§3. SEMICLASSICAL QUANTIZATION

Goal of the next few lectures:
for a function $a(x,\xi)$, $(x,\xi)\in\mathbb{R}^{2n}$,
define the h-dependent family of operators
$$Op_h(a)=a(x,hD_x),\quad D_x=\frac{1}{i}\partial_x$$
& establish properties of Op_h

STANDARD QUANTIZATION:

$$(1)\quad Op_h(a)u(x)=(2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle}a(x,\xi)u(y)\,dy\,d\xi$$

If $a\in\mathcal{S}(\mathbb{R}^{2n})$ and $u\in\mathcal{S}(\mathbb{R}^n)$ then the integral in (1) converges and defines
$$Op_h(a)u\in\mathcal{S}(\mathbb{R}^n).$$
It is useful to rewrite (1) as follows:
$$(2)\quad Op_h(a)u(x)=(2\pi h)^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle}a(x,\xi)\,\hat{u}\left(\frac{\xi}{h}\right)d\xi$$
where $\hat{u}$ is the Fourier transform of u

Using (2) we get better mapping properties:

- $a\in\mathcal{S}(\mathbb{R}^{2n})\Rightarrow Op_h(a):\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$
- $a\in C^\infty(\mathbb{R}^{2n})$, $|a(x,\xi)|\le C\langle x\rangle^N\langle\xi\rangle^N$ for some C,N
  $\Downarrow$
  $Op_h(a)$ can be defined $\mathcal{S}(\mathbb{R}^n)\to\langle x\rangle^N L^\infty(\mathbb{R}^n)$

$\boxed{\text{NOTATION: } \langle x\rangle:=\sqrt{1+|x|^2}}$

EXAMPLES:

- $a = 1 \Rightarrow Op_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} \hat{u}(\xi/h)\, d\xi$

$= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{u}(\xi)\, d\xi = u(x)$ by Fourier inversion

Thus $\boxed{Op_h(1) = I}$ ← identity operator

- $a = a(\xi) \Rightarrow Op_h(a)$ is a Fourier multiplier:

$\widehat{Op_h(a)u}(\xi) = a(h\xi)\hat{u}(\xi)$

$\mathcal{F}_h(Op_h(a)u)(\xi) = a(\xi)\mathcal{F}_h u(\xi)$ semiclassical Fourier tr.

We denote $Op_h(a) = a(hD_x)$

In particular $\boxed{Op_h(\xi_j) = hD_{x_j}}$

- $a = a(x) \Rightarrow Op_h(a)u(x) = a(x)u(x)$

In particular $\boxed{Op_h(x_j) = x_j}$

- $a = \sum_{|\alpha|\le k} a_\alpha(x)\xi^\alpha \Rightarrow Op_h(a) = \sum_{|\alpha|\le k} a_\alpha(x)(hD_x)^\alpha$

these are called semiclassical differential operators of order k

In particular $\boxed{Op_h(|\xi|^2) = -h^2\Delta}$

The above justify the notation

$Op_h(a) = a(x, hD_x)$

though this notation is only formal: x, hD_x do not commute, so we cannot define a function of both...

To recover the symbol a of an operator $Op_h(a)$ we use the following

Theorem (Oscillatory testing) Put

$$e_\xi(x) = e^{\frac{i}{h}\langle x,\xi\rangle} \in S'(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n.$$

Assume that $a \in S(\mathbb{R}^{2n})$ and $A: S'(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous. Then

$$A = Op_h(a) \iff \forall x, \xi \in \mathbb{R}^n, \ (Ae_\xi)(x) = a(x,\xi)e_\xi(x)$$

Remark This lets us recover a by testing $Op_h(a)$ on oscillatory functions e_ξ.

Proof $\Rightarrow$ We have $\forall u \in S'(\mathbb{R}^n)$

$$Au(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\eta\rangle} a(x,\eta)\,\hat{u}\left(\tfrac{\eta}{h}\right) d\eta$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\eta\rangle} a(x,h\eta)\,\hat{u}(\eta)\, d\eta$$

If $u = e_\xi$, then $\hat{u}(\eta) = (2\pi)^n \delta\left(\eta - \frac{\xi}{h}\right)$

delta distribution

So $Au(x) = \int_{\mathbb{R}^n} e^{i\langle x,\eta\rangle} a(x,h\eta)\,\delta\left(\eta - \frac{\xi}{h}\right) d\eta = e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi)$

$\Leftarrow$ Put $B = A - Op_h(a)$, then $B: S'(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous and

$$Be_\xi = 0 \quad \forall \xi \in \mathbb{R}^n.$$

This implies that $B = 0$ (see exercises) $\square$

WEYL QUANTIZATION:

$$\mathrm{Op}_h^w(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle} \underbrace{a\left(\tfrac{x+y}{2},\xi\right)} u(y)\,dy\,d\xi$$

in standard quantization we had $a(x,\xi)$

This is the quantization used in Zworski's book. It has the same mapping properties as Op_h (see exercises) but does not have the Fourier representation (2) or oscillatory testing.

It is not always equal to Op_h:

$$\mathrm{Op}_h^w(1) = \mathrm{Op}_h(1) = I$$

$$\mathrm{Op}_h^w(x_j) = \mathrm{Op}_h(x_j) = x_j$$

$$\mathrm{Op}_h^w(\xi_j) = \mathrm{Op}_h(\xi_j) = hD_{x_j}$$

but

$$\mathrm{Op}_h^w(x_j\xi_j) = x_j(hD_{x_j}) - \frac{ih}{2} = \mathrm{Op}_h\left(x_j\xi_j - \frac{ih}{2}\right)$$

However, the difference between the two quantizations is $O(h)$ and they give the same class of operators.

So for basic applications it does not matter which one to use.