NOTES ON A MEASURE THEORETIC VERSION OF NABER-VALTORTA'S RECTIFIABILITY THEOREM

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Here S typically represents the (k,ϵ)-strata of some geometric object. The monotone quantity typically quantifies how "homogenous" the geometric object is. Assumption 0.2 can usually be proven by a straightforward contradiction argument, using various compactness properties of the geometric object, (upper-semi-)continuity of the monotone quantity, and sharpness of the monotone quantity when it's constant in r. Assumption 0.3 usually follows from the "sharp" form of the monotone quantity. For examples, see [NV17] (harmonic maps), [NV15] (stationary varifolds), [NV16] (approximate harmonic maps), [EE17] (free-boundaries), [FS17] (thin obstacle problem).

We work in $\mathbb{R}^n$. Suppose there is a closed set S, and a non-negative quantity $\Theta(x,r)$, satisfying the following properties:

Assumption 0.1 (monotonicity)**.** *For any* $x \in B_1$, *and* $0 < s < r < 1 - |x|$, *we have*

$$\Theta(x,s) \leq \Theta(x,r) + \epsilon(r),$$

where $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing function satisfying*

$$\int_0^2 \epsilon(r) \frac{dr}{r} < \infty.$$

Assumption 0.2 (quantitative splitting)**.** *For any* ρ, γ, *there is an* $\eta_0 = \eta_0(\gamma,\rho)$ *so that the following occurs. Take* $B_{10r}(x) \subset B_1$, *and* $\eta \leq \eta_0$. *Suppose*

$$\epsilon(10r) \leq \eta, \quad \sup_{B_{3r}(x)} \Theta(z,3r) \leq E.$$

Then at least one of the following occurs:

(1) *we have*

$$S \cap B_r(x) \subset \{z \in B_r(x) : \Theta(z,\gamma r) \geq E - \gamma\}, \quad or$$

(2) *there is an affine* $k-1$*-space* $p + L^{k-1}$, *so that*

$$\{z \in B_r(x) : \Theta(z, 3\eta r) \geq E - \eta\} \subset B_{\rho r}(p+L).$$

Assumption 0.3 (effective beta control)**.** *There is a δ_0 so that the following occurs. Take $B_{10r}(x) \subset B_1$, and μ any finite Borel measure. Suppose*

$$\epsilon(10r) \leq \delta_0, \quad \Theta(x, 8r) - \Theta(x, \delta_0 r) < \delta_0, \quad x \in S.$$

Then

$$\beta^k_\mu(x, r)^2 \leq \frac{c_0}{r^k} \int_{B_r(x)} \Theta(z, 8r) - \Theta(r) + \epsilon(r) d\mu(z).$$

Here $\beta^k_\mu(x, r)$ is the Jones's β number, defined by

$$\beta^k_\mu(x, r)^2 = \inf_{p+V^k} r^{-k-2} \int_{B_r(x)} d(z, p + V)^2 d\mu(z),$$

where the infinimum is over affine k-planes $p + V^k$.

Then it holds that

Theorem 0.4 ([NV17])**.** *Suppose* $\sup_{x\in B_1} \Theta(x, 1) \leq \Lambda$*, and* $\theta \in (0, 1)$*. There is a constant* $c = c(n, \theta, \epsilon(r), c_0, \delta_0, \eta_0)$ *so that* $S \cap B_\theta$ *is* k*-rectifiable, and*

$$\mathcal{H}^k(S \cap B_\theta) \leq c\Lambda^{1+c}.$$

In fact we have the estimate

$$Vol(B_r(S \cap B_\theta)) \leq c\Lambda^{1+c} r^{n-k} \quad \forall r < 1.$$

Remark 0.5. As observed by [NV17], a relatively minor modification of the proof would in fact prove the following. Given any collection of disjoint balls $\{B_{r_a}(a)\}_{a\in\mathcal{A}}$, with $\mathcal{A} \in S \cap B_{1/2}$ and $r_a \leq 1/10$, then there is a subcollection $\mathcal{A}' \subset \mathcal{A}$ satisfying:

$$B_{r_a}(\mathcal{A}) \subset B_{5r_a}(\mathcal{A}'), \quad \sum_{a'\in\mathcal{A}'} r^k_{a'} \leq c,$$

for $c = c(n, \epsilon(r), c_0, \delta_0, \eta_0)$.

We use the following discrete Reifenberg theorem.

Theorem 0.6 ([NV17] or [ENV16])**.** *Let* $\{B_{r_p}(p)\}_p$ *be a finite collection of disjoint balls, with* $r_p \leq 1$. *Let* $\mu = \sum_p r_p^k \delta_p$. *Suppose that*

$$\int_{B_1} \int_0^1 \beta_\mu^k(z,r)^2 dr/r d\mu(z) \leq 1.$$

Then

$$\mu(B_1) \leq c_{dr}(n).$$

And the following rectifiability theorem.

Theorem 0.7 ([AT15] or [ENV16])**.** *Let* $\mu = \mathcal{H}^k \llcorner S$, *for* $S \subset B_1$. *Suppose*

$$\int_0^1 \beta_\mu^k(z,r)^2 dr/r < \infty \quad \mu - a.e. z.$$

Then S *is* k*-rectifiable.*

Lemma 0.8. *There is an* $\eta_1(n, \delta_0, c_0)$ *so that the following holds. Take* $B_{2r}(q) \subset B_1$. *Take* r *sufficiently small so that* $\int_0^r \epsilon dr/r < \eta_1$. *Let* $\{B_{r_p}(p)\}_p$ *be a finite collection of* disjoint *balls, satisfying*

$$\Theta(p, r) - \Theta(p, \delta_0 r_p) \leq \eta_1, \quad p \in S \cap B_r(q), \quad r_p \leq r.$$

Then we have

$$\sum_p r_p^k \leq c_1(n) r^k.$$

Proof. Choose η_1 to be the lesser of

$$\delta_0/5, \quad \frac{1}{16 c_0 20^n (1 + c_{dr})^2}.$$

Write $r_i = 2^{-i} r$. We assume $i \geq 5$, so that $r_i \leq r/32$. Define the packing measure

$$\mu_i = \sum_{r_p \leq r_i} r_p^k \delta_p, \tag{1}$$

and for shorthand write $\beta_i^k = \beta_{\mu_i, 2}^k$.

We make a few remarks about the β_i. Suppose $x \in \mathrm{spt}\mu_i$, and $j \geq i$. Then by disjointness

$$\beta_i(x, r_j) = \begin{cases} \beta_j(x, r_j) & \text{if } x \in \mathrm{spt}\mu_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Since $\epsilon(10 r_i) < 2\eta_1$, and $\Theta(p, 8r_i) - \Theta(\delta_0 r_i) \leq 5\eta_1$, and $p \in S$, we have by our assumption 0.3 that

$$\beta_i(x, r_i)^2 \leq c_0 r_i^{-k} \int_{B_{r_i}(x)} \Theta(y, 8r_i) - \Theta(y, r_i) + \epsilon(8r_i)\, d\mu_i(y). \tag{3}$$

We prove inductively that, for $i \geq 5$, we have the estimate

$$\mu_i(B_{r_i}(x)) \leq c_{dr} r_i^k, \tag{4}$$

where c_{dr} is the constant from discrete Reifenberg. The required estimate will then follow by a obvious packing argument.

(4) holds true for i sufficiently large, so that $r_i \leq \min_p r_p$. By inductive hypothesis, let us assume (4) holds at $i - 1$. By a straightforward packing estimate, we can estimate

$$\mu_j(B_{2r_j}(x)) \leq \Gamma r^k \quad \forall x \in \mathrm{spt}\mu_j, \forall j \geq i,$$

where $\Gamma(n) = 20^n (1 + c_{dr}(n))$.

Then we have

$$\begin{aligned}
&\sum_{r_j\le r_i}\int_{B_{r_i}(x)}\beta_i(z,r_j)^2 d\mu_i(z)\\
&=\sum_{r_j\le r_i}\int_{B_{r_i}(x)}\beta_j(z,r_j)^2 d\mu_j(z)\\
&\le c_0\sum_{r_j\le r_i}\int_{B_{r_i}(x)}\int_{B_{r_j}(z)}\Theta(y,8r_j)-\Theta(y,r_j)+\epsilon(8r_j)d\mu_j(y)d\mu_j(z)\\
&\le c_0\int_{B_{2r_i}(x)}\sum_{r_j\le r_i}\frac{\mu_j(B_{r_j}(y)}{r_j^k}(\Theta(y,8r_j)-\Theta(y,r_j)+\epsilon(8r_j))d\mu_j(y)\\
&\le c_0\Gamma\int_{B_{2r_i}(x)}\sum_{r_y\le r_j\le r_i}\Theta(y,8r_j)-\Theta(y,r_j)+\epsilon(8r_j)d\mu_j(y)\\
&\le 4c_0\Gamma^2 r_i^k\sup_{y\in \mathrm{spt}\mu_i\cap B_{2r_i}(x)}\left(|\Theta(y,8r_i)-\Theta(y,r_y)|+\epsilon(8r_i)+\int_{r_y}^{16r_i}\epsilon dr/r\right)\\
&\le 8c_0\Gamma^2 r_i^k\sup_p\left(|\Theta(p,r)-\Theta(p,\eta_1 r_p)|+\int_0^r\epsilon dr/r\right)\\
&\le 16c_0\Gamma^2\eta r_i^k\\
&\le r_i^k.
\end{aligned}$$

Computations: For $y\in \mathrm{spt}\mu_i$, we have

$$\sum_{r_y\le r_j\le r_i}\epsilon(8r_j)\le\frac{1}{\log 2}\int_{r_y}^{16r_i}\epsilon dr/r.$$

And

$$\sum_{r_y\le r_j\le r_i}\Theta(y,8r_j)-\Theta(y,r_j)\le 2(\Theta(y,8r_i)-\Theta(y,r_y)+\epsilon(8r_i)).$$

Then discrete Reifenberg implies that

$$\mu_i(B_{r_i}(x))\le c_{dr}(n)r_i^k.$$

This proves the inductive claim. □

Theorem 0.9. *For any ρ, there is an $\eta_2(n, c_0, \delta_0, \eta_0, \rho)$ so that the following holds. Take any $\eta \leq \eta_2$, and take $B_{11r}(p) \subset B_1$, with r sufficiently small so that $\int_0^{10r} \epsilon dr/dr < \eta/16$, and fix $R > 0$. Suppose $\sup_{x \in B_{3r}(p)} \Theta(p, 3r) \leq E$, and take $p' \in B_r(p)$, $r' \leq r$.*

There is a collection of balls $\{B_{r_s}(s)\}_{s \in \mathcal{S}} \cup \{B_{r_b}(b)\}_{b \in \mathcal{B}}$ satisfying:

(1) *$\mathcal{B} \cup \mathcal{S} \subset S \cap B_{r'}(p') \cap B_r(p)$ and $r_s \leq r'$, $r_b \leq r'$,*
(2) *$S \cap B_{r'}(p') \subset B_{r_s}(\mathcal{S}) \cup B_{r_b}(\mathcal{B})$,*
(3) *$\sum_{s \in \mathcal{S}} r_s^k + \sum_{b \in \mathcal{B}} r_b^k \leq c_2(n) r'^k$,*
(4) *For every $s \in \mathcal{S}$, we have $r_s = \min\{r', R\}$, and for every $b \in \mathcal{B}$, we have*
$$\{z \in B_{r_b}(b) : \Theta(z, 3\eta r_b) \geq E - \eta/2\} \subset B_{\rho r_b}(q + L^{k-1})$$
for some $(k-1)$-plane $q + L^{k-1}$.

Proof. Choose

$$\gamma = \min\{\delta_0, \eta_1(n, \delta_0, c_0)/2\} \tag{5}$$

$$\eta_2 = \min\{\gamma/10, \eta_0(\gamma, \rho)/10\}. \tag{6}$$

If $R \geq r'$, just let $\mathcal{S} = \{p'\}$, and $r_{p'} \equiv r_s = r'$. Otherwise proceed as follows. By monotonicity, for any ball $x \in B_r(p)$, and $s \leq r$, we have $\Theta(x, s) \leq E + \eta/2$. Then we have by assumption 0.2 that one of the following occurs:

$$\Theta(z, \gamma s) \geq E - \gamma \text{ on } S \cap B_s(x),$$

or

$$\{z \in B_s(x) : \Theta(x, 3\eta s) \geq E - \eta/2\} \subset B_{\rho s}(q + L^{k-1}),$$

for some $(k-1)$-space $q + L^{k-1}$.

For every $x \in B_{r'}(p') \cap B_r(p)$, let

$$r_x = \inf\{r'\} \cup \{s \geq R : \Theta(z, \gamma s) \geq E - \gamma \text{ on } S \cap B_s(x)\}.$$

Trivially $r_x \leq r'$. Then $\{B_{r_x}(x)\}_{x \in S \cap B_{r'}(p') \cap B_r(p)}$ covers $S \cap B_{r'}(p) \cap B_r(p)$. Let $\{B_{r_x}(x)\}_{x \in \mathcal{U}}$ be a Vitali cover, so that the $r_x/5$-balls are disjoint. For every $x \in \mathcal{U}$, we have

$$\Theta(x, r') - \Theta(x, \delta_0 r_x/5) \leq \gamma + \eta/4 \leq \eta_1(n, \delta_0, c_0),$$

and therefore by Theorem 0.6, we have the required packing estimate with $c_2 = 5^k c_1$. Set $\mathcal{S} = \{x \in \mathcal{U} : r_x = R\}$, and $\mathcal{B} = \{x \in \mathcal{U} : r_x > R\}$, and the required properties follow by assumption 0.2. ∎

Theorem 0.10. *There is a $\rho(n)$, $\gamma(n, c_0, \delta_0)$, $\eta_3(n, c_0, \delta_0, \eta_0(\gamma, \rho))$ so that the following holds. Take $B_{11r}(p) \subset B_1$, with r sufficiently small so that $\int_0^{16r} \epsilon dr/r < \eta_3/16$. Suppose $\sup_{y \in B_{3r}(p)} \Theta(y, 3r) \leq E$. Take any $0 < R \leq r$.*

There is a collection of balls $\{B_{r_y}(y)\}_{y \in \mathcal{U}}$ which satisfy the following properties:

1. *$\mathcal{U} \subset S \cap B_r(p)$, and $r_y \leq \max\{r/10, R\}$,*
2. *$S \cap B_r(p) \subset B_{r_y}(\mathcal{U})$,*
3. *$\sum_{y \in \mathcal{U}} r_y^k \leq c_6 r^k$,*
4. *for every $y \in \mathcal{U}$, we have*
$$\text{either} \quad r_y = R \quad \text{or} \quad \sup_{z \in B_{3r_y}(y)} \Theta(y, 3r_y) \leq E - \eta_3/2.$$

Here $c_6 = c_6(n, c_0, \delta_0, \eta_0(\gamma, \rho))$.

Proof. In the following we will fix $\rho = \rho(n)$. Let $\gamma(n, c_0, \delta_0)$ be as in Lemma 0.8, and let $\eta_3 = \min\{\eta_2, \rho\}$.

If $R \geq r/10$, then simply take $\{B_{r_y}(y)\}_{y \in \mathcal{U}}$ to be a Vitali subcover of $\{B_R(y)\}_{y \in S \cap B_r(p)}$, so that the $R/5$-balls are disjoint. Otherwise, we proceed as follows. We build a sequence of covering $\mathcal{S}_i$, $\mathcal{B}_i$ ($i = 0, 1, 2, \ldots$) satisfying the following inductive hypotheses:

1. size control
$$\mathcal{S}_i \cup \mathcal{B}_i \subset B_r(p), \quad \eta_3 R \leq r_s \leq r/10, \quad \eta_3 R \leq r_b \leq 2^{-i} r.$$
2. bad ball packing:
$$\sum_{b \in \mathcal{B}_i} r_b^k \leq c_2(n) r 2^{-i}.$$
3. stop ball packing:
$$\sum_{s \in \mathcal{S}_i} r_s^k \leq c_3(n, \eta_3) r \sum_{j=-1}^{i-1} 2^{-j}.$$
4. covering:
$$S \cap B_r(p) \subset B_{r_s}(\mathcal{S}) \cup B_{r_b}(\mathcal{B}).$$
5. stop ball control: for every $s \in \mathcal{S}_i$, we have
$$r_s \in [\eta_3 R, R] \quad \text{or} \quad \sup_{y \in B_{3r_s}(s)} \Theta(y, 3r_s) \leq E - \eta_2/2.$$
6. bad ball control: for every $b \in \mathcal{B}_i$, we have we have
$$\{z \in B_{r_b}(b) : \Theta(z, 3\eta_3 r_b) \geq E - \eta_3/2\} \subset B_{\rho r_b}(q + L^{k-1})$$
for some $(k-1)$-plane $q + L^{k-1}$.

Apply Theorem 0.9 with $r' = r$, $p' = p$ to obtain a $\mathcal{S}_0$, $\mathcal{B}_0$. These trivially satisfy the required estimates with c_2 in place of c_3. Provided we take $c_3 \geq c_2$, this starts our inductive procedure.

Let us assume we have constructed $\mathcal{S}_i$, $\mathcal{B}_i$ satisfying inductive hypotheses. Fix a $b \in \mathcal{B}_i$. By bad ball control, there is an affine $(k-1)$-space $q + L^{k-1}$ so that

$$\{z \in B_{r_b}(b) : \Theta(z, 3\eta_3 r_b) \geq E - \eta_3/2\} \subset B_{\rho r_b}(q + L).$$

Let $\mathcal{J}_b$ be a maximal ρr_b-net in $B_{10\rho r_b}(q+L) \cap B_{r_b}(b) \cap B_r(p)$. Then

$$\# \mathcal{J}_b \leq c_4(n)\rho^{1-k},$$

and

$$B_{10\rho r_b}(q+L) \cap B_{r_b}(b) \cap B_r(p) \subset \cup_{z \in \mathcal{J}_b} B_{\rho r_b}(z).$$

For each $z \in \mathcal{J}_b$, apply Theorem 0.9 with $r' = \rho r_b$, $p' = z$, to obtain a collection $\mathcal{B}_{b,z}$, $\mathcal{S}_{b,z}$. Then

$$S \cap B_{r_b}(b) \cap B_r(p) \subset B_{r_s}(\mathcal{S}_{b,z}) \cup B_{r_b}(\mathcal{B}_{b,z}),$$

and $r_s = R$ for all $s \in \mathcal{S}_{b,z}$. Define

$$\mathcal{B}_{i+1} = \bigcup_{b \in \mathcal{B}_i} \bigcup_{z \in \mathcal{J}_b} \mathcal{B}_{b,z}.$$

We have

$$\sum_{b \in \mathcal{B}_{i+1}} r_b^k \leq \sum_{b \in \mathcal{B}_i} \sum_{z \in \mathcal{J}_b} c_2 (\rho r_b)^k \tag{7}$$

$$\leq \sum_{b \in \mathcal{B}_i} c_2 c_4 \rho r_b^k \tag{8}$$

$$\leq c_2^2 c_4 \rho 2^{-i} r \tag{9}$$

$$\leq c_2 2^{-i-1} r, \tag{10}$$

provided $\rho = \rho(n)$ is chosen sufficiently small so that $c_2(n)c_4(n)\rho \leq 1/2$ and $\rho \leq 1/10$. This constructs the required $\mathcal{B}_{i+1}$.

We build the required $\mathcal{S}_{i+1}$. For each $b \in \mathcal{B}_i$, let $\mathcal{S}_b$ be a maximal $\eta_3 r_b$-net in $B_{r_b}(b) \cap B_r(p) \setminus B_{10\rho r_b}(q + L^{k-1})$. For every $s \in \mathcal{S}_b$, let $r_s = \eta_3 r_b$. Then

$$\sum_{s \in \mathcal{S}_b} r_s^k \leq c_5(n, \eta_3) r_b^k,$$

and

$$B_{r_b}(b) \cap B_r(p) \setminus B_{10\rho r_b}(q+L) \subset \cup_{s \in \mathcal{S}_b} B_{r_s}(s).$$

Moreover, since $B_{3r_s}(s) \cap B_{\rho r_b}(q+L) = \emptyset$, we have

$$\sup_{y \in B_{3r_s}(s)} \Theta(y, 3r_s) \leq E - \eta_3/2.$$

Let

$$\mathcal{S}_{i+1} = \mathcal{S}_i \cup \bigcup_{b \in \mathcal{B}_i} \left(\mathcal{S}_b \cup \bigcup_{z \in \mathcal{J}_z} \mathcal{S}_{z,b} \right).$$

Then we have

$$\sum_{s \in \mathcal{S}_{i+1}} r_s^k \le c_3 r \sum_{j=-1}^{i-1} 2^{-j} + \sum_{b \in \mathcal{B}_i} \left(\sum_{s \in \mathcal{S}_b} r_s^k + \sum_{z \in \mathcal{J}_z} \sum_{s \in \mathcal{S}_{b,z}} r_s^k \right) \tag{11}$$

$$\le c_3 r \sum_{j=-1}^{i-1} 2^{-j} + c_5 \sum_{b \in \mathcal{B}_i} r_b^k + 2^{-1} \sum_{b \in \mathcal{B}_i} r_b^k \tag{12}$$

$$\le c_3 r \sum_{j=-1}^{i-1} 2^{-j} + (c_5 + 1) c_2 2^{-i} r \tag{13}$$

$$\le c_3 r \sum_{j=-1}^{i} 2^{-j}, \tag{14}$$

provided $c_3 \ge (c_5 + 1)c_2$.

This proves the required estimates, and finishes the construction of the required $\mathcal{B}_i$, $\mathcal{S}_i$.

Notice that for i sufficiently large, we have $\mathcal{B}_i = \emptyset$, and hence $B_{r_s}(\mathcal{S}_i)$ covers $S \cap B_r(p)$. For this i, let $\mathcal{U} = \mathcal{S}_i$, and $r_x = \max\{R, r_s\}$. Then $\mathcal{U}$ satisfies the required conditions, with $c_6 = \eta_3^{-k} c_3 4$. □

Theorem 0.11. *There is an $\eta_4(n, c_0, \delta_0, \eta_0)$ so that the following holds. Take $B_{11r}(p) \subset B_1$, with r chosen sufficiently small so that $\int_0^{10r} \epsilon dr/r < \eta_4$. Suppose $\sup_{y \in B_{3r}(p)} \Theta(y, 3r) \leq E$.*

Then for any $s \leq r$, there is a subset $\mathcal{U} \subset S \cap B_r(p)$, so that

$$S \cap B_r(p) \subset \cup_{y \in \mathcal{U}} B_s(y),$$

and

$$\#\mathcal{U} \leq c_7^{1+2E/\eta_3} s^{-k},$$

where $c_7 = c_7(n, c_0, \delta_0, \eta_0)$.

Proof. Fix $\eta_4 = \eta_3/16$. There is no loss in replacing S with $S \cap B_r(p)$. We construct a sequence of coverings $\mathcal{U}_i$ $(i = 1, 2, \ldots)$ which satisfy the following

(1) covering:
$$S \cap B_r(p) \subset B_{r_x}(\mathcal{U}_i),$$
(2) packing:
$$\sum_{x \in \mathcal{U}_i} r_x^k \leq c_8 \sum_{x \in \mathcal{U}_{i-1}} r_x^k, \quad \sum_{x \in \mathcal{U}_1} r_x^k \leq c_8 r^k,$$
for $c_8 = c_8(n, c_0, \delta_0, \eta_0)$,

(3) energy drop: for every $x \in \mathcal{U}_i$, we have $r_x \geq s$, and either
$$r_x = s \quad \text{or} \quad \sup_{y \in B_{3r_x}(x)} \Theta(y, 3r_x) \leq E - i\eta_3/2.$$
(4) radius control:
$$\sup_{x \in \mathcal{U}_i} r_x \leq \max\{s, 10^{-i} r\}.$$

Apply Theorem 0.10 with $R = s$ to obtain $\mathcal{U}_1$. This starts our inductive process. Suppose we have constructed $\mathcal{U}_i$. We demonstrate how to build $\mathcal{U}_{i+1}$. Fix $x \in \mathcal{U}_i$. If $r_x = s$, then let $\mathcal{U}'_x = \{x\}$. Otherwise, apply Theorem 0.10 with $R = s$, $p = x$, $r = r_x$, $E - i\eta_3/2$ in place of E, to obtain a family $\mathcal{U}'_x$. Define

$$\mathcal{U}_{i+1} = \bigcup_{x \in \mathcal{U}_i} \mathcal{U}'_x.$$

Observe that for $i \geq 1 + 2E/\eta_3$, we must have $r_x = s$ for every $x \in \mathcal{U}_i$, since Θ is a non-negative function. We deduce that this $\mathcal{U}_i$ gives the required covering. ☐

Proof of Theorem 0.4. Choose r_0 sufficiently small so that $11r_0 \leq 1-\theta$, $\int_0^{10r_0} \epsilon dr/r \leq \eta_4$. Let $\{B_{r_0}(x)\}_{x\in\mathcal{U}'}$ be a Vitali cover of $S\cap B_\theta$, so that the $r_0/5$-balls are disjoint. For each $x \in \mathcal{U}'$, apply Theorem 0.11 with $s = r$ and $E = \Lambda + \epsilon(1)$ to obtain a $\mathcal{U}_x$. Define the cover $\mathcal{U} = \cup_{x\in\mathcal{U}'}\mathcal{U}_x$. Then we have

$$S \cap B_\theta \subset \bigcup_{x\in\mathcal{U}} B_r(x),$$

and

$$\#\mathcal{U} \leq c^{1+c(\Lambda+\epsilon(1))}\#\mathcal{U}' \leq c^{1+c\Lambda},$$

where $c = c(n, \theta, \epsilon, c_0, \delta_0, \eta_0)$. This proves the required Minkowski estimate, and hence the required Hausdorff estiamte also.

To prove rectifiable, one uses a minor modification of Theorems 0.9-0.11 with $R = 0$, to break into covers of the form $\mathcal{U}_0 \cup B_{r_y}(\mathcal{U})$, where $\mathcal{U}_0$ is rectifiable, and each $B_{r_y}(y)$ admits a small but definite density drop. We state the theorems below, but leave most of the proofs as a straightforward exercise for the reader. □

In the following, we make the following additional assumption on S, which we just showed follows from Assumptions 0.1, 0.2, 0.3.

Assumption 0.12. *For any $\theta < 1$, there is some constant $\Gamma(\theta)$ so that*

$$\mathcal{H}^k(S \cap B_r(x)) \leq \Gamma r^k \quad \forall x \in B_\theta, r < 1 - |x|,$$

Lemma 0.13 (lemma 0.8'). *Take $B_{11r}(q) \subset B_1$. Take r sufficiently small so that $\epsilon(10r) < \delta_0/16$. Suppose $\mathcal{S}$ is a subset of $S \cap B_r(q)$, satisfying*

$$\Theta(p, 8r) - \Theta(p, 0) \leq \delta_0/16 \quad \forall p \in \mathcal{S}, s \leq r.$$

Then $\mathcal{S}$ is rectifiable.

Proof. Define the Radon measure $\mu = \mathcal{H}^k \llcorner \mathcal{S}$. Let $r_i = 2^{-i}r$. I claim that for any $i \geq 5$, and any $x \in B_r(p)$, we have

$$\sum_{r_j \leq r_i} \int_{B_{r_i}(x)} \beta^k_\mu(z, r_j)^2 d\mu(z) < \infty.$$

Rectifiability will follow from Theorem 0.7.

For every $x \in \mathcal{S}$, we have $\epsilon(10r) < \delta_0$, and $\Theta(x, 8r_i) - \Theta(x, r_i) < \delta_0/16 + \delta_0/8 < \delta_0$, and therefore by assumption 0.3

$$\beta^k_\mu(x, r_i)^2 \leq c_0 r_i^{-k} \int_{B_{r_i}(x)} \Theta(y, 8r_i) - \Theta(y, r_i) + \epsilon(8r_i) d\mu(y).$$

We compute:

$$\begin{aligned}
&\sum_{r_j \leq r_i} \int_{B_{r_i}(x)} \beta^k_\mu(z, r_i)^2 d\mu(z) \\
&\leq c_0 \sum_{r_j \leq r_i} \int_{B_{r_i}(x)} \int_{B_{r_j}(z)} \Theta(y, 8r_j) - \Theta(y, r_j) + \epsilon(8r_j) d\mu(y) d\mu(z) \\
&\leq c_0 \Gamma \int_{B_{2r_i}(x)} \sum_{r_j \leq r_i} \Theta(y, 8r_j) - \Theta(y, r_j) + \epsilon(8r_j) d\mu(y) \\
&\leq c_0 \Gamma^2 \left(\delta_0 + \int_0^{16 r_i} \epsilon dr/r \right) \\
&< \infty.
\end{aligned}$$

□

Theorem 0.14 (theorem 0.9'). *For any ρ, there is an $\eta_2(n, c_0, \delta_0, \eta_0, \rho)$ so that the following holds. Take any $\eta \leq \eta_5$, and take $B_{11r}(p) \subset B_1$, with r sufficiently small so that $\int_0^{10r} \epsilon dr/r < \eta/16$. Suppose $\sup_{x \in B_{3r}(p)} \Theta(p, 3r) \leq E$, and take $p' \in B_r(p)$, $r' \leq r$.*

There is a set $\mathcal{S}_0$, and a collection of balls $\{B_{r_b}(b)\}_{b \in \mathcal{B}}$ satisfying:

1. *$\mathcal{S}_0 \cup \mathcal{B} \subset S \cap B_{r'}(p') \cap B_r(p)$, and $r_b \leq r'$,*

(2) $S \cap B_{r'}(p') \subset \mathcal{S}_0 \cup B_{r_b}(\mathcal{B})$,
(3) $\sum_{b\in\mathcal{B}} r_b^k \leq c_2(n) r'^k$,
(4) $\mathcal{S}$ *is* k*-rectifiable, and for every* $b \in \mathcal{B}$*, we have*

$$\{z \in B_{r_b}(b) : \Theta(z, 3\eta r_b) \geq E - \eta/2\} \subset B_{\rho r_b}(q + L^{k-1}),$$

for some $(k-1)$*-plane* $q + L^{k-1}$.

Proof. Same as Theorem 0.9, except we let $R = 0$, and then define $\mathcal{S}_0 = \{x \in \mathcal{U} : r_x = 0\}$, and use Lemma 0.8' to prove rectifiability (in addition to Lemma 0.8 to prove packing control on the bad balls). □

Theorem 0.15 (theorem 0.10')**.** *There is a* $\rho(n)$, $\gamma(n, c_0, \delta_0)$, $\eta_3(n, c_0, \delta_0, \eta_0(\gamma, \rho))$ *so that the following holds. Take* $B_{11r}(p) \subset B_1$*, with* r *sufficiently small so that* $\int_0^{16r} \epsilon dr/r < \eta_3/16$*. Suppose* $\sup_{y\in B_{3r}(p)} \Theta(y, 3r) \leq E$.

There is a set $\mathcal{U}_0$*, and a collection of balls* $\{B_{r_y}(y)\}_{y\in\mathcal{U}}$*, which satisfy the following properties:*

(1) $\mathcal{U}_0 \cup \mathcal{U} \subset S \cap B_r(p)$*, and* $r_y \leq r/10$,
(2) $S \cap B_r(p) \subset \mathcal{U}_0 \cup B_{r_y}(\mathcal{U})$,
(3) $\sum_{y\in\mathcal{U}} r_y^k \leq c_6 r^k$,
(4) $\mathcal{U}_0$ *is* k*-rectifiable, and for every* $y \in \mathcal{U}$*, we have*

$$\sup_{z\in B_{3r_y}(y)} \Theta(y, 3r_y) \leq E - \eta_3/2.$$

Proof. Same as Theorem 0.10, except with $R = 0$, and in the inductive construction, use that at stage i we have $\mathcal{H}^k(S \setminus \mathcal{S}_0 \cup B_{r_s}(\mathcal{S})) \leq c_2 r 2^{-i} \to 0$. □

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