From Microlocal to Global Analysis

In Honor of Richard Melrose

Microlocal Analysis and Inverse Problems

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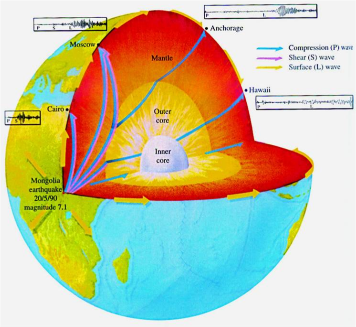
MIT, May 11, 2024

Travel Time Tomography

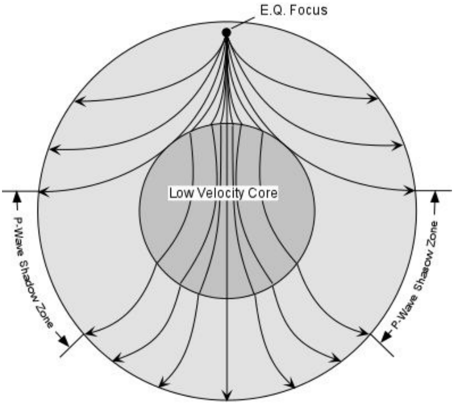
Global Seismology



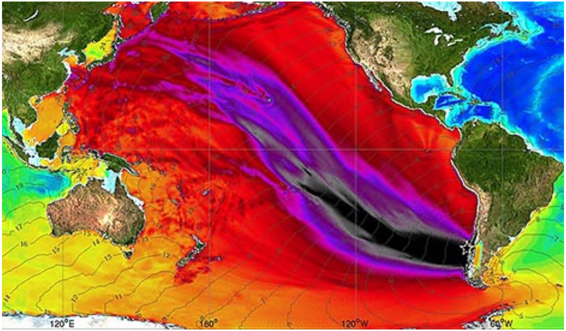
Inverse Problem: Determine inner structure of Earth by measuring travel time of seismic waves.

Travel Time Tomography

Travel time tomography: recover the sound speed of Earth from travel times of earthquakes.



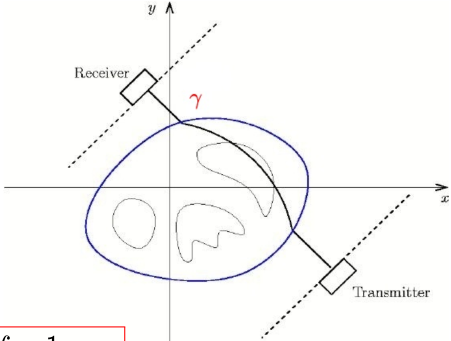
Tsunami of 1960 Chilean Earthquake



Black represents the largest waves, decreasing in height through purple, dark red, orange and on down to yellow. In 1960 a tongue of massive waves spread across the Pacific, with big ones throughout the region.

Human Body Seismology

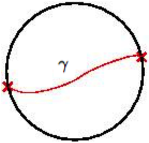
ULTRASOUND TRANSMISSION TOMOGRAPHY(UTT)



$$\boxed{T = \int_\gamma \frac{1}{c(x)}\,ds} \quad = \text{ Travel Time (Time of Flight).}$$

Travel Time Tomography (Transmission)

Motivation:Determine inner structure of Earth by measuring travel times of seismic waves



Herglotz (1905), Wiechert-Zoeppritz (1907)
Sound speed $c(r)$, $r = |x|$

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$$

$T = \int_\gamma \frac{1}{c(r)}$. What are the curves of propagation γ?

Anisotropic Sound Speed

The curves are geodesics of a metric.

$$ds^2 = \frac{1}{c^2(r)} dx^2$$

More generally $ds^2 = \frac{1}{c^2(x)} dx^2$

$$\text{Velocity } v(x,\xi) = c(x), \quad |\xi| = 1 \text{ (isotropic)}$$

Anisotropic case

$$\boxed{ds^2 = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j}$$

$g = (g_{ij})$ is a positive definite symmetric matrix

$$\text{Velocity } v(x,\xi) = \sqrt{\sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j}, \quad |\xi| = 1$$

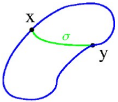
$$g^{ij} = (g_{ij})^{-1}$$

The information is encoded in the boundary distance function

Boundary Rigidity

More general set-up

Let (M, g) be a compact Riemannian manifold with boundary, $g = (g_{ij})$.



$$x, y \in \partial M$$

$$d_g(x,y) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} L(\sigma)$$

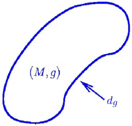
$L(\sigma)$ = length of curve σ

$$L(\sigma) = \int_0^1 \sqrt{\sum_{i,j=1}^n g_{ij}(\sigma(t)) \frac{d\sigma_i}{dt} \frac{d\sigma_j}{dt}} dt$$

Inverse problem: Determine g knowing $d_g(x,y)$ $x, y \in \partial M$

Another Motivation (String Theory)

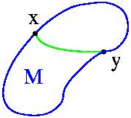
HOLOGRAPHY



Inverse problem: Can we recover (M, g) (bulk) from boundary distance function ?

M. Parrati and R. Rabadan, Boundary rigidity and holography, JHEP 01 (2004) 034
B. Czech, L. Lamprou, S. McCandlish and J. Sully, Integral geometry and holography, JHEP 10 (2015) 175

Non-uniqueness



$dg \Rightarrow g$?

(Boundary rigidity problem)

Answer NO $\psi : M \to M$ diffeomorphism

$$\psi\big|_{\partial M} = \text{Identity}, \quad d_{\psi^* g} = d_g$$
$$\psi^* g = \left(D\psi \circ g \circ (D\psi)^T\right) \circ \psi$$

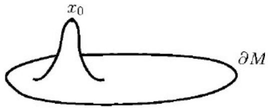
$$L_g(\sigma) = \int_0^1 \sqrt{\sum_{i,j=1}^n g_{ij}(\sigma(t)) \frac{d\sigma_i}{dt}\frac{d\sigma_j}{dt}} dt$$

$$\widetilde{\sigma} = \psi \circ \sigma \quad \boxed{L_{\psi^* g}(\widetilde{\sigma}) = L_g(\sigma)}$$

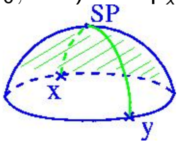
Non-uniqueness

$$d_{\psi^* g} = d_g$$

Only obstruction to determining g from d_g ? No



$$d_g(x_0, \partial M) > \sup_{x,y \in \partial M} d_g(x,y)$$



Can change metric near SP

Boundary Rigidity

Def (M, g) is boundary rigid if $(M, \widetilde{g})$ satisfies $d_{\widetilde{g}} = d_g$. Then $\exists \psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} =$ Identity, so that

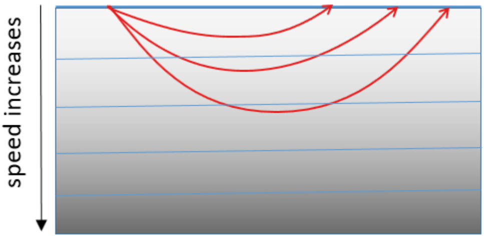
$$\boxed{\widetilde{g} = \psi^* g}$$

Need an a-priori condition for (M, g) to be boundary rigid.

One such condition is that (M, g) is simple

Michel's Conjecture

DEF (M, g) is simple if given two points $x, y \in \partial M$, $\exists !$ minimizing geodesic joining x and y and ∂M is strictly convex



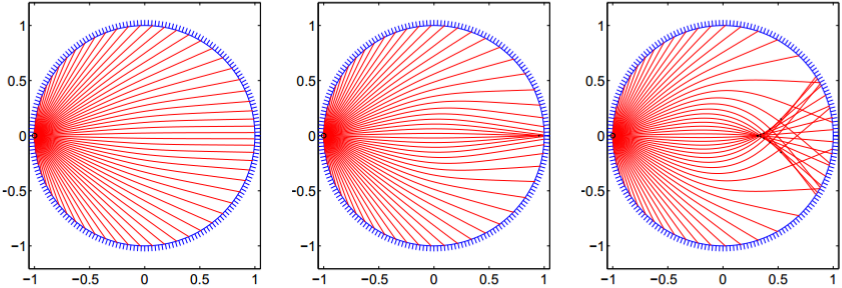
strictly convex

CONJECTURE

(M, g) is simple then (M, g) is boundary rigid ,that is d_g determines g up to the natural obstruction. $(d_{\psi^* g} = d_g)$

(Conjecture posed by R. Michel, 1981)

Metrics Satisfying the Herglotz condition



$k = 0.20$ (simple) $\quad$ $k = 0.49$ (non-simple) $\quad$ $k = 1.23$ (non-simple)

$$g_k(r) = \exp\left(k \exp\left(-\frac{r^2}{2\sigma^2}\right)\right), \quad 0 \le r \le 1, \quad \sigma \text{ fixed}$$

Francois Monard: SIAM J. Imaging Sciences (2014)

Results in Anisotropic Case

(M, g) simple

- R. Michel (1981) Compact subdomains of $\mathbb{R}^2$ or $\mathbb{H}^2$ or the open round hemisphere
- Gromov (1983) Compact subdomains of $\mathbb{R}^n$
- Besson-Courtois-Gallot (1995) Compact subdomains of negatively curved symmetric spaces

(All examples above have constant curvature or special symmetries)

- $\left\{ \begin{array}{c} \text{Stefanov-U (1998)} \\ \text{Lassas-Sharafutdinov-U (2003)} \\ \text{Burago-Ivanov (2010)} \end{array} \right\}$

  $\boxed{dg = dg_0}$, g_0 close to Euclidean

Two Dimensional Case

$\boxed{n = 2}$

- Otal and Croke (1990) $\boxed{K_g < 0}$

THEOREM(Pestov-U, 2005)

Two dimensional Riemannian manifolds with boundary which are simple are boundary rigid ($d_g \Rightarrow g$ up to natural obstruction)

Geodesics in Phase Space

$$g = (g_{ij}(x)) \text{ symmetric, positive definite}$$

Hamiltonian is given by

$$\boxed{H_g(x,\xi) = \frac{1}{2}\Big(\sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j - 1\Big)} \qquad g^{-1} = (g^{ij}(x))$$

$X_g(s, X^0) = (x_g(s, X^0), \xi_g(s, X^0))$ be **bicharacteristics**,

$$\text{sol. of} \quad \boxed{\frac{dx}{ds} = \frac{\partial H_g}{\partial \xi}, \qquad \frac{d\xi}{ds} = -\frac{\partial H_g}{\partial x}}$$

$x(0) = x^0$, $\xi(0) = \xi^0$, $X^0 = (x^0, \xi^0)$, where $\xi^0 \in \mathcal{S}_g^{n-1}(x^0)$

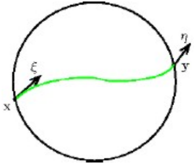
$$\mathcal{S}_g^{n-1}(x) = \{\xi \in \mathbb{R}^n;\ H_g(x,\xi) = 0\}.$$

Geodesics Projections in x: $\boxed{x(s)}$.

Scattering Relation

d_g only measures first arrival times of waves.

We need to look at behavior of all geodesics

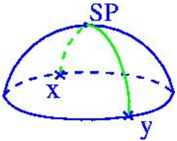


$$\|\xi\|_g = \|\eta\|_g = 1$$

$\alpha_g(x, \xi) = (y, \eta)$, α_g is SCATTERING RELATION

If we know direction and point of entrance of geodesic then we know its direction and point of exit.

Scattering Relation



Scattering relation follows all geodesics.

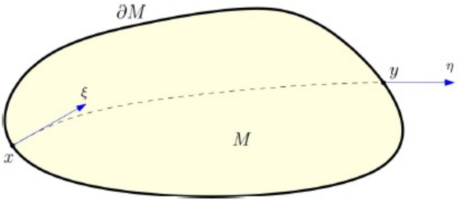
Conjecture Assume (M,g) non-trapping. Then α_g determines g up to natural obstruction.

(Pestov-U, 2005) $n = 2$ Connection between α_g and Λ_g (Dirichlet-to-Neumann map)

(M, g) simple then $d_g \Leftrightarrow \alpha_g$

Lens Rigidity

Define the scattering relation α_g and the length (travel time) function ℓ:



$$\alpha_g : (x, \xi) \to (y, \eta), \quad \ell(x, \xi) \to [0, \infty].$$

Diffeomorphisms preserving ∂M pointwise do not change L, ℓ!

Lens rigidity: *Do α_g, ℓ determine g uniquely, up to isometry?*

Lens Rigidity

No, There are counterexamples for trapping manifolds (Croke-Kleiner).

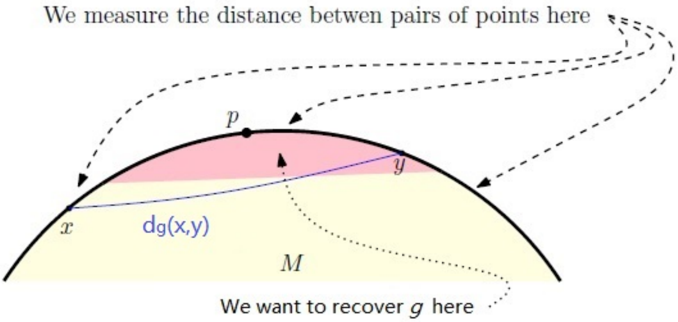
The lens rigidity problem and the boundary rigidity one are equivalent for simple metrics! This is also true locally, near a point p where ∂M is strictly convex.

For non-simple metrics (caustics and/or non-convex boundary), lens rigidity is the right problem to study.

Some results: local generic rigidity near a class of non-simple metrics (Stefanov-U, 2009), lens rigidity for real-analytic metrics satisfying a mild condition (Vargo, 2010), the torus is lens rigid (Croke 2014), stability estimates for a class of non-simple metrics (Bao-Zhang 2014), Stefanov-U-Vasy, 2016 (foliation condition, conformal case); Guillarmou, 2017 (hyperbolic trapping), Stefanov-U-Vasy, 2021 (foliation condition, general case).

Partial Data

Boundary Rigidity with partial data: Does d_g, known on $\partial M \times \partial M$ near some p, determine g near p up to isometry?



Partial Data

Theorem (Stefanov-U-Vasy, 2021)

Let $\dim M \geq 3$. *If* ∂M *is strictly convex near* p *for* g *and* $\widetilde{g}$, *and* $d_g = d_{\widetilde{g}}$ *near* (p,p), *then* $g = \widetilde{g}$ *up to isometry near* p.

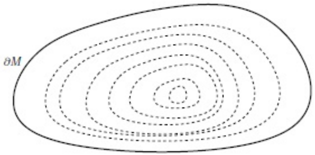
Also stability and reconstruction.

The only results so far of similar nature is for real analytic metrics (Lassas-Sharafutdinov-U, 2003). We can recover the whole jet of the metric at ∂M and then use analytic continuation.

Foliation condition

We could use a layer stripping argument to get deeper and deeper in M and prove that one can determine g (up to isometry) in the whole M.

Foliation condition: M is foliated by strictly convex hypersurfaces if, up to a nowhere dense set, $M = \cup_{t\in[0,T)}\Sigma_t$, where Σ_t is a smooth family of strictly convex hypersurfaces and $\Sigma_0 = \partial M$.



A more general condition: several families, starting from outside M.

Global result under the foliation condition (isotropic case)

Theorem (Stefanov-U-Vasy, 2016)

Let dim $M \geq 3$, let $\widetilde{g} = \beta g$ with $\beta > 0$ smooth on M, let ∂M be strictly convex with respect to both g and $\widetilde{g}$. Assume that M can be foliated by strictly convex hypersurfaces for g. Then if $\alpha_g = \alpha_{\widetilde{g}}, l = \widetilde{l}$ we have $g = \widetilde{g}$ in M.

Examples: The foliation condition is satisfied for strictly convex manifolds of non-negative sectional curvature, simply connected manifolds with non-positive sectional curvature and simply connected manifolds with no focal points.

Foliation condition is an analog of the Herglotz, Wieckert-Zoeppritz condition for non radial speeds.

Revisit the Herglotz and Wiechert & Zoeppritz condition

Example: Herglotz and Wiechert & Zoeppritz showed that one can determine a radial speed $c(r)$ in the ball $B(0,1)$ satisfying

$$\boxed{\frac{d}{dr}\frac{r}{c(r)} > 0.}$$

The uniqueness is in the class of radial speeds.

One can check directly that their condition is equivalent to the following one: the Euclidean spheres $\{|x| = t\}$, $t \leq 1$ are strictly convex for $c^{-2}dx^2$ as well. Then $B(0,1)$ satisfies the foliation condition. Therefore, if $\tilde{c}(x)$ is another speed, not necessarily radial, with the same lens relation, equal to c on the boundary, then $c = \tilde{c}$. There could be conjugate points.

Therefore, speeds satisfying the Herglotz and Wiechert & Zoeppritz condition are conformally lens rigid.

Global Result (general case)

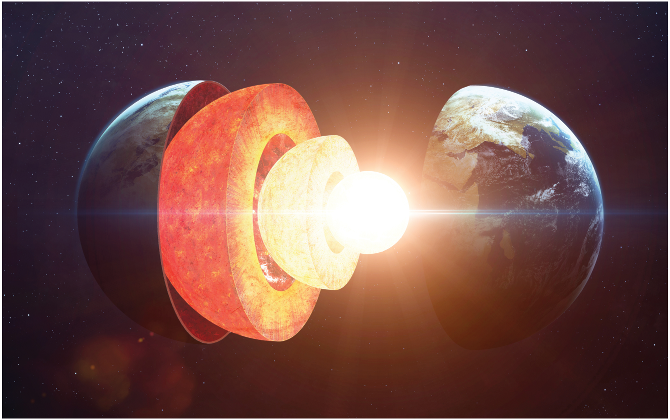
Theorem (Stefanov-U-Vasy, 2021)

Let (M, g) be a compact n-dimensional Riemannian manifold, $n \geq 3$, with strictly convex boundary so that there exists a strictly convex function f on M with $\{f = 0\} = \partial M$. Let $\widetilde{g}$ be another Riemannian metric on M, an assume that ∂M is strictly convex w.r.t. $\widetilde{g}$ as well. If g and $\widetilde{g}$ have the same lens relations, then there exists a diffeomorphism ψ on M fixing ∂M pointwise such that $g = \psi^* \widetilde{g}$.

Examples: This condition is satisfied for strictly convex manifolds of non-negative sectional curvature, simply connected manifolds with non-positive sectional curvature and simply connected manifolds with no focal points.

Travel Time Tomography

Long-awaited mathematics proof could help scan Earth's innards



Nature, Feb, 2017

New Results on Boundary Rigidity

The Boundary Rigidity problem is to recover g from d_g.

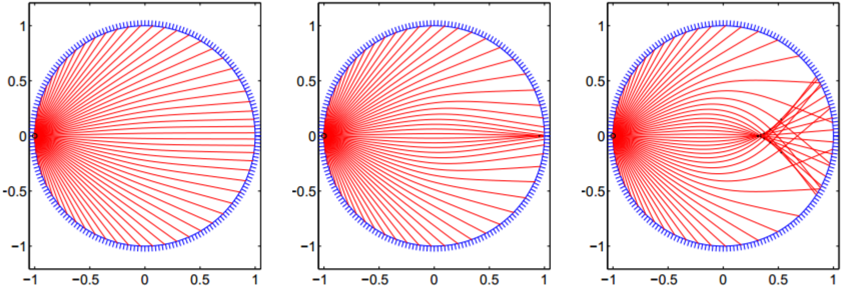
Corollary (New result on boundary rigidity)

Simple manifolds satisfying the foliation condition are boundary rigid.

Example: Simple manifold of non-negative sectional curvature, simple connected manifolds with non-positive sectional curvature and simply connected manifolds with no focal points.

Question: Do simple manifolds satisfy the foliation condition?

Metrics Satisfying the Herglotz condition



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Francois Monard: SIAM J. Imaging Sciences (2014)

The Linear Problem

Let (M, g) be compact with smooth boundary. Linearizing $g \mapsto d_g$ in a fixed conformal class leads to the *ray transform*

$$If(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\, dt$$

where $x \in \partial M$ and $\xi \in S_x M = \{\xi \in T_x M\,;\, |\xi| = 1\}$.

Here $\gamma(t,x,\xi)$ is the geodesic starting from point x in direction ξ, and $\tau(x,\xi)$ is the time when γ exits M. We assume that (M, g) is *nontrapping*, i.e. τ is always finite.

Inversion of X-ray Transform

(M, g) simple

$$If(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(x,t,\xi))dt$$

$$\xi \in S_x M = \{\xi \in T_x M : |\xi| = 1\}$$

where $\gamma(x, t, \xi)$ is the geodesic starting from x in direction ξ, $\tau(x, \xi)$ is the exit time.

Theorem (Guillemin 1975, Stefanov-U, 2004)

(M, g) simple. Then I^*I is an elliptic pseudodifferential operator of order -1.

Inversion of X-ray Transform (Radon 1917)

- $If(x,\theta) = \int f(x+t\theta)dt, \quad |\theta| = 1$
- $(-\Delta)^{1/2} I^* If = cf, \quad c \neq 0$
- $(-\Delta)^{-1/2} f = \int \frac{f(y)}{|x-y|^{n-1}} dy$

I^*I is an elliptic pseudodifferential operator of order -1.

Idea of the Proof in Isotropic Case

The proof is based on two main ideas.

First, we use the approach in a recent paper by U-Vasy (2012) on the linear integral geometry problem.

Second, we convert the non-linear boundary rigidity problem to a "pseudo-linear" one. Straightforward linearization, which works for the problem with full data, fails here.

The Local Linear Problem

U-Vasy result: Consider the inversion of the geodesic ray transform

$$If(\gamma) = \int f(\gamma(s))\, ds$$

known for geodesics intersecting some neighborhood of $p \in \partial M$ (where ∂M is strictly convex) "almost tangentially". It is proven that those integrals determine f near p uniquely. It is a Helgason support type of theorem for non-analytic curves! This was extended recently by H. Zhou for arbitrary curves (∂M must be strictly convex w.r.t. them) and non-vanishing weights.

The main idea in U-Vasy is the following:

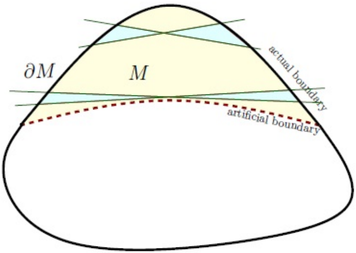
Introduce an artificial, still strictly convex boundary near p which cuts a small subdomain near p. Then use Melrose's scattering calculus to show that the I, composed with a suitable "back-projection" is elliptic in that calculus. Since the subdomain is small, it would be invertible as well.

Artificial Boundary

Consider

$$Pf(z) := I^* \chi I f(z) = \int_{S_z M} x^{-2} \chi I f(\gamma_{z,v}) dv,$$

where χ is a smooth cutoff sketched below (angle $\sim x$), and x is the distance to the artificial boundary.



Inversion of Local Geodesic Transform

$$Pf(z) := I^*\chi If(z) = \int_{S_z M} x^{-2}\chi If(\gamma_{z,v})dv,$$

Main result: P is an elliptic pseudodifferential operator in Melrose's scattering calculus.

There exists A such that $AP = Identity + R$

This is Fredholm and R has a small norm in a neighborhood of p. Therefore invertible near p.

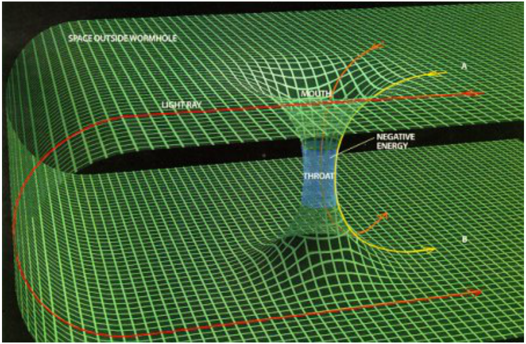
Scattering Calculus

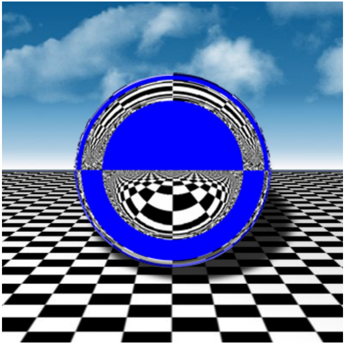
The scattering calculus (Melrose) is a version of the classical one on $\mathbb{R}^n_x$ with a compactification of $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Consider pseudodifferential operators with symbols $a(z, \zeta)$ satisfying symbol-like estimates both w.r.t. z and ζ (Hörmander, Parenti, Shubin)

$$|\partial_z^\alpha \partial_\zeta^\beta a(z, \zeta)| \leq C_{\alpha,\beta} \langle z \rangle^{l-|\alpha|} \langle \zeta \rangle^{m-|\beta|}$$

This defines the class $S^{l.m}(\mathbb{R}^n \times \mathbb{R}^n)$. Lower order means both lower order of differentiaion and a slower growth at infinity. Now compactify both $\mathbb{R}^n_x$ and $\mathbb{R}^n_\xi$ to get the scattering calculus.

Goal: To Determine the Topology and Metric of Space-Time





How can we determine the topology and metric of complicated structures in space-time with a radar-like device?

Figures: Anderson institute and Greenleaf-Kurylev-Lassas-U.

Non-linearity Helps

We will consider inverse problems for non-linear wave equations, e.g.

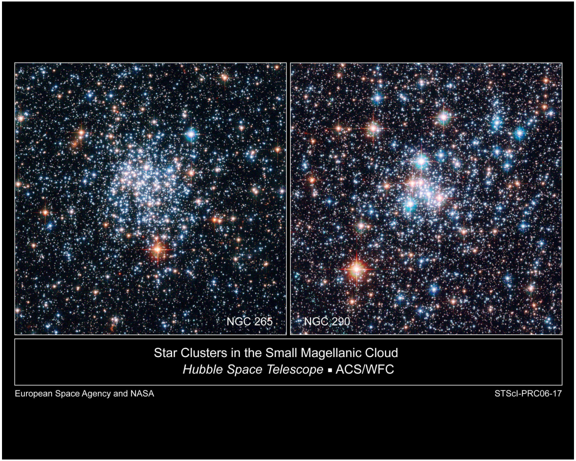
$$\frac{\partial^2}{\partial t^2} u(t,y) - c(t,y)^2 \Delta u(t,y) + a(t,y) u(t,y)^2 = f(t,y).$$

We will show that:

-Non-linearity helps to solve the inverse problem,

-"Scattering" from the interacting wave packets determines the structure of the spacetime.

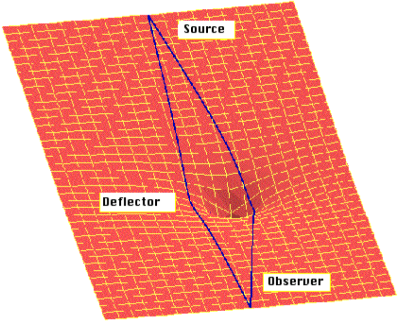
Inverse Problems in Space-Time: Passive Measurements

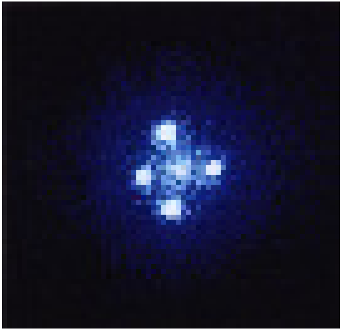


Can we determine the structure of space-time when we see light coming from many point sources varying in time? We can also observe gravitational waves.

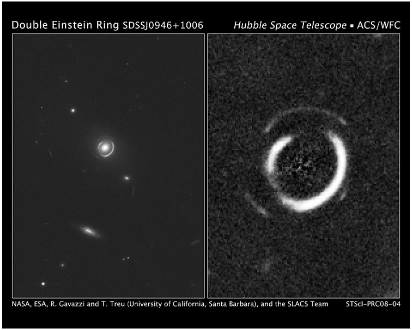
Gravitational Lensing

We consider e.g. light or X-ray observations or measurements of gravitational waves.

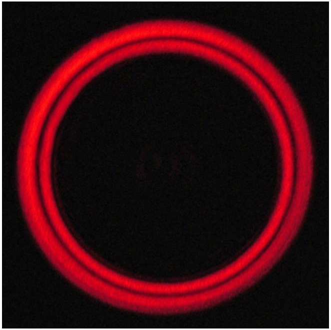




Gravitational Lensing



Double Einstein Ring



Conical Refraction

Vol. 46, No. 3 DUKE MATHEMATICAL JOURNAL © September 1979

MICROLOCAL STRUCTURE OF INVOLUTIVE CONICAL REFRACTION

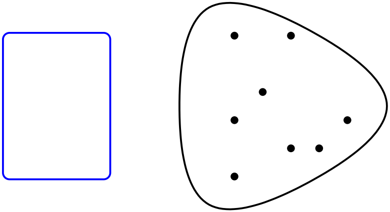
R. B. MELROSE AND G. A. UHLMANN

Duke Math. J. Volume 46, Number 3 (1979), 571-582.

Passive Measurements: Gravitational Waves

NSF Announcement, Feb 11, 2015

Inverse Problem for Passive Measurements



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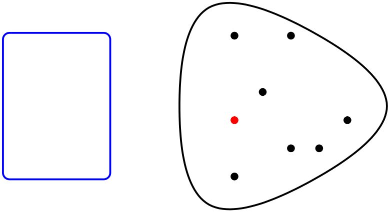

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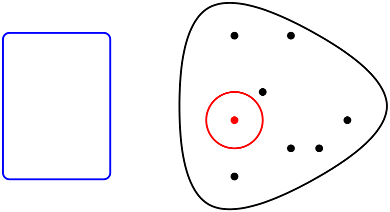

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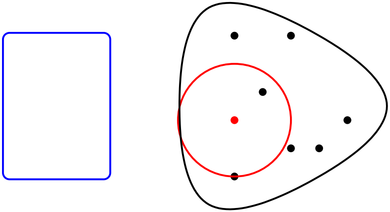

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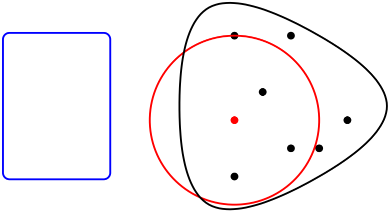

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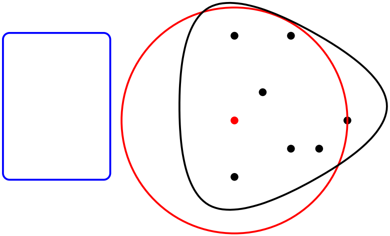

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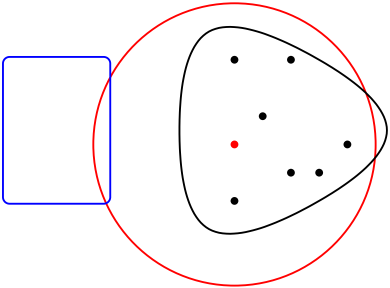

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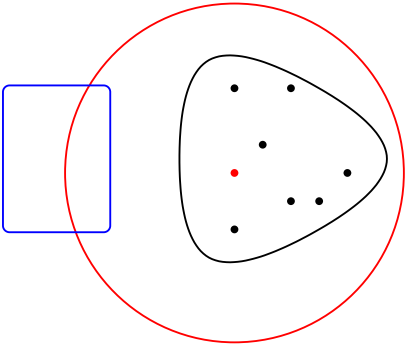

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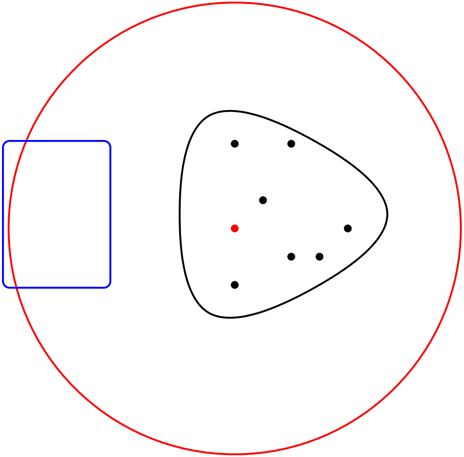

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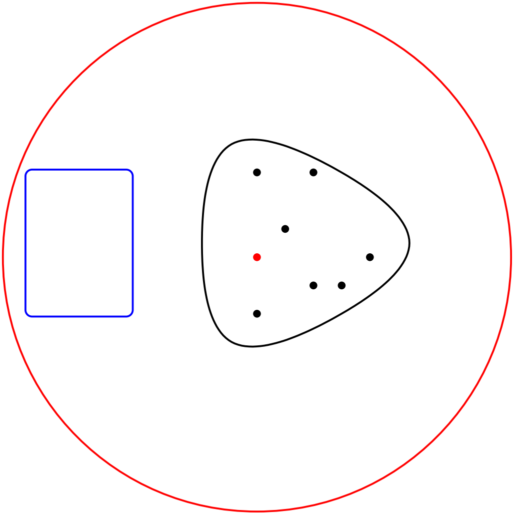

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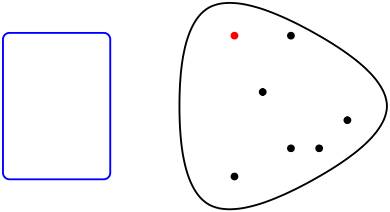

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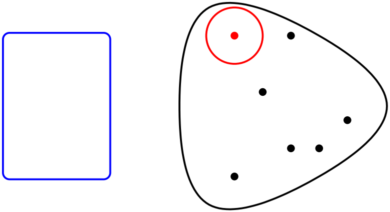

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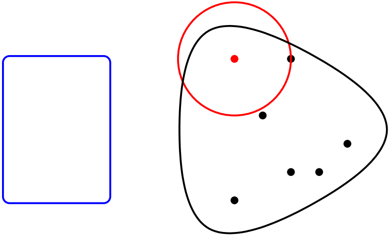

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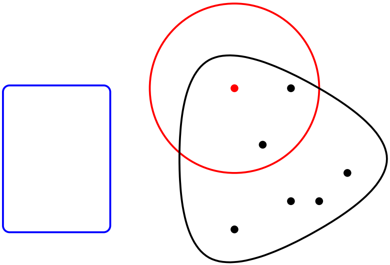

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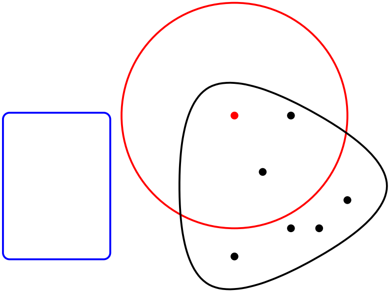

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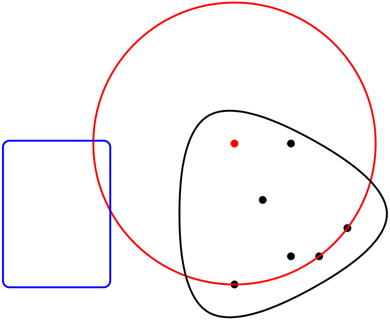

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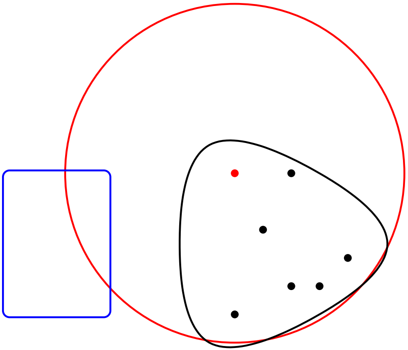

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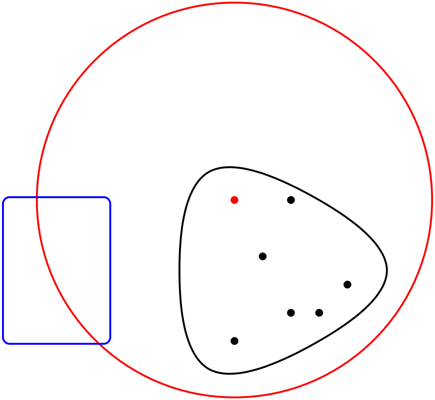


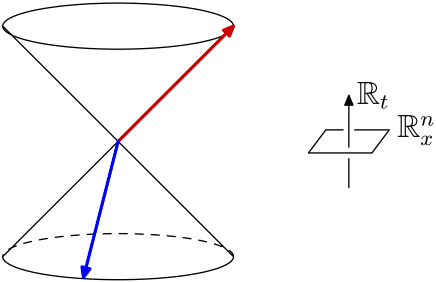
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Lorentzian Geometry

$(n+1)$-dimensional Minkowski space: (M, g)

$$M = \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x, \quad \text{metric: } g = -dt^2 + dx^2.$$

Null/lightlike vectors: $V \in T_q M$ with $g(V, V) = 0$.



$L^{\pm}_q M$: future/past null vectors

Lorentzian Geometry

In general:

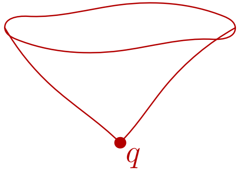
$$M = (n+1)\text{-dimensional manifold}, \; g \text{ Lorentzian } (-,+,\ldots,+).$$

Assume: existence of time orientation.

$$T_qM \cong (\mathbb{R}^{1+n}, \text{Minkowski metric}).$$

Null-geodesics: $\gamma(s) = \exp_q(sV)$, $V \in T_qM$ null.

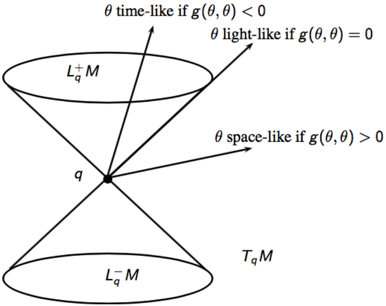
Future light cone: $\mathcal{L}_q^+ = \{\exp_q(V) \colon V \text{ future null}\}$



Lorentzian Manifolds

Let (M, g) be a $1+3$ dimensional time oriented Lorentzian manifold.
The signature of g is $(-,+,+,+)$.
Example: Minkowski space-time $(\mathbb{R}^4, g_m)$, $g_m = -dt^2 + dx^2 + dy^2 + dz^2$.

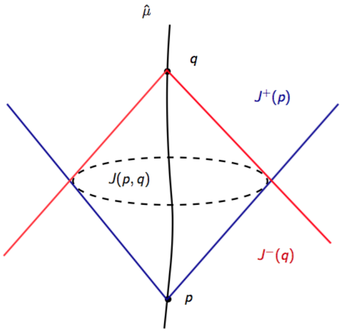
- $L_q^{\pm} M$ is the set of future (past) pointing light like vectors at q.
- Casual vectors are the collection of time-like and light-like vectors.
- A curve γ is time-like (light-like, causal) if the tangent vectors are time-like (light-like, causal).



Causal Relations

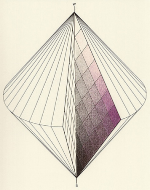
Let $\widehat{\mu}$ be a time-like geodesic, which corresponds to the world-line of an observer in general relativity. For $p, q \in M$, $p \ll q$ means p, q can be joined by future pointing time-like curves, and $p < q$ means p, q can be joined by future pointing causal curves.

- The chronological future of $p \in M$ is $I^+(p) = \{q \in M : p \ll q\}$.
- The causal future of $p \in M$ is $J^+(p) = \{q \in M : q < p\}$.
- $J(p, q) = J^+(p) \cap J^-(q)$, $I(p, q) = I^+(p) \cap I^-(q)$.



Global Hyperbolicity

A Lorentzian manifold (M, g) is globally hyperbolic if



- there is no closed causal paths in M;
- for any $p, q \in M$
  and $p < q$, the set $J(p, q)$ is compact.

Then hyperbolic equations are well-posed on (M, g)
Also, (M, g) is isometric to the product manifold

$$\mathbb{R} \times N \text{ with } g = -\beta(t, y)dt^2 + \kappa(t, y).$$

Here $\beta : \mathbb{R} \times N \to \mathbb{R}_+$ is smooth, N is a 3 dimensional manifold and κ is a Riemannian metric on N and smooth in t.
We shall use $x = (t, y) = (x_0, x_1, x_2, x_3)$ as the local coordinates on M.

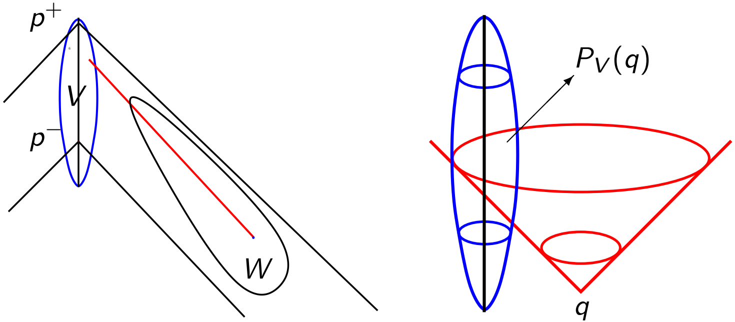
Light Observation Set

Let $\mu = \mu([-1,1]) \subset M$ be time-like geodesics containing p^- and p^+. We consider observations in a neighborhood $V \subset M$ of μ.

Let $W \subset I^-(p^+) \setminus J^-(p^-)$ be relatively compact and open set.

The light observation set for $q \in W$ is

$$P_V(q) := \{\gamma_{q,\xi}(r) \in V;\ r \geq 0,\ \xi \in L_q^+ M\}.$$



Inverse Problems with Passive Measurements

The earliest light observation set of $q \in M$ in V is

$$\mathcal{E}_V(q) = \{x \in \mathcal{P}_V(q) : \text{ there is no } y \in \mathcal{P}_V(q) \text{ and future pointing time like path } \alpha \text{ such that } \alpha(0) = y \text{ and } \alpha(1) = x\} \subset V.$$

In the physics literature the light observation sets are called light-cone cuts (Engelhardt-Horowitz, arXiv 2016)

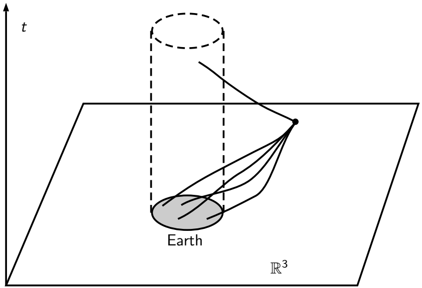
Theorem (Kurylev-Lassas-U 2018, arXiv 2014)

Let (M, g) be an open smooth globally hyperbolic Lorentzian manifold of dimension $n \geq 3$ and let $p^+, p^- \in M$ be the points of a time-like geodesic $\widehat{\mu}([-1,1]) \subset M$, $p^\pm = \widehat{\mu}(s_\pm)$. Let $V \subset M$ be a neighborhood of $\widehat{\mu}([-1,1])$ and $W \subset M$ be a relatively compact set. Assume that we know

$$\mathcal{E}_V(W).$$

Then we can determine the topological structure, the differential structure, and the conformal structure of W, up to diffeomorphism.

Interaction of Nonlinear Waves



Inverse Problem for a Non-linear Wave Equation

Consider the non-linear wave equation

$$\square_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } M^0 = (-\infty, T) \times N,$$
$$\text{supp }(u) \subset J_g^+(\text{supp }(f)),$$

where $\text{supp}(f) \subset V$, $V \subset M$ is open,

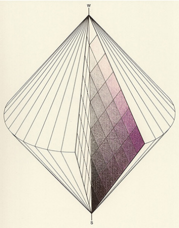
$$\square_g u = -\sum_{p,q=1}^{4} (-\det(g(x)))^{-1/2} \frac{\partial}{\partial x^p} \left((-\det(g(x)))^{1/2} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right),$$

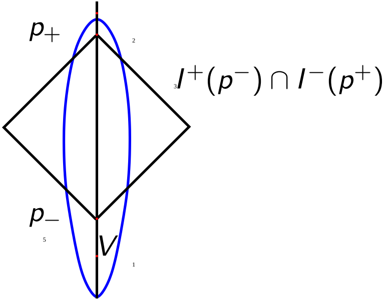
$\det(g) = \det((g_{pq}(x))_{p,q=1}^4)$, $f \in C_0^6(V)$ is a controllable source, and $a(x)$ is a non-vanishing C^∞-smooth function. In a neighborhood $\mathcal{W} \subset C_0^2(V)$ of the zero-function, define the measurement operator by

$$L_V : f \mapsto u|_V, \quad f \in C_0^6(V).$$

Theorem (Kurylev-Lassas-U, 2018)

Let (M, g) be a globally hyperbolic Lorentzian manifold of dimension $(1+3)$. Let μ be a time-like path containing p^- and p^+, $V \subset M$ be a neighborhood of μ, and $a : M \to \mathbb{R}$ be a non-vanishing function. Then $(V, g|_V)$ and the measurement operator L_V determines the set $I^+(p^-) \cap I^-(p^+) \subset M$ and the conformal class of the metric g, up to a change of coordinates, in $I^+(p^-) \cap I^-(p^+)$.

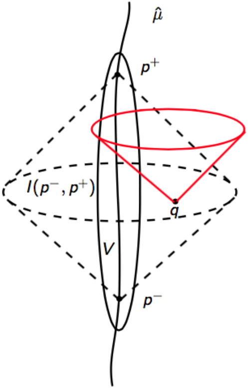




Idea of the Proof in the Case of Quadratic Nonlinearity: Interaction of Singularities

We construct the earliest light observation set by producing artificial point sources in $I(p_-, p_+)$. The key is the singularities generated from nonlinear interaction of linear waves.

- We construct sources f so that the solution u has new singularities.
- We characterize the type of the singularities.
- We determine the order of the singularities and find the principal symbols.



Non-linear Geometrical Optics

Let $u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$ satisfy

$$\begin{aligned} &\square_g u + a u^2 = f, \quad \text{in } M^0 = (-\infty, T) \times N, \\ &u|_{(-\infty,0)\times N} = 0 \end{aligned}$$

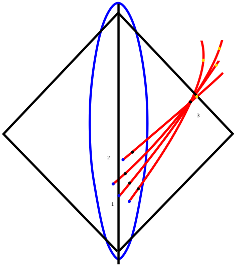
with $f = \varepsilon f_1$. When $Q = \square_g^{-1}$, we have

$$\begin{aligned} w_1 &= Qf, \\ w_2 &= -Q(a\, w_1\, w_1), \\ w_3 &= 2Q(a\, w_1\, Q(a\, w_1\, w_1)), \\ w_4 &= -Q(a\, Q(a\, w_1\, w_1)\, Q(a\, w_1\, w_1)) \\ &\quad -4Q(a\, w_1\, Q(a\, w_1\, Q(a\, w_1\, w_1))), \\ \|E_\varepsilon\| &\leq C\varepsilon^5. \end{aligned}$$

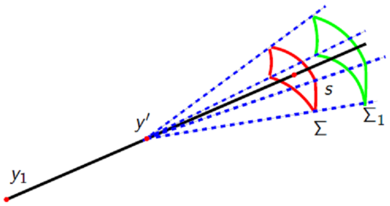
Non-linear Geometrical Optics

The product has, in a suitable microlocal sense, a principal symbol.

There is a lot of technology availale for the interaction analysis of conormal waves: intersecting pairs of conormal distributions (Melrose-U, 1979, Guillemin-U, 1981, Greenleaf-U, 1991).



Pieces of spherical waves



Consider solutions of $\square_g u_1 = f_1$, where f_1 is a conormal distribution that is singular on $\{t_0\} \times \Sigma$. The solution u_1 is a distribution associated to two intersecting Lagrangian manifolds. We can control the width s of the waves.

From $\Box_g u_1 = f_1$ we have

$$u_1 = \Box_g^{-1} f_1.$$

Thus,

$$\mathrm{WF} u_1 \subset \mathrm{WF} f_1 \cup \Lambda_p(\mathrm{WF} f_1)$$

where

$$\Lambda_p(\mathrm{WF} f_1) = \text{forward flow out by } H_p \text{ starting at } \mathrm{WF} f_1 \text{ intersected with } \{p = 0\}.$$

Here $p = \tau^2 - \sum g^{ij}(y)\xi_i\xi_j$.

H_p is the Hamiltonian vector field.

Notice that $\{p = 0\}$ is the light cone.

Lagrangian Intersection and the Cauchy Problem

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AND

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Massachusetts Institute of Technology

Comm. Pure Appl. Math., 32 (1979), no.4, 483-519.

Interaction of Waves in Minkowski Space $\mathbb{R}^4$

Let x^j, $j = 1,2,3,4$ be coordinates such that $\{x^j = 0\}$ are light-like. We consider waves

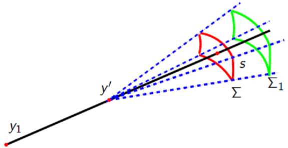
$$u_j(x) = v \cdot (x^j)^m_+, \quad (s)^m_+ = |s|^m H(s), \quad v \in \mathbb{R}, j = 1,2,3,4.$$
$$x^j = t - x \cdot \omega_j, \quad |\omega_j| = 1$$

Waves u_j are conormal distributions, $u_j \in I^{m+1}(K_j)$, where

$$K_j = \{x^j = 0\}, \quad j = 1,2,3,4.$$

The interaction of the waves $u_j(x)$ produce new sources on

$$K_{12} = K_1 \cap K_2,$$
$$K_{123} = K_1 \cap K_2 \cap K_3 = \text{line},$$
$$K_{1234} = K_1 \cap K_2 \cap K_3 \cap K_4 = \{q\} = \text{one point}.$$



Interaction of Two Waves (Second order linearization)

If we consider sources $f_{\vec\varepsilon}(x) = \varepsilon_1 f_{(1)}(x) + \varepsilon_2 f_{(2)}(x)$, $\vec\varepsilon = (\varepsilon_1, \varepsilon_2)$, and the corresponding solution $u_{\vec\varepsilon}$, we have

$$\begin{aligned} W_2(x) &= \frac{\partial}{\partial \varepsilon_1}\frac{\partial}{\partial \varepsilon_2} u_{\vec\varepsilon}(x)|_{\vec\varepsilon=0} \\ &= Q(a\, u_{(1)} \cdot u_{(2)}), \end{aligned}$$

where $Q = \Box_g^{-1}$ and

$$u_{(j)} = Qf_{(j)}.$$

Recall that $K_{12} = K_1 \cap K_2 = \{x^1 = x^2 = 0\}$. Since the normal bundle N^*K_{12} contain only light-like directions $N^*K_1 \cup N^*K_2$,

$$\mathrm{singsupp}(W_2) \subset K_1 \cup K_2.$$

Thus no new interesting singularities are produced by the interaction of two waves (Greenleaf-U, 1991).

Three plane waves interact and produce a conic wave. (Bony, 1996, Melrose-Ritter, 1987, Rauch-Reed, 1982)

Interaction of Three Waves (Third order linearization)

If we consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{3} \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have

$$
\begin{aligned}
W_3 &= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \\
&= \quad 4Q(a\, u_{(1)}\ Q(a\, u_{(2)}\ u_{(3)})) \\
&\quad +4Q(a\, u_{(2)}\ Q(a\, u_{(1)}\ u_{(3)})) \\
&\quad +4Q(a\, u_{(3)}\ Q(a\, u_{(1)}\ u_{(2)})),
\end{aligned}
$$

where $Q = \square_g^{-1}$. The interaction of the three waves happens on the line $K_{123} = K_1 \cap K_2 \cap K_3$.
The normal bundle N^*K_{123} contains light-like directions that are not in $N^*K_1 \cup N^*K_2 \cup N^*K_3$ and hence new singularities are produced.

Interaction of Four Waves (Fourth order linearization)

If we consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^4 \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have following. Consider

$$W_4 = \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} \partial_{\varepsilon_4} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}.$$

Since $K_{1234} = \{q\}$ we have $N^* K_{1234} = T_q^* M$. Hence new singularities are produced and

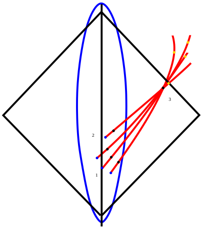
$$\text{singsupp}(W_4) \subset (\cup_{j=1}^4 K_j) \cup \Sigma \cup \mathcal{L}_q^+ M,$$

where Σ is the union of conic waves produced by sources on K_{123}, K_{134}, K_{124}, and K_{234}. Moreover, $\mathcal{L}_q^+ M$ is the union of future going light-like geodesics starting from the point q.

Interaction of Four Waves

The 3-interaction produces conic waves (only one is shown below).

The 4-interaction produces a spherical wave from the point q that determines the light observation set $P_V(q)$.



Active and Passive Measurements

(M, g) $(2+1)$-dimensional, $\square_g u = u^3 + f$.

Idea (Kurylev-Lassas-U 2018, arXiv 2014): Using nonlinearity to create point sources in $I(p_-, p_+)$.

$$f = \sum_{i=1}^{3} \epsilon_i f_i, \quad u_i := \square_g^{-1} f_i.$$

Take f_i = conormal distribution, e.g.

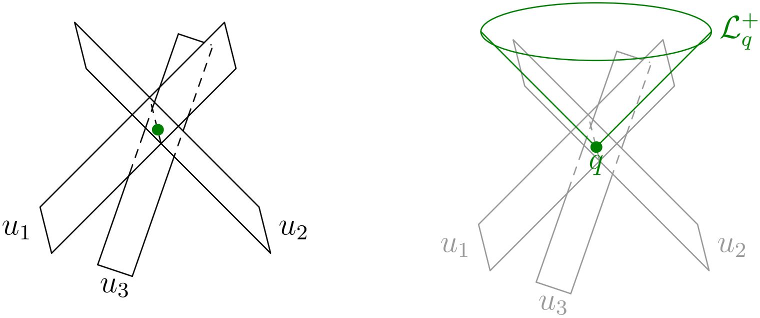
$$f_1(t,x) = (t - x_1)_+^{11} \chi(t,x), \quad \chi \in C_c^\infty(\mathbb{R}^{1+2}).$$

Then

$$u \approx \sum \epsilon_i u_i + 6\epsilon_1\epsilon_2\epsilon_3 \square_g^{-1}(u_1 u_2 u_3).$$

Generating Point Sources

non-linear interaction of conormal waves $u_i = \Box_g^{-1} f_i$: $\Box_g^{-1}(u_1 u_2 u_3)$



$$q = \bigcap_{i=1}^{3} \textit{sing supp } u_i, \quad \mathcal{L}_q^+ = \textit{sing supp } \Box_g^{-1}(u_1 u_2 u_3)$$

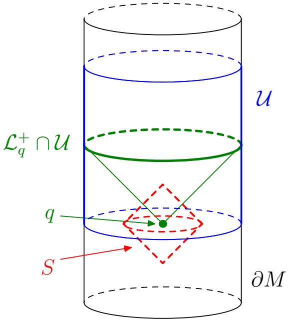
$\Rightarrow$ singularities of $\partial^3_{\epsilon_1 \epsilon_2 \epsilon_3} u$ give light observation sets $\mathcal{L}_q^+$

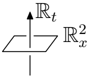
Further Developments

1. Einstein's equations coupled with scalar fields (Kurylev-Lassas-U, 2013; Kurylev-Lassas-Oksanen-U, 2022)
2. Einstein-Maxwell's equations in vacuum (Lassas-U-Wang, 2017)
3. Einstein's equations (U-Wang, 2020)
4. Non-linear elasticity (de Hoop-U-Wang, 2020; U-Zhai, 2021)
5. Yang-Mills (Chen-Lassas-Oksanen-Paternain, 2021, 2022)
6. Inverse Scattering (Sa Barreto-U-Wang, 2022)
7. Semilinear equations (Kurylev-Lassas-U, 2018; Wang-U, 2018; Wang-Zhou, 2019; Hintz-U-Zhai, 2022; Stefanov-Sa Barreto, 2021; U-Zhang 2021; Hintz-U-Zhai, 2022)
8. Non-linear Acoustics (Acosta-U-Zhai, 2023; U-Zhang, 2023)

Boundary Light Observation Set

$M = \{(t, x) \colon |x| < 1\} \subset \mathbb{R}^{1+2}$.





Set of sources $S \subset M^\circ$.

Observations in $\mathcal{U} \subset \partial M$.

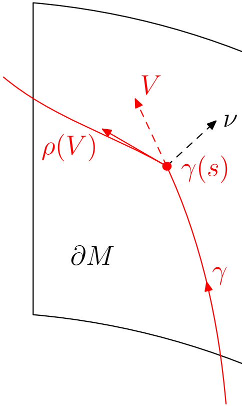
Data: $\mathscr{S} = \{\mathcal{L}_q^+ \cap \mathcal{U} \colon q \in S\}$

Theorem

The collection $\mathscr{S}$ determines the topological, differentiable, and conformal structure $[g|_S] = \{fg|_S \colon f > 0\}$ of S.

Reflection at the Boundary

γ null-geodesic until $\gamma(s) \in \partial M$.



$\rho(V) =$ reflection of V across ∂M. (Snell's law.)

$\rightarrow$ continuation of γ as broken null-geodesic

Null-convexity

Simplest case:

All null-geodesics starting in M° hit ∂M transversally. (1)

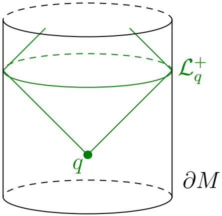
Proposition

(1) *is equivalent to null-convexity of* ∂M*:*

$$II(W, W) = g(\nabla_W \nu, W) \geq 0, \quad W \in T\partial M \text{ null.}$$

Stronger notion: strict null-convexity. ($II(W, W) > 0$, $W \neq 0$.)

Define light cones $\mathcal{L}_q^+$ using broken null-geodesics.



Main Result

Setup:

- (M, g) Lorentzian, dim ≥ 2, strictly null-convex boundary
- existence of $t \colon M \to \mathbb{R}$ proper, timelike
- sources: $S \subset M^\circ$ with $\bar{S}$ compact
- observations in $\mathcal{U} \subset \partial M$ open

Assumptions:

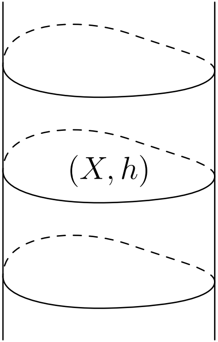
1. $\mathcal{L}^+_{q_1} \cap \mathcal{U} \neq \mathcal{L}^+_{q_2} \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$
2. points in S and $\mathcal{U}$ are not (null-)conjugate

Theorem (Hintz–U, 2019)

The smooth manifold $\mathcal{U}$ and the unlabelled collection $\mathscr{S} = \{\mathcal{L}^+_q \cap \mathcal{U} \colon q \in S\} \subset 2^{\mathcal{U}}$ uniquely determine $(S, [g|_S])$ (topologically, differentiably, and conformally).

Example for (M, g)

(X, h) compact Riemannian manifold with boundary.

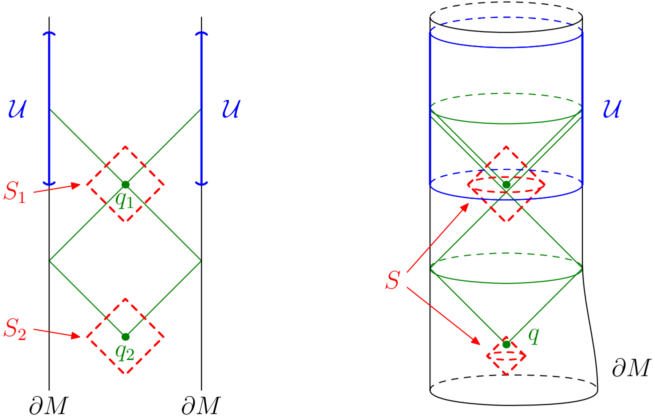


$$M = \mathbb{R}_t \times X, \quad g = -dt^2 + h.$$

(Strict) null-convexity of $\partial M \iff$ (strict) convexity of ∂X

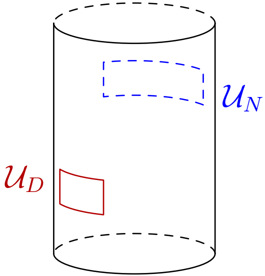
'Counterexamples'

Necessity of assumption 1. ($\mathcal{L}^+_{q_1} \cap \mathcal{U} \neq \mathcal{L}^+_{q_2} \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$)



S_1 and $S_1 \cup S_2$ are indistinguishable from $\mathcal{U}$.

Active Measurements for Boundary Value Problems



(Special case: $\mathcal{U}_N = \mathcal{U}_D$.)

Propagation of singularities: (strict) null-convexity assumption simplifies structure of null-geodesic flow. (Taylor '75, '76, Melrose–Sjöstrand '78, '82.)

Inverse Boundary Value Problem

Assume $M = \mathbb{R} \times N$ is a Lorentzian manifold of dimension $(1+3)$ with time-like boundary.

$$\begin{aligned} \Box_g u(x) + a(x)u(x)^4 &= 0, &&\text{on } M, \\ u(x) &= f(x), &&\text{on } \partial M, \\ u(t,y) &= 0, && t < 0, \end{aligned}$$

Inverse Problem: determine the metric g and the coefficient a from the Dirichlet-to-Neumann map.

The Main Result

Theorem (Hintz-U-Zhai, 2022)

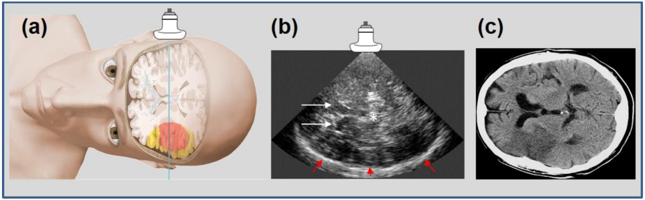
Consider the semilinear wave equations

$$\Box_{g^{(j)}} u(x) + a^{(j)} u(x)^4 = 0, \qquad j = 1, 2,$$

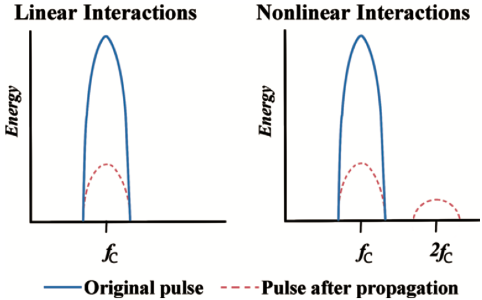
on Lorentzian manifold $M^{(j)}$ *with the same boundary* $\mathbb{R} \times \partial N$*. If the Dirichlet-to-Neumann maps* $\Lambda^{(j)}$ *acting on* $\mathcal{C}^5([0, T] \times \partial N)$ *are equal,* $\Lambda^{(1)} = \Lambda^{(2)}$*, then there exist a diffeomorphism* $\Psi: U_{g^{(1)}} \to U_{g^{(2)}}$ *with* $\Psi|_{(0,T)\times\partial N} = \mathcal{I}d$ *and a smooth function* $\beta \in C^\infty(M^{(1)})$*,* $\beta|_{(0,T)\times\partial N} = \partial_\nu \beta|_{(0,T)\times\partial N} = 0$*, so that, in* $U_{g^{(1)}}$*,*

$$\Psi^* g^{(2)} = e^{-2\beta} g^{(1)}, \quad \Psi^* a^{(2)} = e^{-\beta} a^{(1)}, \quad \Box_g e^{-\beta} = 0.$$

Ultrasound Imaging



Nonlinear interaction: waves at frequency f_C generate waves at frequency $2f_C$:



Inverse Boundary Value Problem

The acoustic waves are modeled by the Westervelt-type equation

$$\begin{aligned}\frac{1}{c^2(x)}\partial_t^2 p(t,x) - \beta(x)\partial_t^2 p^2(t,x) &= \Delta p(t,x), \quad \text{in } (0,T)\times\Omega,\\ p(t,x) &= f, \quad \text{on } (0,T)\times\partial\Omega,\\ p = \frac{\partial p}{\partial t} &= 0, \quad \text{on } \{t=0\},\end{aligned}$$

- c: wavespeed
- β: nonlinear parameter

Inverse problem: recover β from the Dirichlet-to-Neumann map Λ.

Second Order Linearization

Second order linearization and the resulted integral identity:

$$\int_0^T \int_{\partial\Omega} \frac{\partial^2}{\partial\epsilon_1 \partial\epsilon_2} \Lambda(\epsilon_1 f_1 + \epsilon_2 f_2)\Big|_{\epsilon_1=\epsilon_2=0} f_0 \, dS \, dt$$
$$= 2 \int_0^T \int_\Omega \beta(x) \partial_t(u_1 u_2) \partial_t u_0 \, dx \, dt.$$

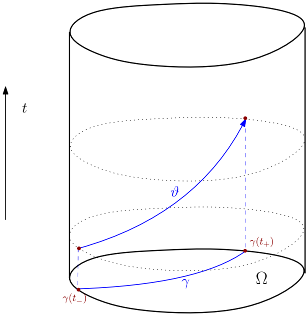
where u_j, $j = 1, 2$ are solutions to the linear wave equation

$$\frac{1}{c^2} \partial_t^2 u_i(t, x) - \Delta u_j(t, x) = 0$$

with $u_j|_{(0,T)\times\partial\Omega} = f_j$, and u_0 is the solution to the backward wave equation with $u_0|_{(0,T)\times\partial\Omega} = f_0$

Reduction to a Weighted Ray Transform

Construct Gaussian beam solutions u_0, u_1, u_2 traveling along the same null-geodesic $\vartheta(t) = (t, \gamma(t))$, where $\gamma(t), t \in (t_-, t_+)$ is the geodesic in (Ω, g) joining two boundary points $\gamma(t_-), \gamma(t_+) \in \partial\Omega$.



Insert into the integral identity, one can extract the Jacobi-weighted ray transform of $f = \beta c^{3/2}$ $\Rightarrow$ invert this weighted ray transform (Paternain-Salo-U-Zhou, 2019; Feizmohammadi-Oksanen, 2020)

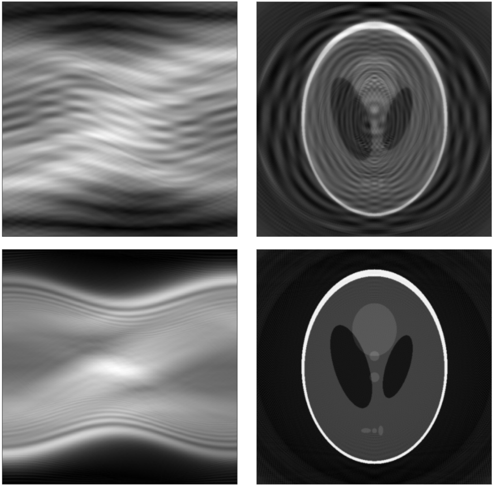


Figure: $L/\lambda = 10$ (top row) and $L/\lambda = 100$ (bottom row) where L is the size of the image and λ is the wavelength.

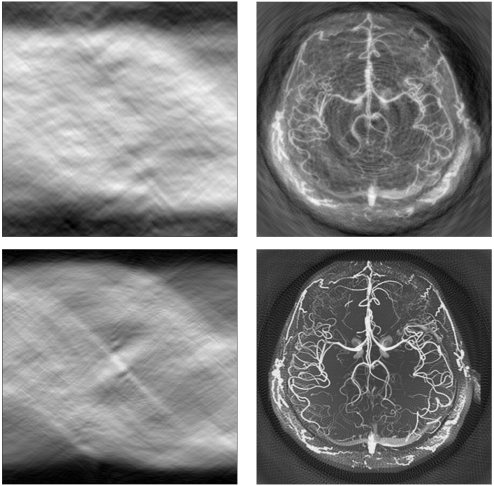


Figure: $L/\lambda = 10$ (top row) and $L/\lambda = 100$ (bottom row) where L is the size of the image and λ is the wavelength.

Belated Happy Birthday, Richard!