

# §7. ELLIPTICITY

## §7.1. ELLIPTIC PARAMETRIX

Now that we know that

$$Op_h(a)Op_h(b) = Op_h(ab) + \dots$$

it is tempting to write

$$Op_h(p)^{-1} = Op_h(p^{-1}) + \dots \quad \text{if } p \neq 0 \text{ everywhere (semiclassical)}$$

This would let us approximately solve elliptic differential equations and this was one of the original motivations for introducing pseudodifferential operators, as approximate inverses of elliptic differential operators.

Theorem Let  $m_1, m_2$  be order functions

and  $a \in S'(m_1)$ ,  $p \in S'(m_2)$ . Assume that

$p$  is elliptic on  $\text{supp } a$  in the following sense:

$$\exists c > 0 \forall h \forall (x, \xi) \in \text{supp } a, |p(x, \xi; h)| \geq c m_2(x, \xi).$$

Then there exist symbols

$$q(x, \xi; h), q'(x, \xi; h) \in S'(m_1/m_2)$$

such that  $\text{supp } q, \text{supp } q' \subset \text{supp } a$ ; (for every  $h$ )

$$a = q \# p + O(h^\infty)_{S'(m_1)}$$

$$a = p \# q' + O(h^\infty)_{S'(m_1)}$$

} Recall  
 $Op_h(q)Op_h(p)$   
 $Op_h(q \# p)$

Remark: Combining this with  $L^2$  boundedness we see that when  $m_1=1$  we have

$$O_{p_h}(a) = O_{p_h}(q) O_{p_h}(p) + O(h^\infty)_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)},$$

$$O_{p_h}(a) = O_{p_h}(p) O_{p_h}(q') + O(h^\infty)_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}.$$

Proof We only construct  $q$ , with  $q'$  obtained similarly.

1. Put  $q_0 := \frac{a}{p}$ . This is well-defined and bounded by  $C \frac{m_1}{m_2}$  by the ellipticity condition.

In fact we have  $q_0 \in S(\frac{m_1}{m_2})$ ,  $\text{supp } q_0 \subset \text{supp } a$ .  
exercise

By the Product Rule we get

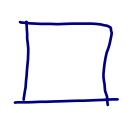
$$a = q_0 \cdot p = q_0 \# p + O(h)_{S(m_1)}.$$

2. Arguing iteratively (see exercises) we construct  $q_1, q_2, \dots \in S(\frac{m_1}{m_2})$

such that  $\text{supp } q_j \subset \text{supp } a$  and  $\forall k$ ,  
 $a = (q_0 + h q_1 + \dots + h^{k-1} q_{k-1}) \# p + O(h^k)_{S(m_1)}$ .

3. Using Borel's Theorem, we now take  $q$  as an asymptotic series:

$q \sim \sum_{j=0}^{\infty} h^j q_j$ . Then  $\forall k$ ,  $a = q \# p + O(h^k)_{S(m_1)}$   
 thus  $a = q \# p + O(h^\infty)_{S(m_1)}$  as needed.



## §7.2. ELLIPTIC ESTIMATE

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Theorem Assume that  $a \in S(1)$ ,  
 $p \in S(m)$ ,  $m \geq 1$ , and  $p$  is elliptic on  $\text{supp } a$   
(that is,  $\exists c > 0: \forall h, \forall (x, \xi) \in \text{supp } a(\cdot; h), |p(x, \xi; h)| \geq cm(x, \xi)$ )

Then  $\exists C > 0$  such that for all  $h \in (0, 1]$   
and all  $u \in L^2(\mathbb{R}^n)$ ,

$$\|O_{p_h}(a)u\|_{L^2} \leq C \|O_{p_h}(p)u\|_{L^2} + O(h^\infty) \|u\|_{L^2}.$$

Proof Using elliptic parametrix, we take

$q \in S(1/m) \subset S(1)$  such that

$$a = q \# p + O(h^\infty)_{S(1)}. \quad \text{Then}$$

$$\begin{aligned} \|O_{p_h}(a)u\|_{L^2} &= \|O_{p_h}(q)O_{p_h}(p)u\|_{L^2} + O(h^\infty) \|u\|_{L^2} \\ &\leq C \|O_{p_h}(p)u\|_{L^2} + O(h^\infty) \|u\|_{L^2}. \quad \square \end{aligned}$$

Application to Schrödinger eigenfunctions in 1D:

$$(-h^2 \partial_x^2 + V(x) - E)u = 0, \quad V \in C^\infty(\mathbb{R})$$

$$O_{p_h}(p-E) \text{ where } p(x, \xi) = \xi^2 + V(x)$$

elliptic estimate works when  $\text{supp } a \cap p^{-1}(E) = \emptyset$   
If  $a(x, \xi) = \chi(x)$ , then we need  $\text{supp } \chi \cap \Omega_E = \emptyset$   
where  $\Omega_E \subset \mathbb{R}$  is the classically allowed region:

$$\Omega_E = \text{projection onto } x \text{ of } p^{-1}(E) = \{x \in \mathbb{R} : V(x) \leq E\}$$

So  $u$  is  $O(h^\infty)$  away from  $\Omega_E$

FOR DETAILS SEE EXERCISES

MATLAB  
DEMO...