EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §2

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Exercise 2.1. Integrating by parts, verify the following identity used in the proof of the method of nonstationary phase: if $U\subset\mathbb{R}^n$ is an open set, $\varphi\in C^\infty(U;\mathbb{R})$ satisfies $d\varphi\neq 0$ on U, and the differential operators L,L^t are defined by

$$La(y)=-i\sum_{j=1}^n\frac{\partial_{y_j}\varphi(y)}{|d\varphi(y)|^2}\partial_{y_j}a(y),\quad L^tb(y)=i\sum_{j=1}^n\partial_{y_j}\Big(\frac{\partial_{y_j}\varphi(y)}{|d\varphi(y)|^2}b(y)\Big)$$

then for all $a,b\in C^\infty(U)$ such that at least one of these functions is compactly supported, we have

$$\int_U(La(y))b(y)\,dy=\int_U a(y)(L^tb(y))\,dy.$$

Exercise 2.2. **(a)** Prove the method of nonstationary phase for the special case $\varphi(y)=y_j$, $j=1,\dots,n$, using the fact that the Fourier transform of the amplitude a is a Schwartz function.

(b) Using a partition of unity on a and the Inverse Mapping Theorem, reduce the general method of nonstationary phase to the case considered in (a).

Exercise 2.3. Following the steps below, get the estimate used in the proof of quadratic stationary phase: for each $a\in\mathscr{S}(\mathbb{R}^n)$, if Q is an invertible symmetric $n\times n$ real matrix and

$$J(h):=\int_{\mathbb{R}^n}e^{\frac{h}{2i}\langle Q^{-1}\eta,\eta\rangle}\hat a(\eta)\,d\eta,$$

then for each N there exists a constant $C_{N,Q}$, depending only on N and Q, such that

$$\Big|J(h)-\sum_{j=0}^{N-1}\frac{1}{j!}\int_{\mathbb{R}^n}\Big(\frac{h}{2i}\langle Q^{-1}\eta,\eta\rangle\Big)^j\hat a(\eta)\,d\eta\Big|\leq C_{N,Q}h^N\max_{|\alpha|\leq 2N+n+1}\|\partial^\alpha a\|_{L^1(\mathbb{R}^n)}.\quad(2.1)$$

(a) Using Taylor expansion of the function $s\mapsto e^{is}$ at $s=0$, show that there exists a constant C_N depending only on N such that

$$\Big|e^{is}-\sum_{j=0}^{N-1}\frac{(is)^j}{j!}\Big|\leq C_N|s|^N\quad\text{for all}\quad s\in\mathbb{R}.\quad(2.2)$$

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(b) Substituting $s := -\frac{h}{2}\langle Q^{-1}\eta, \eta\rangle$ into (2.2), show that the left-hand side of (2.1) is bounded above by

$$C_N \int_{\mathbb{R}^n} \Big|\frac{h}{2}\langle Q^{-1}\eta,\eta\rangle\Big|^N |\hat{a}(\eta)|\,d\eta.$$

(c) Finish the proof, using that for each multiindex α,

$$\|\eta^\alpha \hat{a}(\eta)\|_{L^\infty(\mathbb{R}^n)} = \|\widehat{\partial^\alpha a}\|_{L^\infty(\mathbb{R}^n)} \leq \|\partial^\alpha a\|_{L^1(\mathbb{R}^n)}.$$

Exercise 2.4. Finish the proof of the general method of stationary phase from the lecture, using Morse's Lemma and quadratic stationary phase.

Exercise 2.5. **(a)** Assume that $U \subset \mathbb{R}^n$ is an open set and $\varphi \in C^\infty(U;\mathbb{R})$ has a nondegenerate critical point at y_0. Show that y_0 is an isolated critical point, namely there exists a neighborhood V of y_0 such that $d\varphi \neq 0$ on $V \setminus \{y_0\}$.

(b) Using the method of stationary phase (for a single critical point as stated in lecture) and a partition of unity, derive an asymptotic expansion for

$$I(h) = \int_U e^{\frac{i\varphi(y)}{h}} a(y)\,dy$$

where $\varphi \in C^\infty(U;\mathbb{R})$ is a Morse function (possibly with several critical points) and $a \in C^\infty_{\mathrm{c}}(U)$.

Exercise 2.6.* Recall from §1 the formula for a solution to the Schrödinger equation,

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\frac{i}{h}(x\xi - t\xi^2)}\widehat{\chi}(\xi)\,d\xi, \quad \widehat{\chi} \in C^\infty_{\mathrm{c}}(\mathbb{R}). \tag{2.3}$$

Assume that $\operatorname{supp}\widehat{\chi} = [-1,2]$ and fix $t \geq 0$. Show that

$$\mathrm{WF}_h(u(t,\bullet)) = e^{tH_p}\big(\{(0,\xi) \mid \xi \in [-1,2]\}\big).$$

(One of the inclusions was proved in §1, for the other one use the leading term in the method of stationary phase.)

Exercise 2.7.* Assume that $U, V \subset \mathbb{R}^n$ are open sets and $\varkappa : V \to U$ is a diffeomorphism. Let $\varphi \in C^\infty(U;\mathbb{R})$ be a Morse function and $a \in C^\infty_{\mathrm{c}}(U)$. As in the lecture, define the oscillatory integral

$$I(h) = \int_U e^{\frac{i\varphi(y)}{h}} a(y)\,dy. \tag{2.4}$$

Using the change of variables formula, where $J_\varkappa$ is the Jacobian of $\varkappa$, we get

$$I(h) = \int_V e^{\frac{i\varphi(\varkappa(x))}{h}} a(\varkappa(x)) J_\varkappa(x)\,dx. \tag{2.5}$$

Note that (2.5) is also an oscillatory integral, with phase function $\varphi \circ \varkappa$. Verify that the leading term of the stationary phase expansion for $I(h)$ is the same regardless of whether we use the representation (2.4) or (2.5).

Exercise 2.8.* In practice (including in this course) the expressions in stationary phase often depend on parameters and we need to know that the expansion is uniform in parameters, with all derivatives. Here we study for simplicity a one-dimensional case of this statement. Let

$$I(w;h) = \int_{\mathbb{R}} e^{\frac{i}{h}\varphi_w(y)} a_w(y)\, dy, \quad w \in \mathbb{R}$$

where:

- $\varphi_w(y) = \varphi(y,w)$, where $\varphi \in C^\infty(U;\mathbb{R})$ and $U \subset \mathbb{R}^2$ is an open set containing the origin;
- φ_0 has only one critical point, at $y = 0$, and this critical point is nondegenerate;
- $a_w(y) = a(y,w)$ where $a \in C^\infty_{\mathrm{c}}(U)$.

(a) Show that for w small enough, φ_w has a unique critical point $y_0(w)$ near 0, depending smoothly on w, and $\operatorname{supp} a_w$ contains no other critical points of φ_w.

(b) For w small enough, show the stationary phase expansion

$$I(w;h) \sim e^{\frac{i\varphi_w(y_0(w))}{h}} \sum_{j=0}^{\infty} h^{\frac{1}{2}+j} L_{\varphi,j,w} a_w(y_0(w))$$

with all derivatives, namely for each N and k there exists a constant C depending only on φ, N, k, and a compact set containing $\operatorname{supp} a$, such that for all small w

$$\Big|\partial_w^k \Big(e^{-\frac{i\varphi_w(y_0(w))}{h}} I(w;h) - \sum_{j=0}^{N-1} h^{\frac{1}{2}+j} L_{\varphi,j,w} a_w(y_0(w))\Big)\Big| \leq C_{N,k} h^{\frac{1}{2}+N} \|a\|_{C^{2N+k+2}}.$$

This requires quite a bit of work, if you adapt the proof from the lecture then you need to inspect the proof of Morse's Lemma (Zworski's book, Theorem 3.15). Why did we multiply the expansion by $e^{-\frac{i\varphi_w(y_0(w))}{h}}$ here?