§9. CALCULUS ON MANIFOLDS

For simplicity, we assume that
M is a <u>compact</u> manifold of dimension n.

Want to define the class $\Psi_h^k(M)$, $k \in \mathbb{R}$, of semiclassical pseudodifferential operators of order k on M. These will be h-dependent families of operators

$$A = A(h) : C^\infty(M) \to C^\infty(M),\ \mathcal{D}'(M) \to \mathcal{D}'(M)$$

We use local charts:

<u>Definition 0</u> A cutoff chart (c.c.) on M is a pair (φ, χ) where $\varphi : U \to V$ is a diffeomorphism, $U \subset M$, $V \subset \mathbb{R}^n$ open sets, and $\chi \in C_c^\infty(U)$.

<u>Definition 1</u> We say $A \in \Psi_h^k$ if it has the form

$$A = \sum_j \chi_j \varphi_j^* \mathrm{Op}_h(a_j)(\varphi_j^{-1})^* \chi_j + O(h^\infty)_{\Psi^{-\infty}}$$

where (φ_j, χ_j) is a finite collection of cutoff charts and $a_j \in S^k(\mathbb{R}^{2n})$.

Here $O(h^\infty)_{\Psi^{-\infty}}$ denotes an operator of the form $u \mapsto \int_M K(x,y;h)u(y)\,dy$ where $\forall N$,

$$\|K\|_{C^N(M\times M)} = O(h^N).$$

Definition 2 We say $A \in \Psi_h^k$ if

(2a) A is pseudolocal, i.e. $\forall \chi_1, \chi_2 \in C^\infty(M)$, $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$,

we have $\chi_1 A \chi_2 = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$

(2b) For each cutoff chart (φ, χ) on M, $\exists\, a_{\varphi,\chi} \in S^k(\mathbb{R}^{2n})$ such that

$$(\varphi^{-1})^* \chi A \chi \varphi^* = \mathrm{Op}_h(a_{\varphi,\chi}).$$

Definition 1 ⇒ Definition 2:

use pseudolocality of $\mathrm{Op}_h(a)$, $a \in S^k(\mathbb{R}^{2n})$, and the change of variables theorem.

Definition 2 ⇒ Definition 1:

Take a finite collection of cutoff charts (φ_j, χ_j), $\varphi_j : U_j \to V_j$, such that $\sum_j \chi_j = 1$.

Fix also $\chi_j' \in C_c^\infty(U_j)$, $\chi_j' = 1$ near $\operatorname{supp} \chi_j$

We write

$$A = \sum_j \chi_j A = \sum_j \chi_j A \chi_j' + \underbrace{\sum_j \chi_j A (1 - \chi_j')}_{\mathcal{O}(h^\infty)_{\Psi^{-\infty}} \text{ by (2a)}}$$

can be written as $\chi_j' \varphi_j^* \mathrm{Op}_h(a_j)(\varphi_j^{-1})^* \chi_j'$ using (2b)

See Dyatlov–Zworski book, Proposition E.13 for details

We now want to associate to $A \in \Psi_h^k(M)$ an invariantly defined <u>principal symbol</u>. For that, recall the change of variables formula ($\varphi : U \to V$)

$$\chi \varphi^* \mathrm{Op}_h(a) (\varphi^{-1})^* \chi = \mathrm{Op}_h\big(\chi(x)^2 (a \circ \tilde{\varphi}) + O(h)\big)$$

where $\tilde{\varphi}(x, \xi) = (\varphi(x), d\varphi(x)^{-T} \cdot \xi)$.

If we think of $U, V \subset \mathbb{R}^n$ as manifolds,

then $d\varphi(x) : T_x U \to T_{\varphi(x)} V$,

$d\varphi(x)^{-T} : T_x^* U \to T^*_{\varphi(x)} V$. $\Big\}$ cotangent bundles!

So $\tilde{\varphi} : T^*U \to T^*V$ can be defined for $\varphi : U \to V$, U, V manifolds.

Now, for $A \in \Psi_h^k(M)$ define the <u>principal symbol</u> $\sigma_h(A) \in \dfrac{S^k(T^*M)}{hS^{k-1}(T^*M)}$ $\leftarrow$ quotient space

where $S^k(T^*M)$ is the Kohn-Nirenberg class on T^*M, as follows:

if (φ, χ) is a cutoff chart and

$$(\varphi^{-1})^* \chi A \chi \varphi^* = \mathrm{Op}_h(a_{\varphi,\chi}), \quad a_{\varphi,\chi} \in S^k(\mathbb{R}^{2n})$$

then $\chi(x)^2 \sigma_h(A) = a_{\varphi,\chi} \circ \tilde{\varphi} \mod hS^{k-1}(T^*M)$

See Dyatlov-Zworski, Proposition E.14

We still have nice algebraic properties:

Product Rule

If $A \in \Psi_h^k$, $B \in \Psi_h^\ell$, then $AB \in \Psi_h^{k+\ell}$
and $\sigma_h(AB) = \sigma_h(A)\sigma_h(B)$

Commutator Rule

If $A \in \Psi_h^k$, $B \in \Psi_h^\ell$, then $[A,B] \in h\Psi_h^{k+\ell-1}$
and $\sigma_h(h^{-1}[A,B]) = -i\{\sigma_h(A), \sigma_h(B)\}$
Here the Poisson bracket $\{\cdot,\cdot\}$ is invariantly defined on functions on T^*M

Adjoint Rule

Assume we fix a C^∞ density on M, which fixes an inner product on $L^2(M)$.
If $A \in \Psi_h^k$, then $A^* \in \Psi_h^k$ and $\sigma_h(A^*) = \overline{\sigma_h(A)}$

Sobolev spaces

Can define $H_h^s(M)$, $s \in \mathbb{R}$

If $A \in \Psi_h^k$, then $\|A\|_{H_h^s \to H_h^{s-k}} \le C$ ← h-independent

Sharp Gårding inequality

If $A \in \Psi_h^k$, $\operatorname{Re} \sigma_h(A) \ge 0$, then $\exists C\ \forall u \in H_h^{\frac{k}{2}}$,
$\operatorname{Re}\langle Au, u\rangle \ge -Ch\|u\|_{H_h^{\frac{k-1}{2}}}$ ← note improvement in regularity

Quantization procedure (non-canonical)

Fix a finite collection of cutoff charts

(χ_j, φ_j), $\varphi_j : U_j \to V_j$, $\sum_j \chi_j = 1$,

and take $\chi'_j \in C_c^\infty(U_j)$, $\chi'_j = 1$ near supp χ_j.

For $a \in S^k(T^*M)$, define

$$\mathrm{Op}_h^M(a) = \sum_j \chi'_j \varphi_j^* \mathrm{Op}_h\left((\chi_j a) \circ \tilde{\varphi}_j^{-1}\right) (\varphi_j^{-1})^* \chi'_j \in \Psi_h^k(M)$$

where $\tilde{\varphi}_j(x, \xi) = (\varphi_j(x), d\varphi_j(x)^{-T} \cdot \xi)$

Then $\sigma_h(\mathrm{Op}_h^M(a)) = a \mod h S^{k-1}(T^*M)$

We usually denote Op_h^M by just Op_h