§5. CALCULUS FOR GENERAL SYMBOLS

We now generalize the statements in §4 to a more general class of symbols

NOTATION: $\langle x \rangle := \sqrt{1+|x|^2}$ (Japanese bracket)

Asymptotic to $1+|x|$ & smooth at $x=0$

DEFINITION $m: \mathbb{R}^{2n} \to (0,\infty)$ is called an order function, if $\exists\, C, N: \forall z, w \in \mathbb{R}^{2n}$

$$m(w) \leq C \langle z-w \rangle^N m(z).$$

DEFINITION (Symbol Classes) Let m be an order function and $a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$.

- We say $a \in S(m)$ if $\forall$ multiindex α $\exists\, C_\alpha$: $\forall (x,\xi) \in \mathbb{R}^{2n}$, $|\partial^\alpha_{(x,\xi)} a(x,\xi)| \leq C_\alpha m(x,\xi)$
- If a additionally depends on h, then we require that the constants C_α be h-independent (in order for a to be in $S(m)$).

Caution about notation: $\mathcal{S}(\mathbb{R}^n)$ Schwartz functions
$S(m)$ symbol class

Example: $m(x,\xi) = \langle \xi \rangle^k$, $a(x,\xi) = \sum_{|\beta| \leq k} a_\beta(x) \xi^\beta$

polynomial in ξ with $\partial^\gamma a_\beta$ bounded $\forall \gamma$

$$\mathrm{Op}_h(a) = \sum_{|\beta| \leq k} a_\beta(x)(hD_x)^\beta$$ semiclassical differential operator

§ 5.1. Mapping properties

Recall from §3 that for $a \in S(m)$ we may define $Op_h(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. But this is not good enough to compose operators $Op_h(a)$ with each other. We thus show

<u>Theorem</u> Let $a \in S(m)$ for some m and fix $h \in (0,1]$. Then

$$Op_h(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \; \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

is a continuous operator

<u>Proof</u> Will only show $Op_h(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. For $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ enough to show that $Op_h(a)^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, see exercises.

For notational simplicity we fix $h := 1$ & denote

$$Op_h =: Op$$

<u>1.</u> The integral formula

$$Op(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\langle x-y, \xi\rangle} a(x,\xi) u(y)\, dy\, d\xi.$$

implies that

$$a \in S(\langle \xi \rangle^{-n-1}) \Rightarrow Op(a) : \langle x \rangle^{-n-1} L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$$

In particular, we have $\forall \alpha', \beta'$,

$$Op(a) x^{\alpha'} D_x^{\beta'} : \mathcal{S}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \quad (*)$$

when $a \in S(\langle \xi \rangle^{-n-1})$

$D = \frac{1}{i}\partial$

2. Fix an order function m.
There exists an integer $N \geq 0$ such that

$$m(x,\xi) \leqslant C \langle x \rangle^{2N} \langle \xi \rangle^{2N} \langle \xi \rangle^{-n-1} \quad \forall (x,\xi)$$

Then each $a \in S'(m)$ lies in

$$\langle x \rangle^{2N} \langle \xi \rangle^{2N} S'(\langle \xi \rangle^{-n-1})$$

and thus is a linear combination of symbols of the form

$$x^\gamma \xi^\delta b, \quad b \in S'(\langle \xi \rangle^{-n-1}), \quad |\gamma|, |\delta| \leqslant 2N$$

Thus it suffices to show: $\forall \alpha, \beta, \gamma, \delta$,

$$b \in S'(\langle \xi \rangle^{-n-1}) \implies x^\alpha D_x^\beta Op(x^\gamma \xi^\delta b) : \mathcal{S}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n). \quad (**)$$

3. It remains to show that $(*) \Rightarrow (**)$.
For that we use the identities

$$x_j Op(a) = Op(a) x_j + i Op(\partial_{\xi_j} a)$$
$$D_{x_j} Op(a) = Op(a) D_{x_j} - i Op(\partial_{x_j} a)$$
$$Op(x_j a) = Op(a) x_j + i Op(\partial_{\xi_j} a)$$
$$Op(\xi_j a) = Op(a) D_{x_j}$$

For $a \in \mathcal{S}(\mathbb{R}^{2n})$ these follow from the definition of $Op_h(a)$; for general a they follow by approximation by fns in $\mathcal{S}(\mathbb{R}^{2n})$.
See Exercise 3.5(a).

Iterating the above identities,
we see that $\forall \alpha, \beta, \gamma, \delta$, $\forall b \in S(\langle\xi\rangle^{-n-1})$,

$x^\alpha D_x^\beta \operatorname{Op}(x^\gamma \xi^\delta b)$ = linear combination of

$$\operatorname{Op}(\underbrace{\partial_x^{\gamma'} \partial_\xi^{\delta'} b}_{\substack{\text{still lies in} \\ S(\langle\xi\rangle^{-n-1})}}) x^{\alpha'} D_x^{\beta'}$$

which map $\mathcal{S}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ by (*).
This gives (**). $\square$

§5.2. The calculus

We now give the analogs of the statements in §4. We do not provide proofs, referring to Zworski's book, Theorems 4.17–4.18

Theorem (Composition formula)

Assume $a \in S(m_1)$, $b \in S(m_2)$. Then

$$\operatorname{Op}_h(a)\operatorname{Op}_h(b) = \operatorname{Op}_h(a \# b) \quad \text{where}$$

$$a \# b(x,\xi;h) \sim \sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi;h)\, \partial_x^\alpha b(x,\xi;h)$$

where the expansion is in $S(m_1 \cdot m_2)$, defined similarly to expansions in $\mathcal{S}(\mathbb{R}^n)$ from §4.

Theorem (Adjoint formula)

Assume that $a \in S(m)$. Then

$$Op_h(a)^* = Op_h(a^*) \quad \text{where}$$

$$a^*(x,\xi;h) \sim \sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \overline{a(x,\xi;h)}$$

and the expansion is in $S(m)$.