A Counterexample to a Hadamard Matrix Pivot Conjecture

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Abstract

In the study of the growth factor of completely pivoted Hadamard matrices, it becomes natural to study the possible pivots. Very little is known or provable about these pivots other than a few cases at the beginning and end. For example it is known that the first four pivots must be $1, 2, 2$ and 4 and the last three pivots in backwards order must be $n, n/2$, and $n/2$. Based on computational experiments, it was conjectured by Day and Peterson, that the $n-3$rd pivot must always be $n/4$. This conjecture would have suggested a kind of symmetry with the first four pivots. In this note we demonstrate a matrix whose $n-3$rd pivot is $n/2$ showing that the conjecture is false.

Problems involving Hadamard matrices sound very easy, but they are notoriously difficult to solve. One interesting open problem that has captured the attention of a number of mathematicians is the question of the largest growth factor for an $n \times n$ Hadamard matrix. A weaker version of a conjecture by Cryer [2] is that the largest growth factor is exactly n. This problem remains of theoretical interest to some numerical analysts and probably deserves more attention in the combinatorics community.

A Hadamard matrix H has entries ± 1 and $HH^T = nI$. If $H(k)$ denotes the absolute value of the determinant of the upper left k by k principal submatrix of H $(H(0) \equiv 1)$, then the kth pivot may be defined as

$$p_k \equiv A(k)/A(k-1).$$

The growth factor $g(H) = \max_k p_k$. A matrix is said to be completely pivoted or CP if Gaussian elimination with no pivoting is equivalent to Gaussian elimination with complete pivoting. Mathematically this means that for each k, we have that $A(k)$ is greater than or equal to the absolute value of any other $k \times k$ determinant that includes the first $k-1$ rows and columns. We refer the reader to the paper by Day and Peterson [3] and the book by Higham [9] for background on this problem.

The conjecture that remains open is that $g(H) = n$ for completely pivoted Hadamard matrices H. The only known approach for studying $g(H)$ is to study the pivots $p_k(H)$. Though there are some results, unfortunately, not very much is known or provable about the pivots themselves. The pivots when $n = 4$ or $n = 8$ are easy to compute. When $n = 12$, it already becomes somewhat difficult to prove that there is only one possible pivot pattern [6]. For $n = 16$ we found that there are already at least 34 pivot patters and it is not known how to categorize them. For $n = 20$ or larger, the number of possibilities is already enormous, and little is known about the pivots.

However the first three and the last three pivots are known for any completely pivoted Hadamard matrix H [3]. The first three, p_1, p_2, p_3, are $1, 2, 2$ respectively. The last three, $n/p_n, n/p_{n-1}, n/p_{n-2}$, are also known to follow the sequence $1, 2, 2$ respectively. It is known further that $p_4 = 4$ and there is the implied suggestion by Day and Peterson [3, page 505] that m is a possible value for p_k if and only if m is a possible value for n/p_{n+1-k}. Day and Peterson also make the explicit conjecture that n/p_{n-3} (which was known to possibly be equal to 2 or 4) must also be 4.

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After both of us tried very hard to prove that n/p_{n-3} must be 4, we found a completely pivoted Hadamard matrix whose value for n/p_{n-3} is 2 showing the conjecture to be false. The Hadamard matrix below, where $+$ denotes $+1$ and $-$ denotes -1, has pivot pattern

$$1, 2, 2, 4, 3, 8/3, 2, 4, 4, 4, 4, 8, 8, 8, 8, 16:$$

$$\begin{pmatrix}
+ & + & - & - & - & - & + & - & + & - & + & + & + & + & - & - \\
- & + & + & - & + & + & + & + & - & - & + & - & + & - & - & - \\
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & - & - & + & + & - & - & - & + & + & - & + & + \\
+ & - & - & - & + & - & - & + & - & + & + & - & + & + & - & + \\
- & + & + & + & + & - & - & - & - & - & - & + & + & + & - & + \\
- & - & - & + & + & - & - & + & + & - & + & + & + & - & + & - \\
- & - & - & - & + & + & + & - & + & - & - & - & + & + & + & + \\
- & - & + & - & - & + & - & - & + & + & + & + & + & - & - & + \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\
- & + & - & - & + & - & + & + & + & + & - & + & - & - & - & + \\
+ & - & + & - & + & + & - & + & + & - & - & + & - & + & - & - \\
+ & - & + & + & + & - & + & - & + & - & + & - & - & - & - & + \\
- & - & - & + & - & + & + & + & - & - & + & + & - & + & - & + \\
+ & - & - & + & + & + & + & - & - & + & - & + & + & - & - & - \\
+ & + & - & + & - & + & - & + & + & - & - & - & + & - & - & +
\end{pmatrix}$$

Therefore the general conjecture that the allowable pivots for p_k and the allowable pivots n/p_{n-k+1} are the same is false. The general question of whether $p_k \leq n/2$ for $k = 1, \ldots, n-1$ still remains unsolved.

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