§4. CALCULUS FOR COMPACTLY SUPPORTED SYMBOLS

Recall the standard quantization procedure

$$a(x,\xi) \longmapsto \mathrm{Op}_h(a) = a(x, hD_x)$$

(symbol) (pseudodifferential operator)

We will establish basic algebraic properties of Op_h when $a \in C_c^\infty(\mathbb{R}^{2n})$

smooth compactly supported

They come in the form of asymptotic expansions as $h \to 0$, defined as follows.

Definition Assume that $a_0, a_1, \ldots \in \mathcal{S}(\mathbb{R}^{2n})$ and $a(x,\xi;h) \in \mathcal{S}(\mathbb{R}^{2n})$. We write

$$\boxed{a(x,\xi;h) \sim \sum_{j=0}^{\infty} h^j a_j(x,\xi)}$$

if $\forall N$ $\forall$ multiindices α, β on $\mathbb{R}^{2n}$, $\exists C_{N,\alpha,\beta}$:

$$\forall h \in (0,1],\ \sup_w \left| w^\alpha \partial_w^\beta \left(a(x,\xi;h) - \sum_{j=0}^{N-1} h^j a_j(x,\xi) \right) \right| \le C_{N,\alpha,\beta} h^N$$

(here we denote $w := (x,\xi) \in \mathbb{R}^{2n}$)

Theorem (Borel's Theorem) Let $a_0, a_1, \ldots \in \mathcal{S}(\mathbb{R}^{2n})$. Then there exists $a(x,\xi;h)$ s.t. $a \sim \sum_{j=0}^{\infty} h^j a_j$. Any two such symbols a differ by $\mathcal{O}(h^\infty)_{\mathcal{S}(\mathbb{R}^{2n})}$

Proof See Zworski's book, Theorem 4.15

§4.1. Compositions of Op_h

Theorem Assume that $a, b \in C_c^\infty(\mathbb{R}^{2n})$. Then

$$Op_h(a)\,Op_h(b) = Op_h(a \# b)$$

where $a \# b(x,\xi;h) \in \mathcal{S}(\mathbb{R}^{2n})$ uniformly in h and satisfies the expansion in $\mathcal{S}(\mathbb{R}^{2n})$

$$a \# b(x,\xi;h) \underset{h\to 0}{\sim} \sum_{j=0}^{\infty} (-ih)^j \sum_{\alpha, |\alpha| = j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi)\, \partial_x^\alpha b(x,\xi)$$

$a \# b$ called the Moyal product of a and b

Notation: $\alpha! = \alpha_1! \cdots \alpha_n!$ for $\alpha = (\alpha_1, \dots, \alpha_n)$

EXAMPLE: (here $a, b \notin \mathcal{S}$; will be covered in §5)

$a = \xi_k$, $Op_h(a) = hD_{x_k}$, $b = x_k$, $Op_h(b) = x_k$

$$Op_h(a)\,Op_h(b) = x \cdot (hD_{x_k}) - ih = Op_h(x_k \xi_k - ih)$$

And indeed, $a \# b = x_k \xi_k - ih$ where terms in the expansion

$x_k \xi_k$ comes from $j = 0$

$-ih$ comes from $j = 1$

$j \geq 2$ terms are all equal to 0

COROLLARY 1: PRODUCT RULE

$$a \# b = a \cdot b + O(h)_{\mathcal{S}(\mathbb{R}^n)}$$

Can be also written more informally as

$$Op_h(a)\,Op_h(b) = Op_h(ab) + O(h)$$

Analyzing the $j=0$, $j=1$ terms in the expansion, we next get

COROLLARY 2: COMMUTATOR RULE

$$a\# b - b\# a = -ih\{a,b\} + O(h^2)_{S(\mathbb{R}^{2n})}$$

Can be written informally as

$$[Op_h(a), Op_h(b)] = -ih\, Op_h(\{a,b\}) + O(h^2)$$

commutator: $[A,B] := AB - BA$

Here $\{a,b\} = \sum_{j=1}^{n} \left(\partial_{\xi_j} a \cdot \partial_{x_j} b - \partial_{x_j} a \cdot \partial_{\xi_j} b\right)$ is called the Poisson bracket of a and b

Proof of the Composition Theorem

1. We find $a\# b$ using oscillatory testing: for $(x,\xi) \in \mathbb{R}^{2n}$ and $e_\xi : y \mapsto e^{\frac{i}{h}\langle y,\xi\rangle}$, put

$$a\# b(x,\xi;h) := e^{-\frac{i}{h}\langle x,\xi\rangle}\left(Op_h(a)\, Op_h(b)\, e_\xi\right)(x)$$

Since $Op_h(a)\, Op_h(b) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, once we show $a\# b \in \mathcal{S}(\mathbb{R}^{2n})$ (which follows from the proof of the asymptotic expansion below) we do indeed have

$$Op_h(a)\, Op_h(b) = Op_h(a\# b)$$

2. It remains to show the expansion for $a\#b$. We have by oscillatory testing,

$$(\mathrm{Op}_h(b)e_\xi)(y) = b(y,\xi)e^{\frac{i}{h}\langle y,\xi\rangle} \quad \forall y,\xi$$

Thus by the definition of $\mathrm{Op}_h(a)$ we get

$$a\#b(x,\xi;h) = e^{-\frac{i}{h}\langle x,\xi\rangle}(\mathrm{Op}_h(a)\mathrm{Op}_h(b)e_\xi)(x)$$

$$= e^{-\frac{i}{h}\langle x,\xi\rangle}(2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\eta\rangle}a(x,\eta)b(y,\xi)e^{\frac{i}{h}\langle y,\xi\rangle}\,dy\,d\eta$$

$$= (2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\eta-\xi\rangle}a(x,\eta)b(y,\xi)\,dy\,d\eta$$

$\boxed{\begin{matrix} y=x+z \\ \eta=\xi+\zeta \end{matrix}}$

$$= (2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{h}\langle z,\zeta\rangle}a(x,\xi+\zeta)b(x+z,\xi)\,dz\,d\zeta$$

Since $a,b\in C_c^\infty(\mathbb{R}^{2n})$, we see from the above formula that $a\#b\in C_c^\infty(\mathbb{R}^{2n})$ as well. Moreover, the function under the $\int$ is compactly supported in z,ζ.

3. We now fix (x,ξ) and show the asymptotic expansion as $h\to 0$. One can differentiate under the integral sign to get an expansion in $S_{x,\xi}(\mathbb{R}^{2n})$ (exercise)

We apply <u>quadratic stationary phase</u> (see §2)

We write $a\#b(x,\xi;h) = (2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{2h}\langle Qw,w\rangle}A(w)\,dw$,

where $w=(z,\zeta)$, $Q=\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix}$,

$$A(z,\zeta) = a(x,\xi+\zeta)b(x+z,\xi)$$

We set

$$a\# b(x,\xi;h) \underset{h\to 0}{\sim} \frac{e^{\frac{i\pi}{4}\operatorname{sgn} Q}}{|\det Q|^{1/2}} \sum_{j=0}^{\infty} \frac{(ih)^j}{j!}\left(\frac{\langle Q^{-1}\partial_w,\partial_w\rangle}{2}\right)^j A(0)$$

$= 1$ as $\operatorname{sgn} Q = 0$, $|\det Q| = 1$

We have $Q^{-1} = Q = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$, $\dfrac{\langle Q^{-1}\partial_w,\partial_w\rangle}{2} = -\langle \partial_z,\partial_\zeta\rangle$

Thus the j-th term is

$$\frac{(-ih)^j}{j!}\left(\langle \partial_z,\partial_\zeta\rangle^j\big(a(x,\xi+\zeta)\,b(x+z,\xi)\big)\right)\Big|_{\substack{z=0\\ \zeta=0}}$$

By the multinomial theorem, this is equal to

$$(-ih)^j \sum_{|\alpha|=j}{}' \partial_\xi^\alpha a(x,\xi)\,\partial_x^\alpha b(x,\xi),$$

giving the required expansion. $\square$

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§4.2. Adjoints of Op_h

General fact: if $A: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is given by $Au(x) = \int_{\mathbb{R}^n} K_A(x,y)\,u(y)\,dy$ where $K_A \in \mathcal{S}(\mathbb{R}^{2n})$ then the adjoint A^* is given by the same ~~formula~~ with

$$K_{A^*}(x,y) = \overline{K_A(y,x)}.$$

By adjoint we mean that $\forall u,v \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle Au, v\rangle_{L^2} = \langle u, A^* v\rangle_{L^2}, \quad \langle u,v\rangle_{L^2} := \int_{\mathbb{R}^n} u\cdot\bar{v}\,dx$$

Theorem

Assume that $a \in C_c^\infty(\mathbb{R}^{2n})$. Then

$$Op_h(a)^* = Op_h(a^*)$$

where $a^*(x,\xi;h) \in \mathcal{S}(\mathbb{R}^{2n})$ uniformly in h satisfies the asymptotic expansion in $\mathcal{S}(\mathbb{R}^{2n})$

$$a^*(x,\xi;h) \underset{h\to 0}{\sim} \sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \overline{a(x,\xi)}$$

COROLLARY: ADJOINT RULE

$$a^* = \bar{a} + O(h)_{\mathcal{S}(\mathbb{R}^{2n})}$$

Can be written more informally as

$$Op_h(a)^* = Op_h(\bar{a}) + O(h).$$

Proof of the theorem 1. Recalling that

$$Op_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\eta\rangle} a(x,\eta)u(y)\,dy\,d\eta$$

we have

$$Op_h(a)^* u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\eta\rangle} \overline{a(y,\eta)}\, u(y)\,dy\,d\eta$$

We now write using oscillatory testing $\left(e_\xi(y) = e^{\frac{i}{h}\langle y,\xi\rangle}\right)$

$$Op_h(a)^* = Op_h(a^*) \quad \text{where}$$

$$a^*(x,\xi;h) = e^{-\frac{i}{h}\langle x,\xi\rangle} \left(Op_h(a)^* e_\xi\right)(x)$$

$$= (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\eta-\xi\rangle}\, \overline{a(y,\eta)}\,dy\,d\eta$$

2. We now apply quadratic stationary phase to get the expansion for any fixed x,ξ:

$$a^*(x,\xi;h) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{2h}\langle Qw,w\rangle} A(w)\,dw$$

where $Q = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$, $w = (z,\zeta)$,

$$A(z,\zeta) = \overline{a(x+z,\xi+\zeta)}$$

Similarly to the composition formula we get the expansion

$$a^*(x,\xi;h) \underset{h\to 0}{\sim} \sum_{j=0}^{\infty} \frac{(-ih)^j}{j!} \left(\langle \partial_z, \partial_\zeta\rangle^j \overline{a(x+z,\xi+\zeta)}\right)\Big|_{\substack{z=0\\ \zeta=0}}$$

By the multinomial theorem, this is same as

$$\sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \overline{a(x,\xi)}.$$

3. It remains to deal with asymptotics as $(x,\xi)\to\infty$. Unlike Composition Theorem, a^* is not compactly supported in (x,ξ).

Choose $R \geq 1$ such that $\operatorname{supp} a \subset B_{\mathbb{R}^{2n}}(0,R)$.

Then it suffices to show that $\forall \alpha,\beta$

$$\rho^\alpha \partial_\rho^\beta a^*(\rho;h) = O(h^\infty) \quad \text{where } \rho=(x,\xi)$$

for $|\rho| \geq 2R$.

For the proof of the latter statement, see exercises. $\square$