§7. ELLIPTICITY

§7.1. ELLIPTIC PARAMETRIX

Now that we know that

$$Op_h(a)\, Op_h(b) = Op_h(ab) + \cdots$$

it is tempting to write

$$Op_h(p)^{-1} = Op_h(p^{-1}) + \cdots \quad \text{if } p \neq 0 \text{ everywhere}$$

This would let us approximately solve (semiclassical) elliptic differential equations and this was one of the original motivations for introducing pseudodifferential operators, as approximate inverses of elliptic differential operators.

<u>Theorem</u> Let m_1, m_2 be order functions and $a \in S(m_1)$, $p \in S(m_2)$. Assume that p is <u>elliptic</u> on supp a in the following sense:

$\exists c>0\ \forall h\ \forall (x,\xi) \in \operatorname{supp} a,\ |p(x,\xi;h)| \geq c\, m_2(x,\xi).$

Then there exist symbols

$$q(x,\xi;h),\ q'(x,\xi;h) \in S(m_1/m_2)$$

such that supp q, supp $q' \subset$ supp a; (for every h)

$$a = q \# p + O(h^\infty)_{S(m_1)},$$

$$a = p \# q' + O(h^\infty)_{S(m_1)}.$$

Recall $Op_h(q)\, Op_h(p) = Op_h(q \# p)$

<u>Remark</u>: Combining this with L^2 boundedness we see that when $m_1 = 1$ we have

$$\mathrm{Op}_h(a) = \mathrm{Op}_h(q)\mathrm{Op}_h(p) + O(h^\infty)_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)},$$
$$\mathrm{Op}_h(a) = \mathrm{Op}_h(p)\mathrm{Op}_h(q') + O(h^\infty)_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}.$$

<u>Proof</u> We only construct q, with q' obtained similarly.

<u>1</u>. Put $q_0 := \frac{a}{p}$. This is well-defined and bounded by $C\frac{m_1}{m_2}$ by the ellipticity condition. In fact we have $\underbrace{q_0 \in S\left(\frac{m_1}{m_2}\right)}_{\text{exercise}}$, $\operatorname{supp} q_0 \subset \operatorname{supp} a$.

By the Product Rule we get
$$a = q_0 \cdot p = q_0 \# p + O(h)_{S(m_1)}.$$

<u>2</u>. Arguing iteratively (see exercises) we construct $q_1, q_2, \ldots \in S\left(\frac{m_1}{m_2}\right)$ and $\forall k$, such that $\operatorname{supp} q_j \subset \operatorname{supp} a$
$$a = (q_0 + hq_1 + \cdots + h^{k-1}q_{k-1}) \# p + O(h^k)_{S(m_1)}.$$

<u>3</u>. Using Borel's Theorem, we now take q as an asymptotic series:
$$q \sim \sum_{j=0}^\infty h^j q_j.$$
Then $\forall k$, $a = q\#p + O(h^k)_{S(m_1)}$ thus $a = q\#p + O(h^\infty)_{S(m_1)}$ as needed.

$\square$

§7.2. Elliptic estimate

Theorem Assume that $a \in S(1)$, $p \in S(m)$, $m \geq 1$, and p is elliptic on supp a (that is, $\exists c > 0$: $\forall h$, $\forall (x,\xi) \in \operatorname{supp} a(\cdot\,; h)$, $|p(x,\xi;h)| \geq c\, m(x,\xi)$).
Then $\exists C > 0$ such that for all $h \in (0,1]$ and all $u \in L^2(\mathbb{R}^n)$,

$$\|Op_h(a)u\|_{L^2} \leq C\|Op_h(p)u\|_{L^2} + O(h^\infty)\|u\|_{L^2}.$$

Proof Using elliptic parametrix, we take $q \in S(1/m) \subset S(1)$ such that

$$a = q \# p + O(h^\infty)_{S(1)}.$$

Then

$$\|Op_h(a)u\|_{L^2} = \|Op_h(q)\,Op_h(p)u\|_{L^2} + O(h^\infty)\|u\|_{L^2}$$
$$\leq C\|Op_h(p)u\|_{L^2} + O(h^\infty)\|u\|_{L^2}. \quad \square$$

Application to Schrödinger eigenfunctions in 1D:

$$(\underbrace{-h^2\partial_x^2 + V(x) - E}\,)u = 0, \quad V \in C^\infty(\mathbb{R})$$

$Op_h(p-E)$ where $p(x,\xi) = \xi^2 + V(x)$

Elliptic estimate works when $\operatorname{supp} a \cap p^{-1}(E) = \emptyset$
If $a(x,\xi) = \chi(x)$, then we need $\operatorname{supp}\chi \cap \Omega_E = \emptyset$
where $\Omega_E \subset \mathbb{R}$ is the classically allowed region:
Ω_E = projection onto x of $p^{-1}(E) = \{x \in \mathbb{R} : V(x) \leq E\}$
So $\boxed{u \text{ is } O(h^\infty) \text{ away from } \Omega_E}$

FOR DETAILS SEE EXERCISES

MATLAB DEMO ...