EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §4

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Recall the Composition and Adjoint Theorems: for $a, b \in \mathscr{S}(\mathbb{R}^{2n})$,

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(a\#b), \quad \mathrm{Op}_h(a)^* = \mathrm{Op}_h(a^*)$$

where we have the asymptotic expansions in $\mathscr{S}(\mathbb{R}^{2n})$, as $h \to 0$

$$a\#b(x,\xi;h) \sim \sum_{j=0}^{\infty}(-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi), \tag{4.1}$$

$$a^*(x,\xi;h) \sim \sum_{j=0}^{\infty}(-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial_x^\alpha \partial_\xi^\alpha \overline{a(x,\xi)}. \tag{4.2}$$

Exercise 4.1. (a) Check by hand that an expansion similar to (4.1) holds for $a = \xi_j$, $b = x_j$. (Of course the expansion will no longer be in $\mathscr{S}(\mathbb{R}^{2n})$; the next section will address this.) Check the Product Rule and the Commutator Rule in this case.

(b) Check by hand that an expansion similar to (4.2) holds for $a = x_j\xi_j$.

(c)* By direct computation (using the Leibniz rule) show that expansions of the form (4.1)–(4.2) hold in the case when a, b are polynomials in ξ, and thus $\mathrm{Op}_h(a), \mathrm{Op}_h(b)$ are semiclassical differential operators, see Exercise 3.2.

Exercise 4.2. Verify that the $j = 0, 1$ terms of (4.1) give the Product Rule and the Commutator Rule, and the $j = 0$ term of (4.2) gives the Adjoint Rule.

Exercise 4.3. Using the multinomial theorem, show the following identities used in the proof of the Composition Theorem and the Adjoint Theorem:

$$\frac{1}{j!}\langle \partial_y, \partial_\eta\rangle^j \big(a(x,\eta)b(y,\xi)\big)|_{\substack{y=x\\ \eta=\xi}} = \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi),$$

$$\frac{1}{j!}\langle \partial_x, \partial_\xi\rangle^j \overline{a(x,\xi)} = \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial_x^\alpha \partial_\xi^\alpha \overline{a(x,\xi)}.$$

Exercise 4.4. In lecture, we only established the expansion (4.1) for any fixed (x, ξ). Show that this expansion is valid in $\mathscr{S}(\mathbb{R}^{2n})$, in particular the remainder is controlled

Date: July 28, 2019.

uniformly in (x,ξ) and the expansion can be differentiated. (Here $a\#b$ is compactly supported and thus there is no need to get asymptotics as $(x,\xi)\to\infty$.)

Exercise 4.5.* **(a)** Assume that Q is a $2n\times 2n$ invertible symmetric real-valued matrix, $a\in C_\mathrm{c}^\infty(\mathbb{R}^{2n})$ is supported in the ball $B_{\mathbb{R}^{2n}}(0,R)$ for some $R\geq 1$, and

$$\tilde a(\rho;h):=\int_{\mathbb{R}^{2n}} e^{\frac{i}{2h}\langle Qw,w\rangle} a(\rho+w)\,dw.$$

Show that for each multiindices α,β and each N there exists a constant $C_{\alpha\beta N}$ such that for all $h\in(0,1]$

$$|\rho^\alpha\partial_\rho^\beta\tilde a(\rho;h)|\leq C_{\alpha\beta N}h^N\quad\text{for all}\quad \rho\in\mathbb{R}^{2n},\ |\rho|\geq 2R.$$

(Hint: integrate by parts using the identity $e^{\frac{i}{2h}\langle Qw,w\rangle}=hLe^{\frac{i}{2h}\langle Qw,w\rangle}$ where $L:=-\frac{i}{|w|^2}\langle Q^{-1}w,\partial_w\rangle$.)

(b) Explain how part (a) gives the last part of the proof of the Adjoint Theorem in the lecture.

Exercise 4.6.* Following the proof of the Adjoint Theorem, show the following change of quantization formula: if $a\in C_\mathrm{c}^\infty(\mathbb{R}^{2n})$, then

$$\mathrm{Op}_h^\mathrm{w}(a)=\mathrm{Op}_h(a_\mathrm{w})$$

where $a_\mathrm{w}(x,\xi;h)$ has the asymptotic expansion in $\mathscr{S}(\mathbb{R}^{2n})$

$$a_\mathrm{w}(x,\xi;h)\sim\sum_{j=0}^\infty\Big(-\frac{ih}{2}\Big)^j\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_x^\alpha\partial_\xi^\alpha a(x,\xi).$$

In particular, $a_\mathrm{w}=a+\mathcal{O}(h)_{\mathscr{S}(\mathbb{R}^{2n})}$. For a more general change of quantization statement, see Theorem 4.13 in Zworski's book.