EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §6

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Exercise 6.1. Show the following versions of the Product, Commutator, and Adjoint Rules: if $a, b \in S(1)$ then

$$\operatorname{Op}_h(a)\operatorname{Op}_h(b) = \operatorname{Op}_h(ab) + \mathcal{O}(h)_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)},$$
$$[\operatorname{Op}_h(a), \operatorname{Op}_h(b)] = -ih\operatorname{Op}_h(\{a,b\}) + \mathcal{O}(h^2)_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)},$$
$$\operatorname{Op}_h(a)^* = \operatorname{Op}_h(\overline{a}) + \mathcal{O}(h)_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}.$$

Exercise 6.2. Assume that $a \in S(1)$, the functions $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^n)$ are h-independent, and $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$. Show that

$$\|\chi_1 \operatorname{Op}_h(a)\chi_2\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \mathcal{O}(h^\infty).$$

This is a version of *pseudolocality* of pseudodifferential operators. It is a weaker property than *locality* of differential operators: if a was a polynomial in ξ, then $\chi_1 \operatorname{Op}_h(a)\chi_2 = 0$.

Exercise 6.3. Assume that $a \in S(m)$ where m is an order function and

$$m(w) \to 0 \quad \text{as} \quad w = (x,\xi) \to 0.$$

Fix $\chi \in C_c^\infty(\mathbb{R}^{2n})$ such that $\operatorname{supp}\chi \subset B(0,2)$ and $\chi = 1$ on $B(0,1)$. For $R \geq 1$, define

$$a_R(w) := \chi\Big(\frac{w}{R}\Big)a(w), \quad w \in \mathbb{R}^{2n}.$$

(a) Show that for each multiindex α, we have

$$\sup|\partial^\alpha(a - a_R)| \to 0 \quad \text{as} \quad R \to 0.$$

(b) Using the L^2 boundedness theorem (see Zworski's book, formula (4.5.9)) show that

$$\|\operatorname{Op}_h(a) - \operatorname{Op}_h(a_R)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \to 0 \quad \text{as} \quad R \to 0.$$

Exercise 6.4. For $s \in \mathbb{R}$, define the *semiclassical Sobolev space* $H_h^s(\mathbb{R}^n)$, $\mathscr{S}(\mathbb{R}^n) \subset H_h^s(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$, with the norm

$$\|u\|^2_{H_h^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \langle h\xi\rangle^{2s}|\hat{u}(\xi)|^2\,d\xi, \quad \langle h\xi\rangle := \sqrt{1+|h\xi|^2}.$$

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(a) Show that the norms $\|\bullet\|_{H_h^s(\mathbb{R}^n)}$ are equivalent for fixed s and different values of h, with equivalence constants depending on h.

(b) Show that the norm $\|u\|_{H_h^s(\mathbb{R}^n)}$ is equivalent, with equivalence constants independent of h, to the norm $\|\operatorname{Op}_h(\langle\xi\rangle^s)u\|_{L^2(\mathbb{R}^n)}$.

(c) Assume that $a \in S(\langle\xi\rangle^k)$. Using part (b), the Composition Theorem, and the L^2 Boundedness Theorem, show that for each s there exists a constant C such that for all h

$$\|\operatorname{Op}_h(a)\|_{H_h^s(\mathbb{R}^n)\to H_h^{s-k}(\mathbb{R}^n)} \leq C.$$

Exercise 6.5.* Let $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a bounded operator. Fix a Hilbert basis $\{e_j\}$ of $L^2(\mathbb{R}^n)$ and define the *Hilbert–Schmidt norm* of A by putting

$$\|A\|_{\mathrm{HS}}^2 := \sum_j \|Ae_j\|_{L^2(\mathbb{R}^n)}^2.$$

If $\|A\|_{\mathrm{HS}} < \infty$ then we call A a *Hilbert–Schmidt operator*.

(a) For any other Hilbert basis $\{f_k\}$ show the identities

$$\sum_j \|Ae_j\|_{L^2(\mathbb{R}^n)}^2 = \sum_{j,k} |\langle Ae_j, f_k\rangle_{L^2(\mathbb{R}^n)}|^2 = \sum_k \|A^* f_k\|_{L^2(\mathbb{R}^n)}^2.$$

Use these to show that $\|A\|_{\mathrm{HS}}$ does not depend on the choice of the Hilbert basis and $\|A\|_{\mathrm{HS}} = \|A^*\|_{\mathrm{HS}}$.

(b) Show the inequalities

$$\begin{aligned}
\|A\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} &\leq \|A\|_{\mathrm{HS}},\\
\|AB\|_{\mathrm{HS}} &\leq \|A\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \cdot \|B\|_{\mathrm{HS}},\\
\|AB\|_{\mathrm{HS}} &\leq \|A\|_{\mathrm{HS}} \cdot \|B\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}.
\end{aligned}$$

(c) Show that the space of Hilbert–Schmidt operators is a Hilbert space with the inner product

$$\langle A, B\rangle_{\mathrm{HS}} := \sum_j \langle Ae_j, Be_j\rangle_{L^2(\mathbb{R}^n)}.$$

(d) Assume that A is an integral operator:

$$Au(x) = \int_{\mathbb{R}^n} K_A(x,y)u(y)\,dy, \quad K_A \in L^2(\mathbb{R}^{2n}).$$

Show that A is a Hilbert–Schmidt operator and

$$\|A\|_{\mathrm{HS}} = \|K_A\|_{L^2(\mathbb{R}^{2n})}.$$

You may use the fact that for any two Hilbert bases $\{e_j\}, \{f_k\}$ of $L^2(\mathbb{R}^n)$, if we define $(e_j \otimes f_k)(x,y) = e_j(x)f_k(y)$, then $\{e_j \otimes f_k\}_{j,k}$ is a Hilbert basis of $L^2(\mathbb{R}^{2n})$.

(e) Assume that $a \in L^2(\mathbb{R}^{2n})$. Show that

$$\| \operatorname{Op}_h(a)\|_{\mathrm{HS}} = (2\pi h)^{-\frac{n}{2}} \|a\|_{L^2(\mathbb{R}^{2n})}.$$

Exercise 6.6.* For a bounded operator A on $L^2(\mathbb{R}^n)$, we say it is a *trace class* operator, if it can be written as $A = BC$ where B, C are Hilbert–Schmidt operators. For a trace class operator A, define its *trace* by

$$\operatorname{tr} A = \sum_j \langle Ae_j, e_j\rangle_{L^2(\mathbb{R}^n)}$$

where $\{e_j\}$ is a Hilbert basis of $L^2(\mathbb{R}^n)$.

(a) If $A = BC$ where B, C are Hilbert–Schmidt operators, show that

$$\operatorname{tr} A = \langle C, B^*\rangle_{\mathrm{HS}}.$$

Use this to show that $\operatorname{tr} A$ is independent of the choice of the Hilbert basis.

(b) For $A = BC$ where B, C are Hilbert–Schmidt operators, show that $\operatorname{tr}(BC) = \operatorname{tr}(CB)$ and $\operatorname{tr} A = \operatorname{tr} A^*$.

(c) We use without proof the following fact (see Theorem C.18 in Zworski's book): if A is an integral operator

$$Au(x) = \int_{\mathbb{R}^n} K_A(x, y)u(y)\, dy, \quad K_A \in \mathscr{S}(\mathbb{R}^{2n}),$$

then A is trace class and

$$\operatorname{tr} A = \int_{\mathbb{R}^n} K_A(x, x)\, dx. \tag{6.1}$$

Show the formula (6.1) when $K_A(x, y) = f(x)g(y)$, $f, g \in \mathscr{S}(\mathbb{R}^n)$.

(d) Using (6.1), show that for $a \in \mathscr{S}(\mathbb{R}^{2n})$

$$\operatorname{tr} \operatorname{Op}_h(a) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} a(x, \xi)\, dxd\xi.$$