On a problem of conformal fill in by Poincare Einstein metrics

Sun-Yung Alice Chang, Princeton University
Report of joint works with
Yuxin Ge, University of Toulouse, France

From Microlocal to Global Analysis
A celebration of 75th birthday of Richard Melrose
May 10-12, 2024
MIT
§1. Conformal fill ins by Einstein manifolds

Given a compact manifold \((M^n, h)\), when is it the boundary of a conformally compact Einstein manifold \((X^{n+1}, g^+)\) with \(\rho^2 g^+|_M = h\), where \(\rho\) is a defining function on \(X\)? This problem of finding “conformal fill in” is motivated by:

- The AdS/CFT correspondence in quantum gravity (proposed by Maldacena also Witten, around 1998)
- Geometric considerations to study the structure of non-compact asymptotically hyperbolic manifolds.
Outline of talk

1. Introduction and a brief survey.
2. Set-up of the compactness problem.
3. Compactness results for conformally compact Einstein manifolds of dimension 3+1.
4. Some existence Results.
5. Components of proofs.
§1. Conformally compact Einstein manifolds, Definition

• On a manifold $X$ with boundary $M$, we call $\rho$ a defining function on $X$, if $\rho > 0$ on $X$, $\rho = 0$ on $M$ and $d\rho \neq 0$ on $M$.

• $(X^{n+1}, g^+) \) is conformally compact if $(\bar{X}^{n+1}, \rho^2 g^+)$ is compact. Denote $h = \rho^2 g^+|_M$, we call $(M^n, [h])$ the conformal infinity of $(X^{n+1}, g^+)$, where $[h]$ denotes the conformal class of metrics of $h$, i.e. the collection of metrics $\phi^2 h$ for some function $\phi$ on $M$.

• If $\text{Ric}[g^+] = -n g^+$, we call $(X^{n+1}, M^n, g^+)$ a conformally compact (Poincaré) Einstein (CCE) manifold.

• We remark on a CCE manifold, special $r$ (called the geodesic defining function) can be chosen, with $|\nabla (r^2 g^+) r| \equiv 1$ in an nbhd of $M \times (0, \epsilon)$ for some $\epsilon > 0$, so that $r^2 g^+$ is with totally geodesic boundary.
§1. Examples of CCE manifold

- **Example 1.**
  On \((\mathbb{R}^{n+1}_+, \mathbb{R}^n, g_H)\), where \(g_H = \frac{dx^2 + dy^2}{y^2}\), \(x \in \mathbb{R}^n\), \(y > 0\). Choose \(r = y\), then \((\mathbb{R}^{n+1}_+, dx^2 + dy^2)\) is not compact, but conformal to \(g_H\), with conformal infinity \((\mathbb{R}^n, [dx^2])\).

- **Example 2.**
  On \((\mathbb{B}^{n+1}, S^n, g_H)\), where \((\mathbb{B}^{n+1}, g_H = (\frac{2}{1-|y|^2})^2|dy|^2)\). Choose

  \[
  r := 2 \frac{1 - |y|}{1 + |y|},
  \]

  \[
  g_H = g^+ = r^{-2} \left( dr^2 + \left(1 - \frac{r^2}{4}\right) g_c \right).
  \]

  with \((S^n, [g_c])\) as conformal infinity.
  We remark that \(r = e^{-2t}\), where \(t(y) = dist_{g^+}(0, y)\).
§1. Examples of CCE manifold

- Example 3.

On $\mathbb{S}^1(\lambda) \times \mathbb{S}^2$ with the product metric, when $0 < \lambda < \frac{1}{\sqrt{3}}$, there are at least 3 different "conformal fill ins".

(a) One is when $X$ is $(\mathbb{S}^1(\lambda) \times \mathbb{B}^3)$ with the fill in the hyperbolic metric $g^+ = f(y)dt^2 + g_{\mathbb{H}^3(y)}$.

(b) The other two: $X$ is the AdS-Schwarzchild space $(\mathbb{R}^2 \times \mathbb{S}^2, g_m^+)$, where

$$g_m^+ = V dt^2 + V^{-1} dr^2 + r^2 g_c,$$

$$V = 1 + r^2 - \frac{2m}{r}.$$

It turns out for $\lambda < \frac{1}{\sqrt{3}}$, there are two different choices of $m$. This is the famous "non-unique fill in" example of Hawking-Page '83.
§1. Some earlier existence results, Scattering theory on CCE manifolds

- “Ambient Metric” of Fefferman-Graham ’85. On any compact manifold \((M^n, h)\), \(h\) real analytic, there is a CCE metric on some \(M^{n+1} \times (0, \epsilon)\) of \(M\). Gursky-Székelyhidi ’17, extend to smooth \(h\).

- Graham-Lee ’91: Any \(h\) in a small smooth neighborhood of \(h_c\) on \(S^n\). We remark that the fill in metrics constructed by Graham-Lee \(g^+\) for \(h\) all exist in a small nbhd of the Hyperbolic metric, it turns out they are ”unique” by a later result of C-Ge-Qing, ’21.

- Gursky-Han ’17 and Gursky-Han-Stolz ’18 constructed many examples of boundary conformal classes that do not allow Poincaré-Einstein extensions on specified manifolds \(X^{4k}\) for \(k \geq 2\).

**Theorem** (J, Lee ’95). On CCE manifolds, if \(R(h) > 0\), then \(\lambda_1(-\Delta_{g^+}) \geq \frac{n^2}{4}\).

**Corollary** (J.Qing ’03) On CCE manifolds, if \(R(h) > 0\), then there exists a compactified metric \(g\) with \(g|_{M} = h\) and \(R(g) > 0\).
§1. Scattering theory on CCE manifolds

- Starting point of all
  **Theorem (Mazzeo-Melrose,’ 87)** On an AH manifolds \((X^{n+1}, g^+)\), the essential spectrum of the \(-\Delta_{g^+}\) includes \([\frac{n^2}{4}, \infty)\) and may be a finite points of point spectrum in \((0, \frac{n^2}{4})\).

- **Theorem (J. Lee ’95).** On a CCE manifold, if \(R(h)\) is positive, then \(\lambda_1(-\Delta_{g^+}) \geq \frac{n^2}{4}\).

  In the proof of Lee, he studied solution of the Poisson equation:

  \[
  (*) \quad -\Delta_{g^+} v + (n + 1) v = 0 \quad \text{on } X^{n+1}
  \]

  with asymptotic behavior \(v = r^{-1}(1 + f_2 r^2 + ...)\), where \(r\) denotes the geodesic defining function for \(h\) and when \(R(h) > 0\), he used \(v^{\frac{n}{2}}\) as a testing function to estimate \(\lambda_1(-\Delta_{g^+})\).

- An observation of J. Qing is that in Lee’s proof, \(R(h) > 0\) implies the scalar curvature of metric \(v^{-2}g^+\) is positive.
§2. Compactness of CCE manifolds – the set-up

- An open question: Does the entire class of metrics \((\mathbb{S}^3, h)\) with positive scalar curvature allow CCE fill in \(\mathbb{B}^4\)? The class is path-connected by a result of F. Marques ’12. The index argument for non-existence of Gursky-Han, Gursky-Han-Stolz does not apply.

- We propose to study the “compactness” problem, and as an application some existence result for conformal fill in. More precisely, we ask the question:

Given a sequence of \((M^n, [h_i])\) metrics with positive Yamabe constants, which are conformal infinity of CCE \((X^{n+1}, g_i^+)\); when would

\[
\{[h_i]\} \text{ forms a compact family on } M^n
\]

\[
\implies \{[g_i]\} \text{ forms a compact family on } X^{n+1}?
\]

where \(g_i\) is some compactification of \(\{g_i^+\}\) with \(g_i|_M = h_i\).
§2. Compactness of CCE manifolds – an non-local inverse problem

The difficulty of the problem lies in the existence of an “non-local” term. We will illustrate the case on \((X^4, M^3, g^+)\) CCE manifold with \((M^3, h)\) conformal infinity, recall the asymptotic behavior

\[
g := r^2 g^+ = dr^2 + h + g^{(2)} r^2 + g^{(3)} r^3 + g^{(4)} r^4 + \cdots,
\]

where \(g^{(2)} = -\frac{1}{2} (\text{Ric}_h - \frac{1}{4} R_h h)\) determined by \(h\) (a local term), \(Tr_h g^{(3)} = 0\), while

\[
g^{(3)}_{\alpha,\beta} = -\frac{1}{3} \frac{\partial}{\partial n} (\text{Ric}_g)_{\alpha,\beta}
\]

is a non-local term not determined by \(h\).

We remark that \(h\) together with \(g^{(3)}\) determines the asymptotic behavior of \(g\). \textit{Fefferman-Graham '07, Biquard '08). We remark that \(h\) together with \(g^{(3)}\) determines the asymptotic behavior of \(g\).}
\[ \text{§2. Conformal invariants} \]

**Yamabe constant**

- On \((M^n, h)\), compact closed manifold,
  \[ Y(M, [h]) = \inf_{\tilde{h} \in [h]} \frac{\int_M R[\tilde{h}] \, dvol[\tilde{h}]}{Vol(M, \tilde{h})^{(n-2)/n}}. \]
  We remark \( Y(M, [h]) \) corresponds to the ”isoperimetric constant” of the Sobolev embedding of \( W^{1,2} \) into \( L^{\frac{2n}{n-2}} \).

- On compact manifold with boundary, there are two such constants. \((X^{n+1}, M^n, \bar{g})\)

\[ Y_a(X, M, [\bar{g}]) = \inf_{\tilde{\bar{g}} \in [\bar{g}]} \frac{\int_X R[\tilde{\bar{g}}] \, dvol[\tilde{\bar{g}}] + c_n \int_M H[\tilde{\bar{g}} \, | \, M] \, d\sigma[\tilde{\bar{g}} \, | \, M]}{Vol(X, \tilde{\bar{g}})^{(n-1)/(n+1)}} \]

\[ Y_b(X, M, [\bar{g}]) = \inf_{\tilde{\bar{g}} \in [\bar{g}]} \frac{\int_X R[\tilde{\bar{g}}] \, dvol[\tilde{\bar{g}}] + c_n \int_M H[\tilde{\bar{g}} \, | \, M] \, d\sigma[\tilde{\bar{g}} \, | \, M]}{Vol(M, \tilde{\bar{g}} \, | \, M)^{(n-1)/n}}. \]

\( Y_a \) and \( Y_b \) each corresponds to the (isoperimetric) constants in the Sobolev and Sobolev trace embeddings.
2. Conformal invariants

- As we have mentioned before, it follows from result of J. Lee '95, and the observation by J. Qing, that on CCE setting, $Y(M, [h]) > 0$ implies that $Y_a(X, M, [g]) \geq 0$.

- Combining works of Gursky-Han '17, X. Chen- M. Lai and F. Wang '18, Chang-Ge '21 we established that, there exists some constant $c_n$, such that

$$Y_a(X, M, [g]) \geq C_n Y(M, [h])^{\frac{n}{n+1}}.$$ 

Recall X. Chen-M. Lai and F. Wang

$$Y_b(X, M, [g]) \geq C_n Y(M, [h])^{\frac{1}{2}}.$$
§2. Conformal invariants

- Another conformally invariant quantity is Weyl curvature $W$.
  
  \[ |W|[\tilde{g}] = \rho^{-2} |W|[g], \text{ if } \tilde{g} = \rho^2 g. \]

  Thus $\int_X |W|^\frac{n+1}{2}[g]d\nu_g$ is a conformal invariant.

- On 4-manifold $X$, Bach tensor
  \[
  B_{ij} = \nabla_l \nabla_k W_{kilj} + \frac{1}{2} Ric_{kl} W_{kilj}
  \]

  is a conformally invariant. Bach flat metrics are the critical metric of the functional $g \rightarrow \int_X |W|^2 [g] d\nu_g$. Einstein metrics are Bach flat, hence so are all metrics in the same conformal class of Einstein metric. Thus in a CCE setting $(X, M, g^+)$, all compactified metrics of $[g^+]$ are Bach flat.

- We remark that it turns out we can re-write Bach flat condition as a 4th order system of PDE of elliptic type,
  \[
  \Delta R_{ij} = c \nabla_i \nabla_j R + R_m * Ric,
  \]

  which plays an important role in our estimates of the compactified metrics later. We also remark that for this PDE, the non-local tensor $-3g^{(3)} = \frac{\partial Ric}{\partial n}|_M$ is a natural matching boundary condition.
§2. Compactness of CCE manifolds – the set-up.

• For convenience, we choose \( h = h^Y \in [h] \), the Yamabe metric on \( M \). But what is a good choice of the compactified metric \( g \in [g^+] \)? A first attempt is to choose \( g = g^Y \), a Yamabe metric among compactified metrics of \( g^+ \). The difficulty of this choice is we do not know how to control the behavior of \( g^Y|_M \) in terms of \( h^Y \).

• Instead, following the work of Lee, Graham-Zworski, ’03 we will make a choice of a special representative metric, which we call scalar flat Adapted metrics on \( X \) obtained by solving the Poisson equation \((*)_s\) the boundary metric \( h \) with \( R(h) > 0 \) on \( M \).

\[
(*)_s - \Delta_{g^+} v - s(n-s)v = 0, \; X^{n+1},
\]

with Dirichlet data \( f \equiv 1 \). Choose \( \rho = \sqrt{\frac{1}{n-s}} \) and denote the adapted metric \( g^\ast = \rho^2 g^+ \).

• Properties of \((*)_s\) has been studied in Fefferman-Graham ’02, Chang-Gonzalez ’11, Case-Chang ’16, F. Wang ’21-’22 and S. Lee ’23 and others, Lee’s metric is the adapted metric when \( s = n + 1 \). In the statement of the theorems below, we choose \( s = \frac{n}{2} + 1 \), call it the scalar flat adapted metric.
§2. Properties of the adapted metric

On \((X, M, g_+)\) CCE, for a given metric we have the adapted metric \(g^\ast, g^\ast|_M = h\), with the key properties:

1. \(R[g^\ast] = 0\) on \(X\).
2. \(R[h] > 0\) on \(M\) implies the mean curvature \(H > 0\) on \(M\).
3. Denote \(g^\ast = \rho^2 g^+, |\nabla g^\ast \rho| \leq 1\).
4. Gauss Bonnet formula

\[
8\pi^2 \chi = \int_X (\frac{1}{4} |W|^2 - \frac{1}{2} |E|^2) + \int_M (\frac{4}{3} R[h] H - \frac{2}{27} H^3).
\]

Hence under the assumption \(R[h] > 0\),

\[
\int_X |E|^2 + \int_M H^3 \leq C (\int_X |W|^2 + \int_M (R[h])^3),
\]

where \(E\) denote the traceless Ricci.
§3. A compactness result on 4-manifold

Compactness Theorem (C and Yuxin Ge)

Let \( \{X, M = \partial X, g_i^+\} \) be a family of 4-dimensional CCE manifolds. \( g_i \) is a sequence of adapted metrics. Denote \( h_i = g_i|_M \). Assume

1. The boundary metric \((M, h_i)\) is compact in \( C^{k,\alpha} \) norm with \( k \geq 6 \); and there exists some positive constant \( C_1 > 0 \)

\[
Y(M, [h_i]) \geq C_1;
\]

2. There exists some positive constant \( C_2 > 0 \) such that

\[
\int |W[g_i]|^2 \leq C_2
\]

3. \( H_2(X, \mathbb{Z}) = 0 \) and \( H_1(X, \mathbb{Z}) = 0 \).

Then, the sequence \( g_i \) is compact in \( C^{k,\alpha'} \) norm for any \( \alpha' \in (0, \alpha) \) up to a diffeomorphism fixing the boundary.


§4. An Existence Result

- Recall Graham-Lee ’91: Any $h$ in a small smooth neighborhood of $h_c$ on $S^3$ allows a CCE fill in, which are in a small nbhd of the Hyperbolic metric on $B^4$, thus has the small $L^2$ norm of its Weyl tensor.

On the other hand, we also have the following result:

- When $n = 3$, on a CCE manifold $(X^4, M^3, g^+)$ if $Y(M, [h]) > 0$, and

  $$(*) \int_X |W|^2_{g^+} dv_{g^+} \leq c Y_a^2$$

for some $c \leq \frac{1}{12^2}$, then any metric in some small nbhd of $h$ allows a (unique) CCE fill in.

The natural question we then ask is can one impose conditions on the boundary metric $h$ which will ensure $(*)$ to happen? As an application of our compactness result, we partially answer the question above.
§4. Statement of an existence result

**Existence Theorem** Let $(X = B^4, M = S^3)$ and $h \in C^{6,\alpha}$ be a metric with the positive scalar curvature on $S^3$. Given the positives constants $\bar{C}_4, \delta > 0$, such that

1. $\|h\|_{C^4} \leq \bar{C}_4$,
2. $\Upsilon(M, [h|_M]) \geq \delta$;
3. $\text{vol}(h) = 1$.

Then there exists some constant $C(\bar{C}_4, \delta) > 0$ and some (small) positive constant $\varepsilon$ so that denote $E(h)$ the traceless Ricci of $h$, if

$$\|E(h)\|_2 \leq \varepsilon$$

then for some dimension constant $c_0$, we can find a CCE fill in metric with the conformal infinity $[h]$ satisfying

$$c_0 \|\mathcal{W}\|_2 \leq \sqrt{\varepsilon} C(\bar{C}_4, \delta) \leq \frac{1}{4} Y_a.$$

Moreover, such solution with the above bound is unique.
§5. Some outline of proof of the existence theorem

The strategy of proof is as follows: Denote $g = g^*$, and $S = g^{(3)}$ the non-local term, under assumptions of the theorem.

- **Step 1**: Apply Bach flat equation to $g$, control $\|W\|_2$ by the norm of $S$ and $\hat{E}$, where $\hat{E} = E(h)$. More precisely, We apply the Bach equation to $g$ to obtain

$$
(Y_a - c_0(\|W\|_2 + \|E\|_2))(\|W\|_4^2 + \|E\|_4^2) \leq C \int_{S^3} S \hat{E}.
$$

where $c_0$ and $C$ are some dimension constant.

- **Step 2**: Under assumption (***) $(\frac{5}{18} Y_a - c_0(\|W\|_2 + \|E\|_2) > 0)$,

$$
Y_b \|S\|_3 \leq C(\bar{C}_4, \delta).
$$

(This is the hard step, which we will supplement later.)

Combine step (1) and (2) we have if $\|\hat{E}\|_\frac{3}{2} \leq \epsilon$ then under (***) , we have

$$
c_0(\|W\|_2 + \|E\|_2) \leq \frac{1}{4} Y_a.
$$
§5. Outline of proof of the existence theorem

• **Step 3** We now run a continuity argument connecting $h$ to $h_c$ in $S^3$. Note for metrics close to $h_c$, the fill in metric always exists and $\|W\|_2$ tends to zero so (**) condition is always satisfied. It turns out we can find such a path via the Ricci flow due to some recent work of E. Chen, G. Wei and R. Ye ’24, here we quote a special case $n = 3$ of their work.

**Theorem** On $(S^3, h)$, assume $R(h) > 0$, there exists a constant $\delta(3)$ sufficient small, so that

$$\|\mathcal{E}(h)\|_2^3 + \|R(h) - \bar{R}(h)\|_2^3 \leq \delta(3),$$

where $\bar{R}(h)$ denotes the average of $R(h)$, then along the normalized Ricci flow the family $h(t)$ converges smoothly to $h_c$.

• We remark that under the assumption $\|\mathcal{E}(h)\|_2$ small and $\text{vol}(h) = 1$, the condition in the theorem above is satisfied by an earlier result of Y. Ge and G. Wang ’14.

• Combining the three steps, along this path, under the assumptions of the Existence theorem, (**) is automatic and we reached the estimate in Step 2 and finished the proof of the
§6. More outline of proof of Step 2

• **Step 2**

Estimate of $S$-tensor: Recall $S = \frac{\partial}{\partial n_g} Ric_g$. To estimate $S$, we first recall a fact which was used in the work of S. Bando, A. Kasue, H. Nakajima [BKN]'89 to derive ALE decay of sequence of Einstein metrics. In the special case of 4-manifold, if $g^+$ is an Einstein metric, denote $W^+$ the Weyl tensor of $g^+$, then there is a Kato inequality

$$|\nabla_{g^+} W^+|^2 \geq \frac{5}{3}|\nabla_{g^+} |W^+||^2$$

(1)

From these, one can derive

$$-\Delta_{g^+}|W^+|^{1/3} \leq c|W^+|^{4/3} + \frac{1}{6}R_{g^+}|W^+|^{1/3}$$

In work of [BKN], when scalar curvature $R_{g^+} = 0$, on a region $\int_A |W^+|_{g^+}^2 dv_{g^+}$ is small, [BKN] derive the decay estimate

$$|W^+|^{1/3}(x) \lesssim \frac{1}{|x|^{2-}} \text{ when } x \in A \text{ and } |x| \to \infty$$

Our Lemma is an application of (1) in conformal Einstein setting.
§5. More outline of proof of Step 2

**Lemma 1** Let $g^+$ be CCE, $g = \rho^2 g^+$ be a compactification, define $U = U_g := \left( \frac{|W|_{g^+}}{\rho} \right)^{1/3}$, then

$$- \triangle_g U \leq c |W|_g U + \frac{1}{6} R_g U$$  \hspace{1cm} (2)

**Lemma 2** Denote $\tilde{r}(x) = \text{dist}_g(x, M)$, $x \in X$, $g = g^*$, then

$$|W|_g^2 = e_2 \tilde{r}^2 + e_3 \tilde{r}^3 + O(\tilde{r}^4),$$

where

$$e_2 = 8 |S|^2 + 4 |\hat{C}|^2,$$

$\hat{C}$ is the Cotton tensor on $M^3$, $e_3 = -4 S_{\alpha\beta}(\nabla_\gamma \hat{C}_{\alpha\beta\gamma} + \nabla_\gamma \hat{C}_{\beta\alpha\gamma}) + 4H |S|^2 + \text{some other lower order terms.}$
§5. More outline of proof of Step 2

Lemma 3

\[ U_g^6 = \frac{|W|^2_g}{\rho^2_g} = \frac{|W|^2_g}{r^2} \frac{r^2}{\rho^2_g} \]

where

\[ \rho_g = \tilde{r} - \frac{H}{18} \tilde{r}^2 + O(\tilde{r}^3) \] (3)

\[ U_g^6|_{\partial X} = e_2 \] (4)

\[ \frac{\partial U_g^6}{\partial r} = \frac{1}{9} H e_2 + e_3 \] (5)
§5. More outline of proof of Step 2

We then use the estimates

\[ Y_a \left( \int_X U^{12} \right)^{1/2} \leq \int_X |\nabla U^3|^2 \]  \hspace{1cm} (6)

\[ Y_b \left( \int_{\partial X} U^9 \right)^{1/3} \leq \int_X |\nabla U^3|^2 \]  \hspace{1cm} (7)

while

\[ \frac{5}{9} \int_X |\nabla U^3|^2 d\nu_g = - \int_X (\Delta_g U) U^5 + \frac{1}{6} \int_{\partial X} \frac{\partial U^6}{\partial r} \]  \hspace{1cm} (8)

\[ \leq c \int_X |W|_g U^6 + \frac{1}{6} \int_{\partial X} \frac{\partial U^6}{\partial r} \]  \hspace{1cm} (9)

\[ \leq c \left( \int_X |W|_g^2 \right)^{1/2} \left( \int_X U^{12} \right)^{1/2} + \frac{1}{6} \int_{\partial X} \frac{\partial U^6}{\partial r} \]  \hspace{1cm} (10)
§6. More outline of proof of Step 2

Combine (6) and (7) and estimate in (4) and (5) of $U_g^6$ and $\frac{\partial U_g^6}{\partial r}$ on $\partial X$, we get

$$\left(\frac{5}{18} Y_a - c \| W \|_2 \right) \left( \int_X U^{12} \right)^{1/2} + Y_b \| S \|_3^2$$

(11)

$$\leq \int_X \left| S \hat{\nabla} \hat{C} \right| + \| \hat{E} \|_{3/2}^2 + \| \hat{Ric} \|_2 \| \hat{\nabla} \hat{C} \|_2 + \| \hat{Ric} \|_4^2 \| \hat{\nabla} \hat{C} \|_4^2$$

(12)

Thus under the assumption

$$\left( \star \star \right) \quad \frac{5}{18} Y_a - c \| W \|_2 > 0$$

(13)

we get

$$\| S \|_3 \leq C(\tilde{C}_4, \delta)$$

where $\tilde{C}_4$ is $C^4$ norm of $h$ and $Y(M, [h]) \geq \delta > 0$, since $Y_b \gtrsim \sqrt{\delta}$. 
Congratulations, Richard, for your fantastic life long achievement! May you have many more productive years to come!!