EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §1

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Exercise 1.1. In lecture we had the following formula:

$$u(t,x)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{\frac{i}{h}(x\xi-t\xi^2)}\widehat{\chi}(\xi)\,d\xi,\quad \widehat{\chi}\in C_{\mathrm{c}}^\infty(\mathbb{R}). \tag{1.1}$$

Check that this function does indeed solve Schrödinger's equation

$$-ih\partial_t u(t,x)=h^2\partial_x^2 u(t,x).$$

Exercise 1.2. Go carefully through the proof of the wavefront set theorem we had in lecture: if u is given by (1.1) and $\operatorname{supp}\widehat{\chi}\subset[-1,2]$ then

$$\mathrm{WF}_h(u(t,\bullet))\ \subset\ e^{tH_p}\big(\{(0,\xi)\mid \xi\in[-1,2]\}\big).$$

Exercise 1.3. Assume that $u(x)\in C_{\mathrm{c}}^\infty(\mathbb{R})$ is h-independent. Show that

$$\mathrm{WF}_h(u)\subset\{(x,0)\mid x\in\operatorname{supp}u\}.$$

Exercise 1.4.* Assume that

$$u(x;h)=e^{\frac{i\varphi(x)}{h}}a(x),\quad x\in\mathbb{R}^n \tag{1.2}$$

where $\varphi\in C^\infty(U;\mathbb{R})$, $U\subset\mathbb{R}^n$ is an open set, and $a\in C_{\mathrm{c}}^\infty(U)$. Using the method of nonstationary phase, show that

$$\mathrm{WF}_h(u)\subset\{(x,d\varphi(x))\mid x\in\operatorname{supp}a\}.$$

(Functions of the form (1.2) are a special case of *Lagrangian distributions*, or *WKB states*, which will appear again later. Note that the previous exercise was a special case of this one, with $\varphi\equiv 0$.)

Exercise 1.5.* Consider the Gaussian function

$$u(x;h):=e^{-\frac{|x|^2}{2h}},\quad x\in\mathbb{R}^n.$$

(a) Show that for any (h-independent) $\psi\in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi\cap\{0\}=\emptyset$, we have $\psi u=\mathcal{O}(h^\infty)_{\mathscr{S}(\mathbb{R}^n)}$.

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(b) Show that for any $\psi \in C_c^\infty(\mathbb{R})$ and any $\xi \in \mathbb{R}^n \setminus 0$, we have $\mathcal{F}_h(\psi u)(\xi) = \mathcal{O}(h^\infty)$. You may use the following corollary of the convolution formula for the Fourier transform and the formula for Fourier transform of Gaussians:

$$\mathcal{F}_h(\psi u)(\xi) = (2\pi h)^{-\frac{n}{2}} \widehat{\psi u}\Big(\frac{\xi}{h}\Big) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{-\frac{|\eta|^2}{2h}} \widehat{\psi}\Big(\frac{\xi - \eta}{h}\Big)\, d\eta.$$

(c) Combine parts (a) and (b) to show that

$$\mathrm{WF}_h(u) \subset \{(0,0)\}.$$

Exercise 1.6.* Prove a wavefront set statement similar to the one in the lecture for more general constant coefficient operators, with the original Hamiltonian $p(x,\xi) = \xi^2$ replaced by an arbitrary real-valued polynomial in ξ.