\widetilde{K_B}(x_i,x_j)\right)_{1\le i,j\le k}^{-1} \left(B(x_i,x_j)\right)_{i,j=1}^k$$

with

$$B(x_i,x_j) = \frac{A(x_i,x_j)}{n} + 2(z_c^+/w_c^+-1)\frac{\partial}{\partial\beta}\Big|_{\beta=1}\beta\widetilde{K_B}(\beta x_i,\beta x_j) + o(\frac{1}{n}).$$

The matrix $\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k$ is indeed invertible for any k.

Therefore, up to an error term in the order $o(\frac{1}{n})$ at most,

$$\begin{aligned}&\det(I-\tilde{K}_n)_{L^2(0,s)}\\ &= \det(I-\widetilde{K_B}) + \sum_k \frac{(-1)^k}{k!}\int_{[0,s]^k} \det\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k \operatorname{Tr}\left(G(x_i,x_j)\right)_{i,j=1}^k dx\end{aligned} \tag{17}$$

now if we just consider the term which is linear in (z_c/w_c-1) which will bring the contribution depending on the fourth cumulant we have that the correction is

$$\sum_k \frac{(-1)^k}{k!}\int_{[0,s]^k} \det\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k \operatorname{Tr}(\widetilde{K_B}^{-1}\partial_\beta\beta\widetilde{K_B}(\beta x_i,\beta x_j))_{i,j=1}^k dx|_{\beta=1}$$

$$= \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k \operatorname{Tr}(\log \beta \widetilde{K_B}(\beta x_i, \beta x_j))_{i,j=1}^k dx|_{\beta=1}.$$

As $\widetilde{K_B}$ is trace class, we can write

$$\begin{aligned}
\operatorname{Tr}(\log \beta \widetilde{K_B}(\beta x_i, \beta x_j))_{i,j=1}^k &= \log\det\left(\beta \widetilde{K_B}(\beta x_i, \beta x_j)\right)_{i,j=1}^k \\
&= \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det\left(\beta \widetilde{K_B}(\beta x_i, \beta x_j)\right)_{i,j=1}^k dx|_{\beta=1} \\
&= \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det\left(\beta \widetilde{K_B}(\beta x_i, \beta x_j)\right)_{i,j=1}^k dx|_{\beta=1} \\
&= \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s\beta]^k} \det\left(\widetilde{K_B}(y_i, y_j)\right)_{i,j=1}^k dy_i|_{\beta=1} \\
&= \partial_\beta \det(I - \widetilde{K_B})_{L^2(0,s\beta)}\Big|_{\beta=1}. \qquad (18)
\end{aligned}$$

Hence, since $\det(I-\widetilde{K_B})_{L^2(0,s\beta)}$ is the leading order in the expansion of $\mathbb{P}\left(\lambda_{min} \geq \frac{\alpha s}{n^2}\right)$ plugging (17) into (18) shows that there exists a function g_n^0 (whose leading order is $\det(I - \widetilde{K_B})_{L^2(0,s\beta)}$) so that

$$\mathbb{P}\left(\lambda_{min} \geq \frac{\alpha s}{n^2}\right) = g_n^0(s) + \partial_\beta g_n^0(\beta s)|_{\beta=1} \int dP_n(H)[(\frac{z_c^+}{w_c^+})^2 - 1] + o(\frac{1}{n}) \qquad (19)$$

5.2.3 An estimate for $(\frac{z_c^+}{w_c^+})^2 - 1$

Let

$$X_n(z) = \sum_{i=1}^n \frac{1}{y_i - z} - n m_0(z)$$

where $z \in \mathbb{C}\backslash\mathbb{R}$. Let us express $(z_c^+)^2 - (w_c^+)^2$ in terms of X_n. The critical point z_c^+ of F_n lies in a neighborhood of the critical point w_c^+ of F. So $u_c^+ = (z_c^+)^2$ is in a neighborhood of $v_c^+ = (w_c^+)^2$. These points are the solutions with positive imaginary part of

$$\frac{1}{a^2} + \frac{1}{n}\sum \frac{1}{u_c^+ - y_i} = 0, \quad \frac{1}{a^2} + \int \frac{1}{v_c^+ - y} d\rho(y) = 0.$$

Therefore it is easy to check that

$$\int \frac{u_c^+ - v_c^+}{(v_c^+ - y)^2} d\rho(y) + \frac{1}{n} X_n(v_c^+) = o(\frac{1}{n}, (z_c^+ - w_c^+))$$

which gives

$$(\frac{z_c^+}{w_c^+})^2 - 1 = -\frac{1}{v_c^+ m_0'(v_c^+)} \frac{1}{n} X_n(v_c^+) + o(\frac{1}{n}). \qquad (20)$$

The proof of Theorem 1 is therefore complete. In the next section we estimate the expectation of $X_n(v_c^+)$ to estimate the correction in (19).

5.3 The role of the fourth moment

In this section we compute $\mathbb{E}[X_n(v_c^+)]$, which with Theorem 1, will allow to prove Theorem 3.1.

5.3.1 Central limit theorem estimate

In this section we compute the asymptotics of the mean of $X_n(z)$. Such type of estimates is now well known, and can for instance be found in Bai and Silverstein book [3] for either Wigner matrices or Wishart matrices with $\kappa_4 = 0$. We refer to [3, Theorem 9.10] for a precise statement. In the more complicated setting of F-matrices, we refer the reader to [31]. In the case where $\kappa_4 \neq 0$, the asymptotics of the mean have been computed in [23]. To ease the reading, we here show how this computation can be done, following the ideas from [25] and [3]. We shall prove the following result.

Proposition 1. *Under hypothesis 4, we have*

$$\lim_{n\to\infty} \mathbb{E}[X_n(z)] = A(z) - \kappa_4 B(z)$$

with A independent of κ_4, and if $m_0(z) = \int (x-z)^{-1} d\rho(x)$,

$$B(z) = \frac{m_0(z)^2}{(1+\frac{m_0(z)}{4})^2(z+\frac{zm_0(z)}{2})}\,.$$

Proof. We recall that the entries of W have variance $\frac{1}{4}$. We thus write $WW^* = \frac{1}{4}XX^*$ where X has standardized entries. Let z be a complex number with positive imaginary part and set $\gamma_n = \frac{p}{n}$. We recall [21] that

$$m_0(z) := \lim_{n\to\infty} \frac{1}{n}\mathrm{Tr}(\frac{WW^*}{n} - zI)^{-1}$$

is uniquely defined as the solution with non negative imaginary part of the equation

$$\frac{1}{1+\frac{1}{4}m_0(z)} = -zm_0(z). \tag{21}$$

We now investigate the fluctuations of $m_n(z) := \frac{1}{n}\mathrm{Tr}(\frac{WW^*}{n} - zI)^{-1}$ w.r.t. m_0. We denote for each $k = 1, \ldots, p$ by X_k the kth column of X. Using formula (16) in [20], one has that

$$\begin{aligned} 1 + zm_n(z) &= \gamma_n - \frac{1}{n}\sum_{k=1}^{p} \frac{1}{1+\frac{1}{4n}X_k^* R^{(k)} X_k} \\ &= \gamma_n - \frac{\gamma_n}{1+\frac{1}{4}m_n(z)} + \frac{1}{n}\sum_{k=1}^{p} \frac{\delta_k}{(1+\frac{1}{4}m_n(z)+\delta_k)(1+\frac{1}{4}m_n(z))}, \end{aligned} \tag{22}$$

where $R^{(k)} = \left(\frac{1}{4n}(XX^* - X_k X_k^*) - zI\right)^{-1}$ and

$$\delta_k = \frac{1}{4n} X_k^* R^{(k)} X_k - \frac{1}{4} m_n(z).$$

We next use the fact that the error term δ_k can be written

$$\delta_k = \frac{1}{4n} \sum_{i=1}^{n} \left(|X_{ki}|^2 - 1\right) R_{ii}^{(k)} + \frac{1}{4n} \sum_{i \neq j, i,j=1}^{n} X_{ki} \overline{X_{kj}} R_{ij}^{(k)} + \frac{1}{n} \operatorname{Tr}(R^{(k)} - R).$$

We first show that $\sup_k |\delta|_k \to 0$ a.s. By (4), it it clear that one can fix C large enough so that

$$\mathbb{P}\left(\exists i, j, |X_{ij}| \geq C \ln n\right) \leq \frac{1}{n^2}.$$

Hence, up to a negligible probability set, one can truncate the entries $X_{ij} \to X_{ij} 1_{|X_{ij}| \leq C \ln n}$. Then it can be shown that $\mathbb{E} \sup_k |\delta_k|^6 \leq (C \ln n)^{12} n^{-2}$ so that $\sup_k |\delta_k| \to 0$ as. This follows from Lemma 3.1 in [26].

Plugging the above into (22), we obtain

$$\begin{aligned} 1 + z m_n(z) &= \gamma_n - \frac{\gamma_n}{1 + \frac{1}{4} m_n(z)} + \frac{1}{n} \sum_{k=1}^{p} \frac{\delta_k}{(1 + \frac{1}{4} m_n(z))^2} \\ &\times \left(1 - \frac{\delta_k}{(1 + \frac{1}{4} m_n(z))} + \frac{\delta_k^2}{(1 + \frac{1}{4} m_n(z) + \delta_k)(1 + \frac{1}{4} m_n(z))}\right). \end{aligned} \tag{23}$$

Set now

$$\beta_4 = \mathbb{E}(|X_{ik}|^2 - 1)^2.$$

We are interested in the asymptotics of the expected value of the right hand side of (23) in terms of the fourth moment of the entries of W or equivalently in terms of β_4. First observe that

$$|\mathbb{E}(\delta_k)| = |\mathbb{E} \frac{1}{n} (\operatorname{Tr}(R^{(k)} - R))| \leq \frac{1}{n \Im(z)} \tag{24}$$

by Weyl's interlacing formula. In fact, we have the following linear algebra formula

$$R^{(k)} - R = \frac{1}{4n} R X_k X_k^* R^{(k)}$$

which shows the more precise estimate

$$\mathbb{E}(\delta_k) \simeq \frac{1}{4n^2} \operatorname{Tr}(R^2) + o(\frac{1}{n^2}) = \frac{m_0'(z)}{4n} + o(\frac{1}{n^2})$$

is independent of β_4 at first order. The second moment satisfies

$$\begin{aligned} \mathbb{E}(\delta_k^2) &= \frac{1}{16n^2} \left(\sum_{i=1}^{n} (R_{ii}^{(k)})^2 \beta_4\right) + \mathbb{E} \sum_{i \neq j} \frac{|R_{ij}^{(k)}|^2}{16n^2} |X_{ki} X_{kj}|^2 + \mathbb{E} \frac{1}{n^2} \left(\operatorname{Tr}(R^{(k)} - R)\right)^2 \\ &- \mathbb{E}\left(\frac{1}{2n^2} \Big(\sum_i R_{ii}^{(k)} (|X_{ki}|^2 - 1) + \sum_{i \neq j, i,j=1}^{n} X_{ki} \overline{X_{kj}} R_{ij}^{(k)}\Big) \operatorname{Tr}(R - R^{(k)})\right). \end{aligned} \tag{25}$$

Lemma 5.6. *For all* $z \in \mathbb{C}\backslash\mathbb{R}$

$$\mathbb{E}\frac{1}{n^2}\left(\mathrm{Tr}(R^{(k)}-R)\right)^2 \leq Cte\frac{1}{n^2\Im z^2},$$

and

$$\left|\mathbb{E}\left(\frac{1}{4n^2}\Big(\sum_i R_{ii}^{(k)}(|X_{ki}|^2-1)+\sum_{i\neq j,i,j=1}^{n} X_{ki}\overline{X_{kj}}R_{ij}^{(k)}\Big)\mathrm{Tr}R\right)\right| \leq Cte\frac{1}{n^{\frac{3}{2}}\Im z}.$$

The first inequality follows from (24) whereas the second is based on the use of the same formula together with

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_i R_{ii}^{(k)}(|X_{ki}|^2-1)+\sum_{i\neq j,i,j=1}^{n} X_{ki}\overline{X_{kj}}R_{ij}^{(k)}\right|\right] \\
\leq\ &\mathbb{E}\left[\left|\sum_i R_{ii}^{(k)}(|X_{ki}|^2-1)+\sum_{i\neq j,i,j=1}^{n} X_{ki}\overline{X_{kj}}R_{ij}^{(k)}\right|^2\right]^{\frac{1}{2}} \\
\leq\ &C\mathbb{E}\left[\mathrm{Tr}((R^{(k)})^2)\right]^{\frac{1}{2}} \leq \frac{\sqrt{n}}{\Im z}
\end{aligned}$$

Moreover, as $\forall i=1,\ldots,n$, $|R_{ii}^{(k)}-m_0(z)|$ goes to 0 (as can be checked by concentration inequalities, invariance by permutations of the indices of $\mathbb{E}[R_{ii}^{(k)}]$, and our estimate on m_n), we have

$$\mathbb{E}\sum_{i\neq j}\frac{|R_{ij}^{(k)}|^2}{16n^2} \sim \frac{1}{16n^2}\mathrm{Tr}(RR^*)-\frac{1}{16n}|m_0|^2(z) \sim \frac{\Im m_0(z)}{16n\Im z}-\frac{1}{16n}|m_0|^2(z)$$

Denote by $k_n(z)$ the solution of the equation

$$1+zk_n(z)=\gamma_n-\frac{\gamma_n}{1+\frac{1}{4}k_n(z)},$$

which satisfies $\Im k_n(z)\geq 0$ when $\Im z\geq 0$. Then we have proved that $m_n(z)$ satisfies a similar equation:

$$1+zm_n(z)=\gamma_n-\frac{\gamma_n}{1+\frac{1}{4}m_n(z)}+E_n,$$

where the error term E_n satisfies

$$\mathbb{E}E_n=c_n-\frac{\beta_4 m_0(z)^2}{16n(1+\frac{1}{4}m_0)^3}+o(\frac{1}{n})$$

with

$$c_n = \frac{1}{4n(1+\frac{m_0(z)}{4})^2} m_0'(z) - \frac{1}{16n(1+\frac{m_0(z)}{4})^3} (\frac{\Im m_0(z)}{\Im z} - |m_0(z)|^2) + o(\frac{1}{n})\,.$$

Thus,

$$(m_n(z) - k_n(z)) \left(z + \frac{1-\gamma_n}{4} + \frac{z}{4}(m_n(z) + k_n(z)) \right) = E_n(1 + \frac{1}{4} m_n(z)). \quad (26)$$

From this we deduce that (for the term depending on the fourth cumulant)

$$\mathbb{E}(m_n(z) - k_n(z)) = \frac{c(z)}{n} - \frac{\beta_4}{16n} \frac{m_0(z)^2}{(1+\frac{1}{4}m_0(z))^2} \frac{1}{z + z m_0(z)/2} + o(\frac{1}{n})$$

where $c(z)$ is independent of β_4. Since

$$\begin{aligned} &\beta_4 = \mathbb{E}(|X_{ij}|^2 - 1)^2 = \mathbb{E}(4|W_{ij}|^2 - \sigma^2)^2 \\ &= 4^2(\kappa_4 + 1/16) \end{aligned}$$

we have completed the proof of Proposition 1, since $k_n - m_0(z)$ is of order $\frac{1}{n}$ and independent of κ_4.

□

5.3.2 Estimate at the critical point

We deduce from Proposition 1 that for $z = v_c^+$,

$$n\mathbb{E}[m_n(v_c^+) - m_0(v_c^+)] = \quad c(v_c^+) - \frac{\beta_4}{16} \frac{a^{-4}}{(1+\frac{1}{4}a^{-2})^2} \frac{1}{v_c^+(1+\frac{1}{2}a^{-2})} + o(1).$$

Moreover we know that $m_0(v_c^+) = a^{-2}$, and that

$$v_c^+ = -\frac{a^4}{\frac{1}{4} + a^2} = -\frac{4a^4}{1+4a^2}.$$

Also by (21), after taking the derivative, we have

$$m_0'(z) = -\frac{m_0(z)(1 + m_0(z)/4)}{z(1+m_0(z)/2)},$$

so that at the critical point we get

$$\begin{aligned} m_0'(v_c^+) &= \frac{(4a^2+1)^2}{16a^6(a^2+\frac{1}{2})}, \\ v_c^+ m_0'(v_c^+) &= -\frac{(1+4a^2)}{4(\frac{1}{2}+a^2)} = -\frac{a^{-2}(1+\frac{1}{4a^2})}{1+\frac{1}{2a^2}}. \end{aligned}$$

Therefore, with the notations of Theorem 1, we find constants C independent of β_4 (and which may change from line to line) so that

$$\begin{aligned}\int dP_n(H)[\Delta_n(H)] &= -\frac{1}{v_c^+ m_0'(v_c^+)}\mathbb{E}[n(m_n(v_c^+)-m_0(v_c^+))]+o(1)\\ &= -\frac{1+\frac{1}{2a^2}}{a^{-2}(1+\frac{1}{4a^2})}\frac{\beta_4}{16}\frac{a^{-4}}{(1+\frac{1}{4a^2})^2}\frac{1}{1+\frac{1}{2a^2}}\frac{1+4a^2}{4a^4}+C+o(1)\\ &= -\frac{\beta_4}{(1+4a^2)^2}+C+o(1)=-\frac{\kappa_4+1/16}{(\frac{1}{4}+a^2)^2}+C+o(1).\end{aligned}$$

Rescale the matrix M by dividing it by σ so as to standardize the entries. We have therefore found that the deviation of the smallest eigenvalue are such that

$$\mathbb{P}\left(\lambda_{\min}(\frac{MM^*}{n}\sigma^2)\geq\frac{s}{n^2}\right)=\mathbf{g}_n(s)+\frac{\gamma}{2n}s\mathbf{g}_n'(s)+o(\frac{1}{n}),$$

where γ is the kurtosis defined in Definition (1). At this point $\mathbf{g}_n$ is identified to be the distribution function at the Hard Edge of the Laguerre ensemble with variance 1, as it corresponds to the case where $\gamma=0$.

6 Deformed GUE in the bulk

Let $W=(W_{ij})_{i,j=1}^n$ be a Hermitian Wigner matrix of size n. The entries W_{ij} $1\leq i<j\leq n$ are i.i.d. with distribution μ. The entries along the diagonal are i.i.d. real random variables with law μ' independent of the off diagonal entries. We assume that μ has sub exponential tails and satisfy

$$\int x d\mu(x)=0,\int |x|^2 d\mu(x)=1/4,\int x^3 d\mu(x)=0.$$

The same assumptions are also assumed to hold true for μ'. Let also V be a GUE random matrix with i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$ entries and consider the rescaled matrix

$$M_n=\frac{1}{\sqrt{n}}(W+aV).$$

We denote by $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n$ the ordered eigenvalues of M_n. By Wigner's theorem, it is known that the spectral measure of M_n

$$\mu_n=\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i}$$

converges weakly to the semi-circle distribution with density

$$\sigma_{sc}(x)=\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}1_{|x|\leq 2\sigma};\quad \sigma^2=1/4+a^2. \tag{27}$$

This is the Deformed GUE ensemble studied by Johansson [19]. In this section, we study the localization of the eigenvalues λ_i with respect to the quantiles of the limiting semi-circle distribution. We study the $\frac{1}{n}$ expansion of this localization, showing that it depends on κ_4, and prove Theorem 3.3.

The route we follow is similar to that we took in the previous section for Wishart matrices: we first obtain a $\frac{1}{n}$ expansion of the correlation functions of the Deformed GUE. The dependency of this expansion in the fourth moment of μ is then derived.

6.1 Asymptotic analysis of the correlation functions

Let ρ_n be the one point correlation function of the Deformed GUE. We prove in this subsection the following result, with $z_c^\pm, w_c^\pm$ critical points similar to those of the last section, which we will define precisely in the proof.

Proposition 6.1. *For all $\varepsilon > 0$, uniformly on $u \in [-2\sigma+\varepsilon, 2\sigma-\varepsilon]$, we have*

$$\rho_n(u) = \sigma_{sc}(u) + \mathbb{E}[\left(\frac{\Im z_c^+(u)}{\Im w_c^+(u)} - 1\right)]\sigma_{sc}(u) + \frac{C'(u)}{n} + o(\frac{1}{n}),$$

where the function $u \mapsto C'(u)$ does not depend on the distribution of the entries of W whereas z_c^+ depends on the eigenvalues of W.

Proof of Proposition 6.1: Denote by $y_1 \le y_2 \le \cdots \le y_n$ the ordered eigenvalues of $W/\sqrt{n}$. [19, (2.20)] proves that, for a fixed $W/\sqrt{n}$, the eigenvalue density of M_n induces a determinantal process with correlation kernel given by

$$K_n(u,v;y(\frac{W}{\sqrt{n}})) = \frac{n}{(2i\pi)^2}\int_\Gamma dz \int_\gamma dw e^{n(F_v(w)-F_v(z))}\frac{1-e^{\frac{(u-v)zn}{a^2}}}{z(u-v)} g_n(z,w),$$

where

$$F_v(z) = \frac{(z-v)^2}{2a^2} + \frac{1}{n}\sum \ln(z-y_i),$$

and

$$g_n(z,w) = F_u'(z) + z\frac{F_v'(z)-F_v'(w)}{z-w}.$$

The contour Γ has to encircle all the y_i's and γ is parallel to the imaginary axis.

We now consider the asymptotics of the correlation kernel in the bulk, that is close to some point $u_0 \in (-2\sigma+\delta, 2\sigma-\delta)$ for some $\delta > 0$ (small). We recall that we can consider the correlation kernel up to conjugation: this follows from the fact that $\det(K_n(x_i,x_j;y)) = \det\left(K_n(x_i,x_j;y)\frac{h(x_i)}{h(x_j)}\right)$, for any non vanishing function h. We omit some details in the next asymptotic analysis as it closely follows the arguments of [19] and those of Subsection 5.2.

Let then u, v be points in the bulk with

$$u = u_0 + \frac{\alpha x}{n}, v = u_0 + \frac{\alpha\tilde{x}}{n}; u_0 = \sqrt{1+4a^2}\cos(\theta_0), \theta_0 \in (2\epsilon, \pi-2\epsilon). \quad (28)$$

The constant α will be fixed afterwards. Then the approximate large exponential term to lead the asymptotic analysis is given by

$$\tilde{F}_v(z) = \frac{(z-v)^2}{2a^2} + \int \ln(z-y)d\rho(y),$$

where ρ is the semi-circle distribution with support $[-1,1]$. In the following we note $R_0 = \sqrt{1+4a^2} = 2\sigma$.

We recall the following facts from [19], Section 3. Let $u_0 = \sqrt{1+4a^2}\cos(\theta_0)$ be a given point in the bulk.

- The approximate critical points, i.e. the solutions of $\tilde{F}'_{u_0}(z) = 0$ are given by
$$w_c^{\pm}(u_0) = (R_0 e^{i\theta_c} \pm \frac{1}{R_0 e^{i\theta_c}})/2.$$
The true critical points satisfy $F'_{u_0}(z) = 0$. Among the solutions, we disregard the $n-1$ real solutions which are interlaced with the eigenvalues $y_1, \ldots, y_n$. The two remaining solutions are complex conjugate with non zero imaginary part and we denote them by $z_c^{\pm}(u_0)$. Furthermore [19] proves that
$$|z_c(u_0)^+ - w_c(u_0)^+| \leq n^{-\xi}$$
for any point u_0 in the bulk of the spectrum.

- We now fix the contours for the saddle point analysis. The steep descent/ascent contours can be chosen as :
$$\begin{aligned}\gamma = \ & z_c^+(v) + it, t \in \mathbb{R}, \\ \Gamma = \ & \{z_c^{\pm}(r), r = R_0\cos(\theta), \theta \in (\epsilon, \pi - \epsilon)\} \bigcup \{z_c^{\pm}(R_0\cos(\epsilon)) + x, x > 0\} \\ & \bigcup \{z_c^{\pm}(-R_0\cos(\epsilon)) - x, x > 0\}.\end{aligned}$$
It is an easy computation (using that $\Re F''_{u_0}(w) > 0$ along γ) to check that the contribution of the contour $\gamma \cap |w - z_c^{\pm}(v)| \geq n^{1/12-1/2}$ is exponentially negligible. Indeed there exists a constant $c > 0$ such that
$$\Big| \int_{\gamma\cap|w-z_c^{\pm}(v)|\geq n^{1/12-1/2}} e^{n\Re(F_{u_0}(w) - F_{u_0}(z_c^+(v)))} dw \Big| \leq e^{-cn^{1/6}}.$$
Similarly the contribution of the contour $\Gamma \cap |w - z_c^{\pm}(v)| \geq n^{1/12-1/2}$ is of order $e^{-cn^{1/6}}$ that of a neighborhood of $z_c^{\pm}(v)$.

For ease of notation, we now denote $z_c(v) := z_c^+(v)$. We now modify slightly the contours so as to make the contours symmetric around $z_c^{\pm}(v)$. To this aim we slightly modify the Γ contour: in a neighborhood of width $n^{1/12-1/2}$ we replace Γ by a straight line through $z_c^{\pm}(v)$ with slope $z'_c(v)$. This slope is well defined as

$$z'_c(v) = \frac{1}{F''_v(z_c(v))} \neq 0,$$

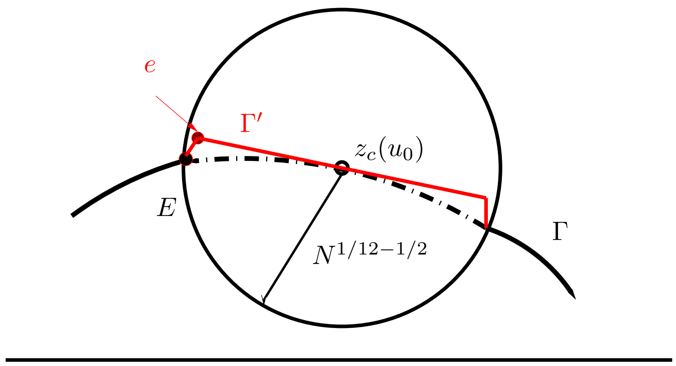

Figure 6: Modification of the Γ contour

using that $|z_c^{\pm}(v) - w_c^{\pm}(u_0)| \leq n^{-\xi}$. We refer to Figure 6.1, to define the new contour Γ' which is more explanatory.

Denote by E the leftmost point of $\Gamma \cap \{w, |w - z_c(v)| = n^{1/12-1/2}\}$. Then there exists v_1 such that $E = z_c(v_1)$. We then define e by $e = z_c(v) + z_c'(v)(v_1 - v)$. We then draw the segment $[e, z_c(v)]$ and draw also its symmetric to the right of $z_c(v)$. Then it is an easy fact that

$$|E - e| \leq Cn^{1/12-1/2}, \text{ for some constant C.}$$

Furthermore, as e, E both lie within a distance $n^{1/12-1/2}$ from $z_c(v)$, it follows that

$$\forall z \in [e, E], \quad \Big|\Re\Big(nF_v(z) - nF_v(E)\Big)\Big| \leq Cn^{3(1/12-1/2)} = Cn^{\frac{1}{4}} << n^{1/6}.$$

This follows from the fact that $|F_v'(z)| = O(n^{1/12-1/2})$ along the segment $[e, E]$. This is now enough as $\Re nF_v(E) > \Re nF_v(z_c) + cn^{1/6}$ to ensure that the deformation has no impact on the asymptotic analysis.

We now make the change of variables $z = z_c^{\pm}(v) + \frac{t}{\sqrt{n}}, w = z_c^{\pm}(v) + \frac{s}{\sqrt{n}}$ where $|s|, |t| \leq n^{1/12-1/2}$. We examine the contributions of the different terms in the integrand. We first consider g_n. We start with the combined contribution of equal critical points, e.g. z and w close to the same critical point. In this case, using (28), we have that

$$\begin{aligned}
\frac{g_n(w,z)}{z} &= F_v''(z_c(v)) + \frac{1}{\sqrt{n}}\left(\frac{F_v^{(3)}(z_c(v))}{2}(s+t) + z_c(v)^{-1}F_v''(z_c(v))t\right) \\
&\quad + \frac{1}{n}\left(\frac{F_v^{(4)}(z_c(v))}{3!}(s^2+t^2+st) - \frac{F_v''(z_c(v))t^2}{z_c(v)^2} + \frac{1}{2}\frac{F_v^{(3)}(z_c(v))}{z_c(v)}t^2\right) \\
&\quad + \frac{\alpha(x-\tilde{x})}{2a^2nz_c(v)} + o(\frac{1}{n}).
\end{aligned}$$

On the other hand when w and z lie in the neighborhood of different critical points, one gets that

$$\frac{g_n(w,z)}{z} = \frac{\alpha(x-\tilde{x})}{nz_c^{\pm}} + \frac{F_v''(z_c(v))t}{z_c^{\pm}\sqrt{n}} + \frac{F_v^{(2)}(z_c^{\pm}(v))t - F_v^{(2)}(z_c^{\mp}(v))s}{2(z_c^{\pm}-z_c^{\mp})\sqrt{n}} + O(\frac{1}{n}),$$

where the $O(\frac{1}{n})$ depends on the third derivative of F_v only. One also has that

$$\exp\{nF_v(z)\} = \exp\{nF_v(z_c^{\pm}) + F_v''(z_c^{\pm})t^2/2 + F_v^{(3)}(z_c^{\pm})\frac{t^3}{3!\sqrt{n}} + o(1/\sqrt{n})\}.$$

Consider for instance the contribution to $\frac{1}{n}K_n(u,v;y(\frac{W}{\sqrt{n}}))$ of contours close to the same critical points $z,w\simeq z_c(v)$: this yields

$$\begin{aligned}&\frac{1}{(2i\pi)^2}\int ds\int dt\left(F_v''(z_c(v)) + \frac{1}{\sqrt{n}}\left(\frac{F_v^{(3)}(z_c(v))}{2}(s+t) + \frac{F_v''(z_c(v))t}{z_c(v)}\right) + O(1/n)\right)\\ &\exp\{F_v''(z_c(v))(s^2-t^2)/2 + F_v^{(3)}(z_c(v))\frac{s^3-t^3}{3!\sqrt{n}} + O(1/n)\}\frac{1-e^{\frac{n(u-v)z_c(v)+t\sqrt{n}(u-v)}{a^2}}}{n(u-v)}\\ &= \frac{\pm 1}{2i\pi}\frac{1-e^{\frac{(u-v)z_c(v)n}{a^2}}}{n(u-v)} + O(1/n),\end{aligned} \tag{29}$$

where we used the symmetry of the contours on s,t to obtain that the $O(1/\sqrt{N})$ vanishes. Note that $n(u-v)$ is of order 1. We next turn to the remaining term in the integrand (which is not exponentially large) and which depends on z only, namely

$$1-e^{\frac{(u-v)zn}{a^2}}.$$

One has that

$$1-e^{(x-\tilde{x})\alpha a^{-2}z_c^{\pm}} = 1-e^{(x-\tilde{x})\alpha a^{-2}\Re z_c^{+}}e^{i\pm(x-\tilde{x})\alpha a^{-2}\Im z_c}.$$

We do the same for the contribution of non equal critical points. One may note in addition that $F_v(z_c^-) = \overline{F_v(z_c^+)}$. Due to the fact that g_n vanishes at different critical points, we see that the contribution from different critical terms is in the order of $1/N$. Furthermore it only depends on $z_c(v)$.
Combining the whole, apart from constants, one has that

$$\frac{\alpha}{n}K_n(u,v;y(\frac{W}{\sqrt{n}})) = \frac{e^{(x-\tilde{x})\frac{\alpha}{a^2}\Re z_c^+}}{2i\pi(x-\tilde{x})}\left(e^{i(x-\tilde{x})\frac{\alpha}{a^2}\Im z_c^+} - e^{-i(x-\tilde{x})\frac{\alpha}{a^2}\Im z_c^+}\right) + \frac{C(x,\tilde{x})}{n} + o(\frac{1}{n}).$$

The function $C(x,\tilde{x})$ does not depend on the detail of the distributions μ,μ' of the entries of W. We now choose $\alpha = \sigma_{sc}(u_0)^{-1}$ where σ_{sc} is the density of the semi-circle distribution defined in (27). It has been proved in [19] that $\Im w_c^+(u_0) = \pi a^2\sigma_{sc}(u_0)$. Setting then

$$\beta := \Im z_c^+(u_0)/\Im(w_c^+(u_0))$$

we then obtain that

$$\frac{\alpha}{n} K_n(u, v; y(\frac{W}{\sqrt{n}})) e^{-(x-\tilde{x})\frac{\alpha}{a^2}\Re z_c^+} = \frac{\sin \pi\beta(x-\tilde{x})}{\pi(x-\tilde{x})} + \frac{C'(x,\tilde{x})}{n} + o(\frac{1}{n}).$$

The constant $C'(x,\tilde{x})$ does not depend on the distribution of the entries of W. This proves Proposition 6.1 since by taking the limit where $\tilde{x} \to x$ e.g.

$$\begin{aligned}
\rho_n(x) &= \mathbb{E}[\frac{1}{n} K_n(u, u; y(\frac{W}{\sqrt{n}}))] \\
&= \frac{1}{\alpha}\mathbb{E}[\beta] + \frac{C'(x,x)}{\alpha n} + o(\frac{1}{n}) \\
&= \sigma_{sc}(u_0) + \sigma_{sc}(u_0)\mathbb{E}[(\beta-1)] + \frac{C'(x,x)}{\alpha n} + o(\frac{1}{n}).
\end{aligned}$$

6.2 An estimate for $z_c - w_c$ and the role of the fourth moment

We follow the route developed for Wishart matrices, showing first that the fluctuations of $z_c^{\pm}(u_0)$ around $w_c^{\pm}(u_0)$ depend on the fourth moment of the entries of W.

We fix a point u in the bulk of the spectrum.

Proposition 6.2. *There exists a constant $C_n = C_n(u)$ independent of the distribution μ and $l = l(u) \in \mathbb{R}$ such that $nC_n \to l$ such that*

$$\mathbb{E}[z_c(u) - w_c(u)] = \frac{C_n + \beta_4 m_0(w_c)^4/(16n)}{(a^{-2} + m_0'(w_c))(w_c + m_0(w_c)/2)} + o(\frac{1}{n}).$$

As a consequence, for any $\varepsilon > 0$ uniformly on $u \in [-2\sigma+\varepsilon, 2\sigma-\varepsilon]$,

$$\rho_n(u) = \sigma_{sc}(u) + \frac{C'(u)}{n} + \kappa_4\frac{D(u)}{n} + o(\frac{1}{n}), \tag{30}$$

where $D(u)$ is the term uniquely defined by

$$n\frac{\mathbb{E}[\Im(z_c(u) - w_c(u))]}{\pi a^2} = n\mathbb{E}[(\beta(u)-1)]\sigma_{sc}(u) = C'(u) + \kappa_4 D(u) + o(1), \tag{31}$$

where $C'(u)$ is a constant independent of μ.

Proof of Proposition 6.2: We first relate critical points z_c and w_c to the difference of the Stieltjes transforms $m_n - m_0$. The true and approximate critical points satisfy the following equations:

$$\frac{z_c - u}{a^2} - m_n(z_c) = 0; \quad \frac{w_c - u}{a^2} - m_0(w_c) = 0.$$

Hence,

$$(\frac{1}{a^2} - m_0'(w_c))(z_c - w_c) = m_n(w_c) - m_0(w_c) + o(\frac{1}{n}) \tag{32}$$

where we have used that $m_n - m_0$ is of order $\frac{1}{n}$. Indeed, the estimate will again rely on the estimate of the mean of the central limit theorem for Wigner matrices, see [3, Theorem 9.2]. For the sake of completeness we recall the main steps. Using Schur complement formulae (see [1] Section 2.4 e.g.) one has that

$$m_n(z) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{-z + W_{ii}n^{-1/2} - h_i^* R^{(i)}(z) h_i},$$

where h_i is the ith column of $W/\sqrt{n}$ with ith entry removed and $R^{(i)}$ is the resolvent of the $(n-1)\times(n-1)$ matrix formed from $W/\sqrt{n}$ by removing column and row i. Copying the proof of Subsection 5.3.1, we write

$$m_n(z) + \frac{1}{z + \frac{1}{4}m_n(z)} = \frac{1}{n}\sum_{i=1}^{n} \frac{\delta_n}{(z + \frac{1}{4}m_n(z))(z + \frac{1}{4}m_n(z) + \delta_n)} =: E_n,$$

where $\delta_n = W_{ii}n^{-1/2} + \frac{1}{4}m_n - h_i^* R^{(i)(z)} h_i$. Again a Central Limit Theorem can be established from the above. We do not give the details as this uses the same arguments as in Subsection 5.3.1. One then finds that

$$\mathbb{E}[E_n] = c_n + \frac{\beta_4 m_0(z)^2}{16n(z + \frac{1}{4}m_0)^3} + o(\frac{1}{n}), \tag{33}$$

where the sequence $c_n = c_n(z)$ is given by

$$c_n = \frac{1}{4n(z + \frac{m_0(z)}{4})^2} m_0'(z) - \frac{1}{16n(z + \frac{m_0(z)}{4})^3}(\frac{\Im m_0(z)}{\Im z} - |m_0(z)|^2) + o(\frac{1}{n}).$$

We recall that the limiting Stieltjes transform satisfies

$$m_0(z) + \frac{1}{z + \frac{1}{4}m_0(z)} = 0.$$

As a consequence , we get

$$(m_n(z) - m_0(z))(z + \frac{1}{4}(m_n(z) + m_0(z)) = E_n(z + \frac{1}{4}m_n(z)), \tag{34}$$

from which we deduce (using that $m_n(z) - m_0(z) \to 0$ as $n \to \infty$) that

$$\begin{aligned}\mathbb{E}[(m_n(z) - m_0(z))](z + m_0(z)/2) &\sim \mathbb{E}[E_n](z + \frac{1}{4}m_0(z)) \\ &= [c_n + \frac{\beta_4 m_0(z)^4}{16n}](z + \frac{1}{4}m_0(z)) + o(\frac{1}{n}).\end{aligned} \tag{35}$$

Combining (33), (34) and (35) and using the fact that $|z_c(v)^+ - w_c(v)^+| \leq n^{-\xi}$ for any point v in the bulk of the spectrum, we deduce the first part of Proposition 6.2. Using Proposition 6.1, the expansion for the one point correlation function follows.

6.3 The localization of eigenvalues

We now use (30) to obtain a precise localization of eigenvalues in the bulk of the spectrum. A conjecture of Tao and Vu (more precisely Conjecture 1.7 in [27]) states that (when the variance of the entries of W is $\frac{1}{4}$), there exists a constant $c > 0$ and a function $x \mapsto C'(x)$ independent of κ_4 such that

$$\mathbb{E}(\lambda_i - \gamma_i) = \frac{1}{n\sigma_{sc}(\gamma_i)} \int_0^{\gamma_i} C'(x)dx + \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + O(\frac{1}{n^{1+c}}) \quad (36)$$

where γ_i is given by $N_{sc}(\gamma_i) = i/n$ if $N_{sc}(x) = \int_{-\infty}^{x} d\sigma_{sc}(u)$. We do not prove the conjecture but another version instead. More precisely we obtain the following estimate. Fix $\delta > 0$ and an integer i such that $\delta < i/n < 1 - \delta$. Define also

$$N_n(x) := \frac{1}{n}\sharp\{i, \lambda_i \leq x\}, \text{ with } \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n; \quad (37)$$

Let us define the quantile $\hat{\gamma}_i$ by

$$\hat{\gamma}_i := \inf\left\{y, \int_{-\infty}^{y} \rho_n(x)dx = \frac{i}{n}\right\}.$$

By definition $\mathbb{E}N_n(\hat{\gamma}_i) = i/n$. We prove the following result.

Proposition 6.3. *There exists a constant $c > 0$ and a function $x \mapsto C'(x)$ independent of κ_4 such that*

$$\hat{\gamma}_i - \gamma_i = \frac{1}{n\sigma_{sc}(\gamma_i)} \int_0^{\gamma_i} C'(x)dx + \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + O(\frac{1}{n^{1+c}}) \quad (38)$$

The main step to prove this proposition is the following.

Proposition 6.4. *Assume that $i \geq n/2$ without loss of generality. There exists a constant $c > 0$ such that*

$$\hat{\gamma}_i - \gamma_i - \hat{\gamma}_{[n/2]} + \gamma_{[n/2]} = \frac{1}{\sigma_{sc}(\gamma_i)} \int_{\gamma_{[n/2]}}^{\gamma_i} [\rho_n(x) - \sigma_{sc}(x)]dx + O(\frac{1}{n^{1+c}}). \quad (39)$$

Note here that $\gamma_{[n/2]} = 0$ when n is even.

Proof of Proposition 6.4: The proof is divided into Lemma 1 and Lemma 2 below.

Lemma 1. *For any $\varepsilon > 0$, there exists $c > 0$ such that uniformly on $i \in [\varepsilon N, (1 - \varepsilon)N]$*

$$\gamma_i - \hat{\gamma}_i = \mathbb{E}(N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i)) \frac{1}{\sigma_{sc}(\gamma_i)} + O(\frac{1}{n^{1+c}}). \quad (40)$$

Proof of Lemma 1: Under assumptions of sub exponential tails, it is proved in [12] (see also Remark 2.4 of [27]) that given $\eta > 0$ for n large enough

$$\mathbb{P}\left(\max_{\varepsilon N \leq i \leq (1-\varepsilon)n} |\gamma_i - \lambda_i| \geq n^{\eta - 1} \right) \leq n^{-\log n}. \tag{41}$$

Note that the λ_i have all finite moments, see e.g. [1, 2.1.6]. In particular this implies that

$$\max_{\varepsilon N \leq i \leq (1-\varepsilon)n} |\gamma_i - \hat{\gamma}_i| \leq n^{\eta - 1}. \tag{42}$$

From the fact that $\mathbb{E} N_n(\hat{\gamma}_i) = N_{sc}(\gamma_i)$, we deduce that

$$\begin{aligned} \mathbb{E} N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i) &= N_{sc}(\gamma_i) - N_{sc}(\hat{\gamma}_i) \\ &= N'_{sc}(\gamma_i)(\gamma_i - \hat{\gamma}_i) - \int_{\gamma_i}^{\hat{\gamma}_i} \int_{\gamma_i}^{u} N''_{sc}(s) ds. \end{aligned} \tag{43}$$

Using that $N'_{sc}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2} 1_{|x| \leq 2\sigma}$ and that both γ_i and $\hat{\gamma}_i$ lie within $(-2\sigma + \epsilon, 2\sigma - \epsilon)$ for some $0 < \epsilon < 2\sigma$, we deduce that

$$\mathbb{E} N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i) = \sigma_{sc}(\gamma_i)(\gamma_i - \hat{\gamma}_i) + O(\gamma_i - \hat{\gamma}_i)^2.$$

We now make the following replacement.

Lemma 2. *Let $\varepsilon > 0$. There exist a constant $c > 0$ such that uniformly on $i \in [\varepsilon n, (1-\varepsilon)n]$,*

$$\mathbb{E}\left(N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i)\right) = \mathbb{E}\left(N_n(\gamma_i) - N_{sc}(\gamma_i)\right) + O\left(\frac{1}{n^{1+c}}\right). \tag{44}$$

Proof of Lemma 2: We write that

$$\begin{aligned} &\mathbb{E}\left(N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i)\right) \\ &= \mathbb{E}\left(N_n(\gamma_i) - N_{sc}(\gamma_i)\right) + \mathbb{E}\left(N_n(\hat{\gamma}_i) - N_n(\gamma_i) - N_{sc}(\hat{\gamma}_i) + N_{sc}(\gamma_i)\right). \end{aligned} \tag{45}$$

We show that the second term in (45) is negligible with respect to n^{-1}. In fact, for $\varepsilon > 0$, there exists $\delta > 0$ such that for any $i \in [\varepsilon n, (1-\varepsilon)n]$,

$$\begin{aligned} \left|\mathbb{E}\left(N_n(\hat{\gamma}_i) - N_n(\gamma_i) - N_{sc}(\hat{\gamma}_i) + N_{sc}(\gamma_i)\right)\right| &\leq \left|\int_{\gamma_i}^{\hat{\gamma}_i} (\rho_n(x) - \sigma(x)) dx\right| \\ &\leq n^{\eta - 1} \frac{1}{n^{\eta + \frac{1-\eta}{2}}} \leq \frac{1}{n^{1 + \frac{1-\eta}{2}}}. \end{aligned} \tag{46}$$

In the last line, we have used (30). This finishes the proof of Lemma 2.

Combining Lemma 1 and Lemma 2 yields Proposition 6.4:

$$\begin{aligned} \gamma_i - \hat{\gamma}_i - \gamma_{[n/2]} + \hat{\gamma}_{[n/2]} &= \frac{1}{\sigma_{sc}(\gamma_i)} \int_{\gamma_{[n/2]}}^{\gamma_i} [\rho_n(x) - \sigma_{sc}(x)] dx + O\left(\frac{1}{n^{1+c}}\right) \\ &= \frac{1}{n\sigma_{sc}(\gamma_i)} \int_{\gamma_{[n/2]}}^{\gamma_i} (C'(x) + \kappa_4 D(x)) dx + O\left(\frac{1}{n^{1+c}}\right) \end{aligned}$$

$$= \frac{1}{n\sigma_{sc}(\gamma_i)} \int_0^{\gamma_i} (C'(x) + \kappa_4 D(x))dx + O(\frac{1}{n^{1+c}}) \tag{47}$$

where we used that $\gamma_{[n/2]}$ vanishes or is at most of order $1/n$. This formula will be the basis for identifying the role κ_4 in the $\frac{1}{n}$ expansion of $\hat{\gamma}_i$. We now write for a point x in the bulk $(-R(1-\delta), R(1-\delta))$ that

$$x = \sqrt{1+4a^2}\cos\theta.$$

We also write that $\gamma_i = \sqrt{1+4a^2}\cos\theta_0$. We then have that

$$w_c(x) = \frac{\cos\theta}{R} + \frac{2a^2}{R}e^{\pm i\theta};\ m_0(w_c(x)) = \pm i\pi\sigma_{sc}(x) - \frac{2}{1+4a^2}x.$$

By combining Proposition 6.2 and (31), we have that

$$C(x) = \Im\left(\frac{m_0(w_c(x))^4}{16(w_c(x) + m_0(w_c(x)))\pi}(1+o(1))\right). \tag{48}$$

When $a \to 0$, we then have the following estimates

$$x \sim \cos\theta;\ m_0(w_c(x)) \sim -2e^{-i\theta};\ \sigma(x) \sim \frac{2}{\pi}\sin\theta; w_c + m_0(w_c)/2 \sim i\sin\theta.$$

Using (47) and identifying the term depending on κ_4 in the limit $a \to 0$, we then find that

$$\begin{aligned}
&\gamma_i - \hat{\gamma}_i - \gamma_{[n/2]} + \hat{\gamma}_{[n/2]} \\
&= \frac{1}{n\sigma_{sc}(\gamma_i)} \int_0^{\gamma_i} (C'(x) + \kappa_4 D(x))dx + O(\frac{1}{n^{1+c}}) \\
&= \frac{1}{n\sigma_{sc}(\gamma_i)} \int_0^{\gamma_i} C'(x)dx + \frac{\kappa_4}{n}\frac{\pi}{2\sin\theta_0}\int_{\theta_0}^{\pi/2}\frac{\cos(4\theta)}{\pi}d\theta + O(\frac{1}{n^{1+c}}) \\
&= \frac{1}{n\sigma_{sc}(\gamma_i)} \int_0^{\gamma_i} C'(x)dx - \frac{\kappa_4}{2n}\cos\theta_0(2\cos^2\theta_0 - 1) + O(\frac{1}{n^{1+c}}),
\end{aligned} \tag{49}$$

where in the last line we used that $\frac{1}{4}\sin(4\theta) = \sin\theta\cos\theta\cos(2\theta)$. Thus we have that

$$\begin{aligned}
&\gamma_i - \hat{\gamma}_i - \gamma_{[n/2]} + \hat{\gamma}_{[n/2]} \\
&= \frac{1}{n\sigma_{sc}(\gamma_i)} \int_0^{\gamma_i} C'(x)dx - \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + O(\frac{1}{n^{1+c}}).
\end{aligned} \tag{50}$$

We finally show that

$$\lim_{n\to\infty} n\left(-\gamma_{[n/2]} + \hat{\gamma}_{[n/2]}\right) = 0$$

which completes the proof of Proposition 6.3.

To that end, let us first notice that for any C^8 function f which is supported in $[-\frac{1}{2}, \frac{1}{2}]$, we have

$$\lim_{n\to\infty} \mathbb{E}[\sum_{i=1}^{n} f(\lambda_i)] = m(f) + \kappa_4 \int_{-1}^{1} f(t) T_4(t) \frac{dt}{\sqrt{1-t^2}} := m_{\kappa_4}(f)\,. \tag{51}$$

with T_4 the fourth Tchebychev polynomials and $m(f)$ a linear form independent of κ_4. This is an extension of the formulas found in [3, Theorem 9.2, formula (9.2.4)] up to the normalization (the variance is $\frac{1}{4}$ here) to C^8 functions. We can extend the convergence (51) to functions which are only C^8 by noticing that the error in (35) still goes to zero uniformly on $\Im z \geq n^{-1/7}$ and then using that for f C^8 compactly supported, we can find by [1, (5.5.11)] a function Ψ so that $\Psi(t,0) = f(t)$ compactly supported and bounded by $|y|^8$ so that for any probability measure μ

$$\Re \int_0^\infty dy \int dx \Psi(x,y) \int \frac{1}{t-x-iy} d\mu(t) = \int \Psi(t,0) d\mu(t)$$

Hence,

$$\mathbb{E}[\sum f(\lambda_i)] - n\sigma_{sc}(f) = \Re \int_0^\infty dy \int dx \Psi(x,y) n (m_n(x+iy) - m_0(x+iy))\,.$$

Applying the previous estimate for $y \geq n^{-1/7}$ and on $y \in [0, n^{-1/7}]$ simply bounding $|n(m_n(x+iy) - m_0(x+iy))| \leq 2ny^{-2}$ s well as $|\Psi|(x,y) \leq 1_{x\in[-M,M]} y^8$ provide the announced convergence (51).

Next we can rewrite (51) in terms of the quantiles $\hat{\gamma}_i$ as

$$\begin{aligned} m_{\kappa_4}(f) &= n \int f(x) \rho_n(x) dx + o(1) \\ &= \sum_i f(\hat{\gamma}_i) + \sum_i f'(\hat{\gamma}_i)(\hat{\gamma}_{i+1} - \hat{\gamma}_i) + o(1) \end{aligned}$$

where we used that $\hat{\gamma}_{i+1} - \hat{\gamma}_i$ is of order n^{-1} by (50). Now, again by (50)

$$\sum_i f(\hat{\gamma}_i) = \sum_i f(\gamma_i) + \sum_i f'(\gamma_i)(\hat{\gamma}_i - \gamma_i) + O(\frac{1}{n^{-1+2\eta}})$$

where we used that $\gamma_{[n/2]} - \hat{\gamma}_{[n/2]} = O(n^{\eta-1})$ by (42). Moreover

$$\sum_i f(\gamma_i) = n \int f(x) \sigma_{sc}(x) dx - \sum_i f'(\gamma_i)(\gamma_{i+1} - \gamma_i) + o(1)$$

Noting that the first term in the right hand side vanishes we deduce that

$$m_{\kappa_4}(f) = \sum_i f'(\gamma_i)[\hat{\gamma}_{i+1} - \hat{\gamma}_i - \gamma_{i+1} + \gamma_i + \hat{\gamma}_i - \gamma_i] + o(1)$$

where $\hat{\gamma}_{i+1} - \hat{\gamma}_i - \gamma_{i+1} + \gamma_i$ is at most of order n^{-2} by (50). Hence, we find that

$$\begin{aligned}
-m_{\kappa_4}(f) &= \sum f'(\gamma_i)(\gamma_i - \hat{\gamma}_i) + o(1) \\
&= \frac{1}{n}\sum_i f'(\gamma_i)[n(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]})] + \frac{1}{n}\sum_i \frac{f'(\gamma_i)}{\sigma_{sc}(\gamma_i)}\int_0^{\gamma_i} C'(x)dx \\
&\quad + \frac{\kappa_4}{2n}\sum_i f'(\gamma_i)(2\gamma_i^3 - \gamma_i) + o(1) \\
&= \int f'(x)\sigma_{sc}(x)dx[n(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]})] + \int f'(x)\int_0^x C'(y)dydx \\
&\quad + \frac{\kappa_4}{2}\int f'(x)(2x^3 - x)\sigma_{sc}(x)dx + o(1)\,.
\end{aligned}$$

We finally take f' even, that is f odd in which case the last term in κ_4 vanishes, as well as the term depending on κ_4 in m_{κ_4} as T_4 is even and f odd. Hence, we deduce that there exists a constant independent of κ_4 such that

$$\lim_{n\to\infty} n(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]}) = C\,.$$

In fact, this constant must vanish as in the case where the distribution is symmetric, and n even, both $\gamma_{[n/2]}$ and $\hat{\gamma}_{[n/2]}$ vanish by symmetry.

Acknowledgments We gratefully acknowledge the early work with Po-Ru Lu that investigated real versions of these results. We are indebted to Bernie Wang for so very much: the careful computations in Julia, incredibly helpful conversations, and his checking the consistency of various formulations. We thank Peter Forrester for helping us track down results for the smallest singular value. We also thank Folkmar Bornemann for teaching us how to use the beautiful codes in [7] for drawing some of these distributions. We gratefully acknowledge the support of the national Science Foundation: DMS-1312831, DMS-1016125, DMS-1016086, DMS-1307704 and of the Simons foundation.

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