Analysis of gravitational instantons

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From Microlocal to Global Analysis @ MIT
an occasion to celebrate Richard Melrose's 75th birthday

Joint work with Rafe Mazzeo, Yu-Shen Lin, Sid Soundararajan

Gravitational instantons

- Calabi–Yau manifolds: Kähler, Ricci flat, holonomy in $SU(n)$
- In real dimension 4, every Calabi–Yau manifold is hyperKähler:
  - Three complex structures satisfying quaternion relations
  - Holonomy in $Sp(1)$
- Non-compact analogue: gravitational instantons
- Definition: non-compact complete hyperkähler 4-manifold with curvature decaying fast enough at infinity
- They arise as moduli spaces of gauge equations and building blocks for gluing constructions
- The infinity structure gives different gravitational instantons

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Volume growth at infinity

- Well-known types of infinity structure of gravitational instantons: ALE, ALF, ALG, ALH
- The order of volume growth a infinity: 4,3,2,1
- Classification: under curvature condition of $|Rm| = \mathcal{O}(r^{-2-\epsilon})$, any non-flat gravitational instanton is one of the above [Minerbe, '09-'10] [Chen–Chen, '15]
- The curvature decay condition above cannot be weakened for the above classification
- Up to diffeomorphism, two additional types: ALH^* and ALG^* [Sun–Zhang, '21]
- ALH^*: volume growth between ALG and ALH
- Tian–Yau metrics: $|Rm| = \mathcal{O}(r^{-2})$, volume growth $\sim r^{4/3}$
- Different types of gravitational instantons are related [Auvray, '13, '18], [Lin–Soundararajan–Z, in progress]

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Gibbons–Hawking ansatz

- Gibbons–Hawking ansatz gives 4-dimensional hyperkähler structures with an $\mathbb{S}^1$ symmetry
- Setup: U a flat 3-manifold with $h_U = e_1^2 + e_2^2 + dz^2$ and a circle bundle $\pi : M^4 \to U^3$
- Take V as a harmonic function on U such that $[\frac{1}{2\pi} * dV] \in H^2(U, \mathbb{Z})$
- The Gibbons–Hawking ansatz is given by

$$g = V\pi^* h_U + V^{-1}\Theta \otimes \Theta$$

- $U = \mathbb{R}^3$, $V = 1/2r \Rightarrow M = \mathbb{R}^4$ and π is the Hopf fibration (ALE)
- $U = \mathbb{R}^3$, $V = \sigma + 1/2r \Rightarrow M$ is a Taub-NUT space (ALF)
- $U = \mathbb{T}^2 \times \mathbb{R}$, $V = 1 \Rightarrow M$ is the flat product $\mathbb{T}^3 \times \mathbb{R}$ (ALH)
- Infinity structure of ALH*: $U = \mathbb{T}^2 \times \mathbb{R}_z$, $V = 2\pi bz/A$

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Relation to K3

- Gravitational instantons arises as bubbles in the degeneration of K3 surfaces
- Literature: [Gross–Wilson, '00] [Foscolo, '16] [Chen–Chen, '18] [Hein–Sun–Viaclovsky–Zhang, '19] [Chen–Viaclovsky–Zhang, '21] [Ozuch, '22] [Esfahani–Li, '24]
- A new way to generate information for K3 using gravitational instantons

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ALH*: Doubly warped at the infinity

- $U = \mathbb{T}^2_{y_1 y_2} \times \mathbb{R}_z$, $V = z$, and

$$g = z\pi^* h_U + z^{-1}\Theta \otimes \Theta = z(dy_1^2 + dy_2^2 + dz^2) + \frac{1}{z}(d\theta + y_1 dy_2)^2$$

- Change $z = 1/x$

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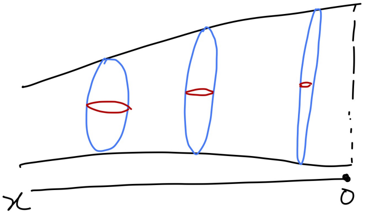

Figure: A schematic picture of the Gibbons-Hawking ansatz towards $x = 0$

The Calabi ansatz

- Calabi ansatz gives the (asymptotic) complex structure of Tian–Yau metrics
- $D = \mathbb{T}^2$ and $L \to D$ a line bundle with a canonical hermitian metric h
- View $M \to \mathbb{T}^2$ as a subset of L with $0 < |\xi|_h < 1$
- The Calabi ansatz:

$$\omega_C = \frac{2}{3} i\partial\bar{\partial}(-\log|\xi|_h^2)^{3/2}$$

- General construction applies to higher dimensions [Sun–Zhang, '20]
- Diffeomorphic to a Gibbons–Hawking ansatz in complex dimension 2 with

$$(-\log|\xi|_h^2)^{1/2} = \frac{1}{x}.$$

Tian–Yau metrics

Theorem (Tian–Yau, '90)

There is a complete Ricci-flat Kähler metric on the complement of a smooth anti-canonical divisor D in a smooth Fano manifold M.

- Relation to Calabi ansatz:
  - $M \to D$ is a subset of an ample line bundle over an anti-canonical divisor
- Solving a Monge-Ampere equation to get $\omega_{TY} = \omega_C + i\partial\bar{\partial}\phi$ where ϕ decays exponentially
- More precise analysis: [Hein, '12]

Collapsing of K3 surfaces

One motivation to study this type of metrics:

- Used as a building block for codimension-3 collapsing K3 surfaces [Hein–Sun–Viaclovsky–Zhang, '19]
- Rescale the collapsing to get the "bubbles": Tian–Yau metrics and multi-Taub–NUT metrics

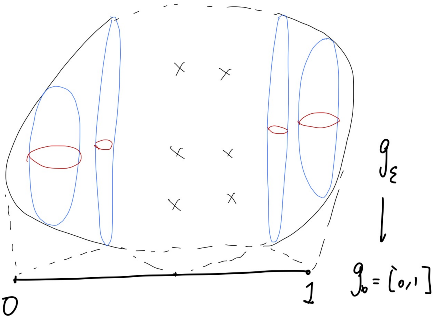

Figure: A family of K3 surfaces degenerating to a line

More appearance of gravitational instantons

More appearance of ALH^*:

- One component in the SYZ symmetry [Collins–Jacob–Lin, '19, '20]
- Mirror symmetry pair between Tian–Yau metrics and (generalized) semi-flat metrics
- Algebraic side of the story: Torelli theorem and weak del Pezzo surfaces [Collins–Jacob–Lin, '21] [Hein–Sun–Viaclovsky–Zhang, '21] [Lee–Lin, '21]
- Physics interpretation: monopole moduli spaces with certain boundary conditions [Cherkis, '14]

The Laplace operator

- In the gluing construction, one of the keys is the study of the Laplace operator
- Mapping properties and Liouville theorem [Hein–Sun–Viaclovsky–Zhang, '19] [Sun–Zhang, '21]
- Also needed for Hodge theory and deformation theory
- The singularity structure can be studied systematically

Main result

Theorem (Mazzeo-Z)

Given a Tian–Yau (or more general, ALH*) metric g,

- *The Laplace operator Δ_g is Fredholm mapping between suitable Sobolev spaces;*
- *The L^2 harmonic forms are identified with suitable weighted cohomology and intersection cohomology;*
- *There is a full asymptotic expansion of the metric near infinity;*
- *The local perturbation of the metric is identified with boundary and interior data.*

The fibration structure

- Recall $U = \mathbb{T}^2_{y_1 y_2} \times \mathbb{R}_z$, $V = z$, and

$$g = z\pi^* h_U + z^{-1}\Theta \otimes \Theta = z(dy_1^2 + dy_2^2 + dz^2) + \frac{1}{z}(d\theta + y_1 dy_2)^2$$

- Change $z = 1/x$

$$g = \frac{dx^2}{x^5} + \frac{dy_1^2 + dy_2^2}{x} + x(d\theta^2 + y_1 dy_2^2)$$

- For each fixed x, the slice is a Heisenberg nilmanifold

$$\mathbb{S}^1 \to Nil \to \mathbb{T}^2$$

- The metric is spanned by

$$x^{-5/2}dx, x^{-1/2}dy_1, x^{-1/2}dy_2, x^{1/2}(d\theta + y_1 dy_2).$$

- Corresponding unit tangent vectors

$$x^{5/2}\partial_x, x^{1/2}\partial_{y_1}, x^{1/2}(\partial_{y_2} - y_1\partial_\theta), x^{-1/2}\partial_\theta.$$

The Laplace operator

- The Laplace operator is

$$\Delta \;=\; x^5\partial_x^2 + x(\partial_{y_1}^2 + \partial_{y_2}^2) + x^{-1}\partial_\theta^2 + \left(-2xy_1\partial_{y_2}\partial_\theta + xy_1^2\partial_\theta^2\right)$$

That is

$$x\Delta = \left(x^6\partial_x^2 + x^2(\partial_{y_1}^2 + \partial_{y_2}^2) + \partial_\theta^2\right) + \left(-2x^2y_1\partial_{y_2}\partial_\theta + x^2y_1^2\partial_\theta^2\right)$$

Three layers of decomposition

$$x\Delta = \left(x^6\partial_x^2 + x^2(\partial_{y_1}^2 + \partial_{y_2}^2) + \partial_\theta^2\right) + \left(-2x^2 y_1\partial_{y_2}\partial_\theta + x^2y_1^2\partial_\theta^2\right)$$

- Vector fields: $x^3\partial_x, x\partial_{y_1}, x\partial_{y_2}, \partial_\theta$
- Analogous to fibred boundary, fibred cusp structures [Mazzeo-Melrose, '98] [Vaillant, '01] [Hausel-Hunsicker-Mazzeo, '04]
- An example of **a**−structure [Grieser–Hunsicker, '09, '13]
- Decompose the function space using Π_θ ($\mathbb{S}^1$ harmonic) and Π_y ($\mathbb{T}^2$ harmonic)
- Three parts:
  a fully elliptic **a**-operator + $x^2 *$ a cusp operator + $x^4 *$ a b-operator

Generalized inverse

- Find a generalized inverse/parametrix G for Δ_g so that $G\Delta - Id$ and $\Delta G - Id$ are compact
- The Schwartz kernel of G and Δ_g live on M^2
- Problem: singularity at infinity means G and Δ_g does not behave well on M^2
- The double space where the parametrix is defined is obtained from blowing up M^2
- Resolution by separate singularities at different levels

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Fredholm theory

Theorem (Mazzeo-Z, in progress)

There exists a parametrix G such that

$$G\Delta - Id, \ \Delta G - Id$$

are both compact. And Δ is Fredholm between suitable weighted Sobolev spaces.

- Construct three parametrices separately then piece together
- Each parametrix is obtained by iteratively inverting the projected operator

L^2 cohomology

- L^2 cohomology: using L^2 forms instead of smooth forms

$$\cdots \to L^2\Omega^p(M,g) \to L^2\Omega^{p+1}(M,g) \to L^2\Omega^{p+2}(M,g) \to \cdots$$

- L^2 cohomology defined accordingly
- Goal: identify L^2 harmonic forms
- Why: use L^2 cohomology to detect the behavior at infinity
- Examples:

$$[(0,1), \frac{dx^2}{x^2(1-x)^2}]$$

- $H^0_{(2)} = 0$
- $H^1_{(2)} =$ infinite dimension

$$[(0,1), dx^2]$$

- $H^0_{(2)} = \mathbb{R}$
- $H^1_{(2)} = 0$

L^2 harmonic forms and stratified spaces

- There has been many works on this topic [Albin, Bei, Carron, Dai, Gell-Redman, Hunsicker, Kottke, Mazzeo, Melrose, Piazza, Rochon, Sher, Vertman...] [Hausel–Hunsicker– Mazzeo, '04]
- Geometric/topological information of stratified spaces
- Intersection cohomology: chains with constraints on how they intersect the boundary strata ("perversity")
  [Goresky–MacPherson, '80, '83]
  [Cheeger–Goresky–MacPherson, '82]

Identification of harmonic forms

Theorem

For ALH spaces, there is an isomorphism between the L^2 harmonic forms and the intersection cohomology.*

$$L^2\mathcal{H}^k(M) \to \begin{cases} H^k(X,B) & k=0,1 \\ IM(H^2(X,B) \to IH_0^2(X,B)) & k=2 \\ IH_0^3(X,B) & k=3 \\ IM(IH_0^4(X,B) \to H^4(X-B)) & k=4 \end{cases}$$

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- Identify L^2 harmonic forms with the weighted cohomology (analysis) , then with the intersection cohomology (geometry)
- Dirac operator $D = d + \delta$ for the metric is Fredholm on suitable spaces
- Key: control of boundary terms of the harmonic forms
- The nontrivial fibration adds off-diagonal terms, need to use twisted differential and explicit computations [Bismut–Cheeger, '89]
- "Perversity" is related to the behavior of the metric at infinity
- Interpretation of the Hodge theory similar to ALE and ALF case?

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Full asymptotic expansion

- Analogue:
  [Conlon–Rochon–Mazzeo, '15] on asymptotic cylindrical Calabi-Yau manifolds
  [Rochon–Zhang, '11] on Kähler-Einstein metrics on quasiprojective manifolds
- Strategy: solving the complex Monge–Ampere equation

$$\frac{(\omega_C + i\partial\bar{\partial}u)^n}{\omega_C^n} = e^f$$

  with a small right hand side, where ω_C is the Calabi ansatz
- We show that u is polyhomogeneous by iteratively approximation
- At each step we use the invertibility of the Laplace operator

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Perturbation of hyperKähler metrics

- Torelli theorem and period domain for ALH* [Collins–Jacob–Lin, '21] [Lee–Lin, '21]
- Would like to understand from the local perturbation point of view
- Setup: deformation of "hyperKähler triples" [Donaldson, '06] [Schroers–Singer, '21] [Sun–Zhang, '21]
- A system of equations on three 2-forms are reduced to 4 nonlinear scalar equations
- Linearization: 12 copies of scalar Laplace operator
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Moduli space of gravitational instantons

- The moduli space of ALH_b^* is parametrized by end structures and interior data
- The space is noncompact
- Several possible ways of degenerations - similar to how K3 surfaces can degenerate [Lin–Takahashi, '23]
- A gravitational instanton can degenerate into another type
- "Modularity conjecture": moduli spaces of different gravitational instantons fit together
- One example: ALF to ALF [Auvray, '13,'18]

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Degenerating sequence of ALH*

- We constructed the following sequence of degenerations

$$ALH \to ALH_1^* \to ALH_2^* \to \cdots \to ALH_9^*$$

- Can be viewed as a bubble tree structure
- Intuition: the number $9 - b$ corresponds to number of monopoles on the manifold; when $b = 0$ this is ALH
- Degenerations happen when monopoles travel to infinity
- When all the monopoles disappear into infinity, we get an ALH_9^* manifold
- It corresponds to a tree structure of del Pezzo surface degenerations [Cherkis, '14]
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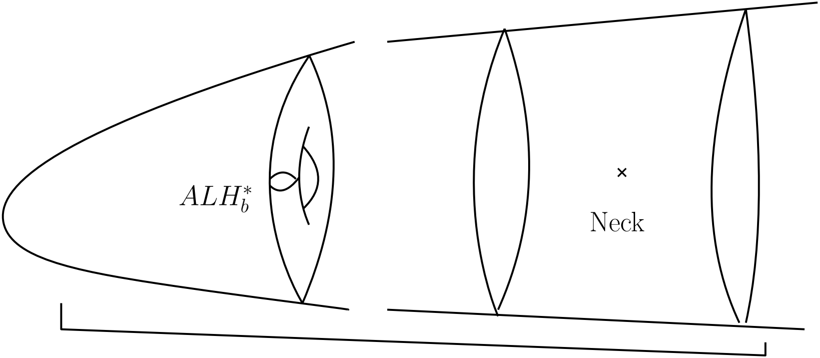
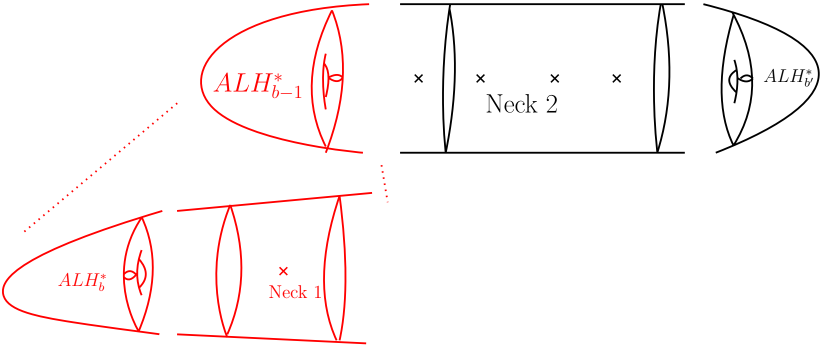
The geometry



$$\sim ALH^*_{b-1}$$

Going to the compact picture

- For the actual proof we glue an approximate model solution with four regions and two levels of scalings
- Similar to the construction of K3 degeneration in the compact case



Partial compactification

Theorem (Lin–Soundararajan–Z)

For any $0 \leq b \leq 8$, there exists a sequence of ALH^*_{b-1} gravitational instantons such that their Gromov-Haursdorff limit is an ALH^*_b manifold.

- Can be viewed as a partial compactification of the *ALH* and ALH^* moduli spaces
- There is a similar result relating ALG and ALG* [Lin–Soundararajan–Z]
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