§ 10. WAVEFRONT SETS

Wavefront set: a microlocal version of the notion of (essential) support

We will consider L^2 functions on $\mathbb{R}^n$. For the case of manifolds, see exercises.

<u>Theorem-Definition</u> Assume that $u = u_h \in L^2(\mathbb{R}^n), h \in (0,1]$ is <u>h-tempered</u> in the following sense:

$$\exists C, N \quad \forall h \quad \|u_h\|_{L^2} \leq C h^{-N}.$$

Let $(x_0, \xi_0) \in \mathbb{R}^{2n}$. <u>The following are equivalent</u>:

① $\exists \chi \in C_c^\infty(\mathbb{R}^n)$, $V \subset \mathbb{R}^n$ neighborhood of ξ_0:

$$\sup_{\xi \in V} |\mathcal{F}_h(\chi u_h)(\xi)| = O(h^\infty)$$

$\boxed{\mathcal{F}_h u(\xi) = (2\pi h)^{-\frac{n}{2}} \hat{u}\left(\frac{\xi}{h}\right) \text{ Semiclassical Fourier transform}}$

② $\exists a \in \mathcal{S}(\mathbb{R}^{2n})$ and c such that $\forall h$, $|a(x_0, \xi_0; h)| \geq c > 0$ and

$$\|\mathrm{Op}_h(a) u_h\|_{L^2(\mathbb{R}^n)} = O(h^\infty)$$

③ $\exists U \subset \mathbb{R}^{2n}$ neighborhood of (x_0, ξ_0) such that $\forall b \in C_c^\infty(\mathbb{R}^{2n})$

$$\operatorname{supp} b \subset U \Rightarrow \|\mathrm{Op}_h(b) u\|_{L^2} = O(h^\infty)$$

(!) In the above χ, V, c, U need to be h-independent; a, b can depend on h but should be bounded in $\mathcal{S}(\mathbb{R}^{2n})$ uniformly in h

If ①-③ hold, we say $(x_0, \xi_0) \notin WF_h(u)$.

This defines the closed set $WF_h(u) \subset \mathbb{R}^{2n}$

<u>Note</u>: ① is the definition we had before

②-③ adapt to the setting of manifolds: $u = u_h \in L^2(M) \Rightarrow$

$\Rightarrow WF_h(u) \subset T^*M$

Proof (1) $\Rightarrow$ (2): Fix $\psi \in C_c^\infty(V)$ h-independent, $\psi(\xi_0) \neq 0$, and put

$$a := \psi(\xi) \# \chi(x), \text{ so that } Op_h(a) = \overbrace{\psi(hD_x)}^{\text{Fourier multiplier}} \chi(x)$$

We have $|a(x_0, \xi_0)| = |\psi(\xi_0)\chi(x_0)| + O(h) \geq c > 0$ for small enough h. Next,

$$\|Op_h(a)u\|_{L^2} = \|\mathcal{F}_h(Op_h(a)u)\|_{L^2} = \|\psi \cdot \mathcal{F}_h(\chi u)\|_{L^2} = O(h^\infty)$$

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(2) $\Rightarrow$ (3): Since $\|a\|_{C^1}$ is bounded uniformly in h, there exists a neighborhood U of (x_0, ξ_0) such that

$$|a(x, \xi; h)| \geq \frac{c}{2} > 0 \text{ for all } (x, \xi) \in U.$$

Now take $b \in C_c^\infty(\mathbb{R}^{2n})$, $\operatorname{supp} b \subset U$.

Then a is elliptic on $\operatorname{supp} b$.

Thus by the elliptic estimate,

$$\|Op_h(b)u\|_{L^2} \leq \underbrace{C\|Op_h(a)u\|_{L^2}}_{O(h^\infty) \text{ by } (2)} + \underbrace{O(h^\infty)\|u\|_{L^2}}_{O(h^\infty) \text{ as } u \text{ is } h\text{-tempered}}$$

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(3) $\Rightarrow$ (1): Fix $\chi, \psi \in C_c^\infty(\mathbb{R}^n)$, $\chi(x_0) \neq 0$, $\psi(x_0) \neq 0$ s.t. $\operatorname{supp}\chi \times \operatorname{supp}\psi \subset U$. Fix V nbhd of ξ_0: $|\psi| \geq c > 0$ on V.

For each α, $\|\partial_\xi^\alpha \mathcal{F}_h(\chi u)\|_{L^2(V)} \leq C\|\psi \cdot \partial_\xi^\alpha \mathcal{F}_h(\chi u)\|_{L^2(\mathbb{R}^n)}$
$= h^{-|\alpha|}\|\psi(hD_x)x^\alpha \chi u\|_{L^2} = O(h^\infty)$ since $\psi(hD_x)x^\alpha\chi = Op_h(b)$ where $b = \psi(\xi) \# x^\alpha \chi(x)$, $\operatorname{supp} b \subset U$.

By Sobolev embedding, $\|\mathcal{F}_h(\chi u)\|_{L^\infty(V)} = O(h^\infty)$. $\square$

Fundamental example

$$u(x;h) = \int_{\mathbb{R}^k} e^{\frac{i}{h}\Phi(x,\theta)} a(x,\theta;h)\, d\theta, \quad x \in \mathbb{R}^n$$

where:
- $\Phi \in C^\infty(U;\mathbb{R})$, $U \subset \mathbb{R}^n_x \times \mathbb{R}^k_\theta$ open
  ! No relation between n and k !
- $a \in C_c^\infty(U)$, supported in an h-independent compact set, and $\forall \alpha$, $\sup|\partial^\alpha a| = O(1)$ as $h \to 0$

To find $WF_h(u)$, take $b \in C_c^\infty(\mathbb{R}^{2n})$ & compute

$$Op_h(b)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n+k}} e^{\frac{i}{h}(\langle x-y,\xi\rangle + \Phi(y,\theta))} b(x,\xi) a(y,\theta;h)\, dy\, d\xi\, d\theta$$

Phase $= \langle x-y,\xi\rangle + \Phi(y,\theta)$

Critical points:
$$\partial_\xi = 0 \iff x = y$$
$$\partial_y = 0 \iff \xi = \partial_x \Phi(x,\theta)$$
$$\partial_\theta = 0 \iff \partial_\theta \Phi(x,\theta) = 0$$

By the method of nonstationary phase,

$Op_h(b)u = O(h^\infty)$ if $\operatorname{supp} b \cap \Lambda_\Phi = \emptyset$

where $\Lambda_\Phi = \{(x, \partial_x \Phi(x,\theta)) \mid \partial_\theta \Phi(x,\theta) = 0\}$

So $\boxed{WF_h(u) \subset \Lambda_\Phi}$

Under certain nondegeneracy assumptions on Φ, $\Lambda_\Phi \subset \mathbb{R}^{2n}$ is a Lagrangian manifold (see exercises)