Fredholm theory for scattering on asymptotically conic spaces and applications

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May 2024, Paris Saclay

We consider generalizations of Euclidean resolvent estimates, in a Fredholm framework... relevant for e.g. wave equation asymptotics. Indeed, one motivation is understanding waves on Kerr spacetimes.

These are already interesting in explaining the Euclidean phenomena: can one phrase the 'limiting absorption principle' as a Fredholm problem?

Structure:

- Euclidean problems/geometric generalization
- Microlocal Fredholm analysis
- Low energy limit

As an illustration, consider Kerr spacetime, which is an asymptotically Minkowski Lorentzian spacetime.

Kerr spacetime is a family, depending on two parameters (m, a), of Lorentzian metrics on

$$\mathbb{R}_{\mathfrak{t}} \times (0, \infty)_r \times \mathbb{S}^2$$

satisfying Einstein's equation with vanishing cosmological constant. There is a black hole event horizon (cf. domain of dependence) at $r = r_-$ for a suitable r_- (concretely $r_- = 2m$ if $a = 0$), and one is most interested in the behavior of waves in $r > r_- - \epsilon$, $\epsilon > 0$ fixed. Also, $\mathfrak{t}$ is a suitable modification of the standard 'time' function t that is equal to the standard t for r large.

The basic stability question is what happens if one perturbs initial data for the (nonlinear!) Einstein's equation around those of Kerr. There is a subtlety as these need to satisfied compatibility conditions, called the constraint equations.

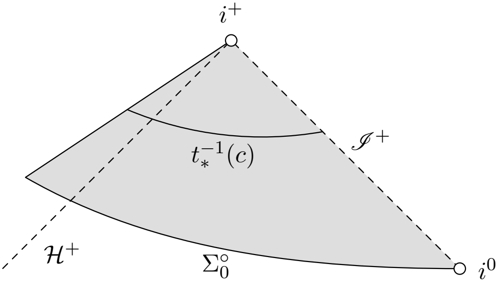


Figure: Part of the Penrose diagram of a Kerr spacetime: the event horizon $\mathcal{H}^+$, null infinity $\mathscr{I}^+$, timelike infinity i^+ and spacelike infinity i^0. We show the domain $\{\mathfrak{t} \geq 0\}$ inside of M° in gray, the Cauchy surface $\Sigma_0^\circ = \mathfrak{t}^{-1}(0)$, and a level set of t_*; $t_* = t - (r + 2m\log(r - 2m))$, r large. In this picture r is an angular variable around i^+; $r = r_-$ is the horizon $\mathcal{H}^+$, $r = \infty$ is $\mathscr{I}^+$, $r = r_- - \epsilon$ is the solid boundary near $\mathcal{H}^+$.

While for positive cosmological constant Λ the analogous nonlinear stability statement has been proved in joint work with Hintz in 2016, the strongest $\Lambda = 0$ nonlinear black hole result is the very recent work of Klainerman and Szeftel under polarized axial symmetry assumptions, plus the even more recent Dafermos-Holzegel-Rodnianski-Taylor stability result when the limiting spacetime is Schwarzschild.

Thus, in $\Lambda = 0$ in full generality (no additional symmetry assumptions) only linearized results are available. These include linearized Schwarzschild, plus Teukolsky in the slowly rotating case: Dafermos, Holzegel and Rodnianski (2016, 2017), as well as the linearized stability result of Andersson, Bäckdahl, Blue and Ma (2019) also in the slowly rotating case, with also a more restricted general result, under a strong asymptotic assumption.

Recall that at the linearized level pullbacks by diffeomorphisms correspond to Lie derivatives along vector fields.

Theorem (Linearized stability of the Kerr family for small a; informal version, Häfner-Hintz-V., arXiv 2019, Inv. Math. 2021)

Let $b = (m, a)$ be close to $b_0 = (m_0, 0)$; let $\alpha \in (0, 1)$. Suppose $\dot{h}, \dot{k} \in C^\infty(\Sigma_0^\circ; S^2 T^*\Sigma_0^\circ)$ satisfy the linearized constraint equations, and decay according to $|\dot{h}(r, \omega)| \leq Cr^{-1-\alpha}$, $|\dot{k}(r, \omega)| \leq Cr^{-2-\alpha}$, together with their derivatives along $r\partial_r$ and ∂_ω (spherical derivatives) up to order 8. Let $\dot{g}$ denote a solution of the linearized Einstein vacuum equations on Ω which attains the initial data $\dot{h}, \dot{k}$ at Σ_0°. Then there exist linearized black hole parameters $\dot{b} = (\dot{m}, \dot{a}) \in \mathbb{R} \times \mathbb{R}^3$ and a vector field V on Ω, lying in a 6-dimensional space, consisting of generators of spatial translations and Lorentz boosts, such that

$$\dot{g} = \dot{g}_b(\dot{b}) + \mathcal{L}_V g_b + \dot{g}',$$

where for bounded r the tail $\dot{g}'$ satisfies the bound $|\dot{g}'| \leq C_\eta \mathfrak{t}^{-1-\alpha+\eta}$ for all $\eta > 0$.

The key analytic ingredients in the proof of this theorem are the main subject of this talk. The reduction is via a Fourier transform along the orbits of the 'time' Killing vector field $\partial_{\mathfrak{t}}$.

In fact, it is better to consider $t_* = t - (r + 2m\log(r - 2m))$ for r large as the input variable of the Fourier transform (rather than $\mathfrak{t}$); this will be reflected in our discussion below.

While I shall discuss scalar problems, the extension to systems is straghtforward. What is *not* straightforward is to deal with the additional degeneracies caused by the highly nontrivial 0-energy nullspace. Indeed, that is why this Kerr paper is much longer than the sum of the two papers on which the main part of this talk is based!

We consider $H = \Delta_g + V$ where

- g_0 be the Euclidean metric,
- g metric on $\mathbb{R}^n$ with $g - g_0 \in S^{-\delta}$, $\delta > 0$ (i.e. $g_{ij} - (g_0)_{ij} \in S^{-\delta}$), g positive definite,
- $V \in S^{-\delta}$, real. (Im $V \in S^{-1-\delta}$ is OK for Fredholm.)

The space of symbols $S^m(\mathbb{R}^n_z)$, which is also used to capture asymptotically Euclidean behavior in geometric problems is: $\forall\alpha$

$$|D_z^\alpha a(z)| \leq C_\alpha \langle z\rangle^{m-|\alpha|}, \qquad \langle z\rangle = (1+|z|^2)^{1/2}.$$

Different way of writing it (away from origin): $\forall k$

$$|W_1 \dots W_k a(z)| \leq C_k \langle z\rangle^m,$$

where

$$W_r = z_{i_r} D_{z_{j_r}}.$$

Let

- g_0 be the Euclidean metric,
- g metric on $\mathbb{R}^n$ with $g - g_0 \in S^{-\delta}$, $\delta > 0$ (i.e. $g_{ij} - (g_0)_{ij} \in S^{-\delta}$), g positive definite,
- $V \in S^{-\delta}$, real.

Then

$$H = \Delta_g + V$$

is self-adjoint on $L^2(\mathbb{R}^n)$, so $H - \lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is invertible, e.g. as a map

$$H - \lambda : H^{s,l} \to H^{s-2,l}, \ s, l \in \mathbb{R}.$$

Moreover, the spectrum in $(-\infty, 0)$ is discrete, with 0 a possible accummulation point (e.g. Coulomb-like potentials); $[0, \infty)$ the essential spectrum.

Here: $H^{s,l} = \langle z \rangle^{-l} H^s$, H^s standard Sobolev space.

While $H-\lambda$ will no longer be invertible between the weighted Sobolev spaces when $\lambda>0$, the limiting absorption principle states that

$$(H-(\lambda\pm i0))^{-1}=\lim_{\epsilon\to 0}(H-(\lambda\pm i\epsilon))^{-1}$$

exist e.g. as limits in $\mathcal{L}(H^{s-2,l},H^{s,l'})$, $l>\frac{1}{2}$, $l'<-\frac{1}{2}$ (so $l-l'>1$).

Under stronger assumptions (Coulomb!), $V\in S^{-2-\delta}$, $\delta>0$, 0 is not an accummulation point of the spectrum, and under stronger restrictions on l,l', in particular $l-l'>2$, $(H-(\lambda\pm i0))^{-1}$ is uniformly bounded between the weighted Sobolev spaces as $\lambda\to 0$ if there are no 0-energy bound states (L^2 nullspace of H) or half-bound states (to be discussed). (Jensen, Kato,...,Fournais, Skibsted, Wang, Derezinski, Bony, Häfner, Rodnianski, Tao and N-body analogues, e.g. Wang, Skibsted, Tamura, as well as geometric microlocal analysis parametrix construction: Guillarmou, Hassell, Sikora)

What kind of structure of Euclidean space is involved? One way to address is via geometric generalizations.

A conic metric, with cross section a Riemannian manifold (Y, h), is the metric $g_0 = dr^2 + r^2 h$ on $\mathbb{R}^+_r \times Y$. Thus $\mathbb{R}^n \setminus \{0\}$ is a cone over $\mathbb{S}^{n-1}$; the results presented here generalize to asymptotic cones, where 'asymptotic' is in a similar sense as in Euclidean space.

Notice that general conic cross sections can be locally identified with subsets of $\mathbb{S}^{n-1}$ thus all function and (e.g. differential) operator spaces can be transported from $\mathbb{R}^n$ to general asymptotic cones.

Can one make the function spaces more precise? For instance, can one fit these into a Fredholm (here typically invertible) statement? Such frameworks are necessarily sharp in a sense.

Here one necessarily must have different domains/target spaces for $H-\lambda$ for the cases producing the inverses $(H-(\lambda\pm i0))^{-1}$. What are these?

It is useful to write the spectral parameter as $\lambda=\sigma^2$, with $\operatorname{Im}\sigma>0$ corresponding to $\lambda\in\mathbb{C}\setminus[0,\infty)$, and let

$$P(\sigma)=H-\sigma^2=\Delta_g+V-\sigma^2.$$

We then are interested in

$$P(\sigma+i0)^{-1}=\lim_{\epsilon\to 0}P(\sigma+i\epsilon)^{-1};$$

note that the limits with $\lambda\pm i0$ become $\pm\sigma+i0$ for $\sigma\in\mathbb{R}$.

Recall:

$$P(\sigma + i0)^{-1} = \lim_{\epsilon \to 0} P(\sigma + i\epsilon)^{-1}$$

exist e.g. as limits in $\mathcal{L}(H^{s-2,l}, H^{s,l'})$, $l > \frac{1}{2}$, $l' < -\frac{1}{2}$ (so $l - l' > 1$).

This is *not* a sharp estimate, even though it is sharp on the standard scale of weighted Sobolev spaces; the point is that this is *not a satisfactory scale*.

One way to see this is the $1+\epsilon$, $\epsilon > 0$, order of loss of decay (cf. one derivative loss in hyperbolic PDE relative to elliptic ones); one expects the loss of 1 order if *done right*.

Moreover, one would like to have a more precise description of the output of $P(\sigma)^{-1}$ for well-behaved inputs: should have the outgoing spherical wave form $e^{i\sigma/x}(\ldots)$ where $x = r^{-1}$, r the 'radius'. I will often use $\rho = r$ below to avoid notational conflicts.

The reason for the non-optimality is that phase space behavior is not taken into account.

To see what this looks like, consider scattering pseudodifferential operators of which $P(\sigma)$ is an example. These have a symbol calculus both in the position z and in the momentum ζ.

The class $\Psi^{m,l}$ of scattering pseudodifferential operators of Melrose, going back to Shubin and Parenti in non-geometric settings, arises from quantizing symbols $S^{m,l}$ satisfying

$$|D_z^\alpha D_\zeta^\beta a(z,\zeta)| \leq C_{\alpha\beta}\langle z\rangle^{l-|\alpha|}\langle\zeta\rangle^{m-|\beta|};$$

here m is the differential and l is the decay order.

One quantizes these symbols as

$$(\mathrm{Op}(a)u)(z) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(z-z')\cdot\zeta} a(z,\zeta)\, u(z')\, dz'\, d\zeta,$$

interpreted as an oscillatory integral, though...

...in this case it converges if u is Schwartz and is written as

$$(\mathrm{Op}(a)u)(z) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(z-z')\cdot\zeta} a(z,\zeta)\,(\mathcal{F}u)(\zeta)\,d\zeta.$$

The symbols of differential operators are polynomials in ζ, and in particular differential operators with symbolic coefficients

$$\sum_{|\alpha|\leq m} a_\alpha(z) D_z^\alpha, \qquad a_\alpha \in S^l(\mathbb{R}^n),$$

correspond to

$$a(z,\zeta) = \sum_{|\alpha|\leq m} a_\alpha(z)\zeta^\alpha.$$

These pseudodifferential operators form a filtered *-algebra, so

- $\Psi^{m,l} \circ \Psi^{m',l'} \subset \Psi^{m+m',l+l'}$,
- $(\Psi^{m,l})^* = \Psi^{m,l}$.

Just as importantly, one can compute the composition and adjoints modulo 'trivial' operators in $\Psi^{-\infty,-\infty}$, which map any weighted Sobolev space to any other.

For instance, to leading order, both in ζ and in z decay, $\mathrm{Op}(a)\,\mathrm{Op}(b)$ is given by $\mathrm{Op}(ab)$! One calls $[a]$, the class of a in $S^{m,l}/S^{m-1,l-1}$, the principal symbol $\sigma_{m,l}(A)$ of $A = \mathrm{Op}(a)$. A is elliptic if $\sigma_{m,l}(A)$ is invertible.

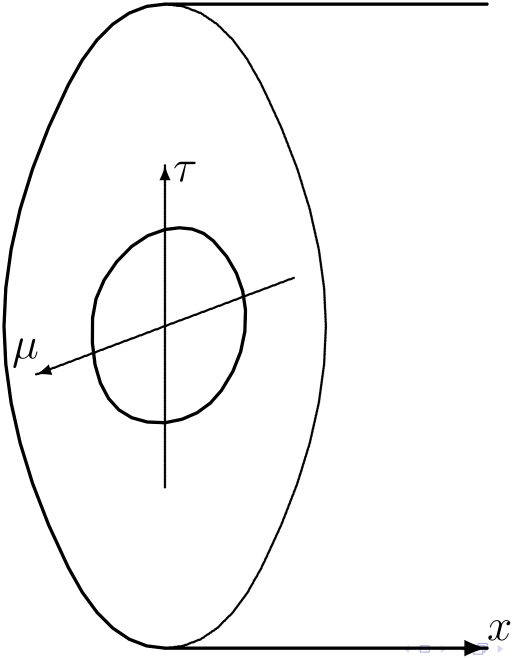
Moreover, $\Psi^{m,l} \subset \mathcal{L}(H^{s,r}, H^{s-m,r-l})$ for all $s, r \in \mathbb{R}$. Indeed, one could *define* $H^{s,r}$ as consisting of tempered distributions u for which $Au \in L^2$ for some *elliptic* $A \in \Psi^{s,r}$.

This transports to asymptotic cones just like the Sobolev spaces.

Now, $\Delta - \sigma^2$ has principal symbol $|\zeta|^2 - \sigma^2$, which vanishes for real σ at certain (finite) ζ; the issue is that this *persists* as $|z| \to \infty$, so in the spatial decay sense this operator is *not* elliptic.

Note that if we conjugate $\Delta - \sigma^2$ by the Fourier transform, this *is* a statement about the standard principal symbol since ζ becomes the 'base', and $-z$ the 'dual' variable.

Since the principal symbol is real, within the characteristic set (where the principal symbol vanishes) we expect propagation as for the wave equation.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To see this more precisely, use conic coordinates, $x = \rho^{-1} = |z|^{-1}$, y coordinates on the cross-section, so the conic metric is

$$d\rho^2 + \rho^2 h = \frac{dx^2}{x^4} + \frac{h}{x^2},$$

h a metric on the cross section. So, if we write covectors as

$$\zeta\, dz = \tau \frac{dx}{x^2} + \mu \frac{dy}{x},$$

the dual metric function is

$$|\zeta|^2 = \tau^2 + |\mu|_h^2;$$

this, minus σ^2, is the principal symbol of $P(\sigma)$:

$$p(\sigma) = \tau^2 + |\mu|_h^2 - \sigma^2.$$

As in wave propagation, within the characteristic set, $p(\sigma) = 0$, the key issue is the behavior of the Hamilton flow. This is the flow of H_p, or more precisely of

$$\mathsf{H}_p = \rho H_p = x^{-1} H_p.$$

- p is homogeneous of degree zero in the spatial (x) dilations, so H_p is homogeneous of degree -1; the above rescaling gives a homogeneous degree zero vector field.
- Here the H_p-flow has a source/sink structure, with $\tau = \pm\sigma$, $\mu = 0$ being the source ($+$ if $\sigma > 0$) or sink ($-$ if $\sigma > 0$).
- $(\sigma, 0)$ is the phase-space location of $e^{-i\sigma/x}$, and $(-\sigma, 0)$ of $e^{i\sigma/x}$:

$$(x^2 D_x \mp \sigma) e^{\mp i\sigma/x} = 0, \;\; x D_y e^{\mp i\sigma/x} = 0,$$

  and the principal symbols of these two operators are

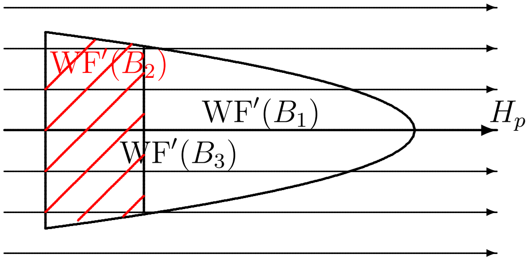
$$\tau \mp \sigma, \;\; \mu.$$

- $e^{\mp i\sigma/x}$ are the incoming ($-$) and outgoing ($+$) spherical wave phase functions.

Away from the source/sink, Hörmander's propagation of singularities theorem applies (as extended by Melrose, but this particular case reduces to the standard one via the Fourier transform). This gives estimates like

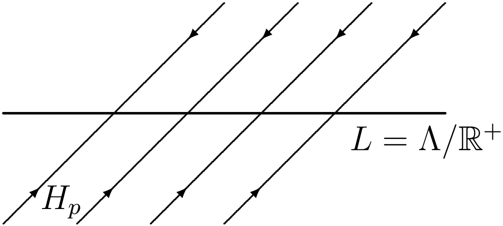
$$\|B_1 u\|_{H^{*,r}} \leq C(\|B_2 u\|_{H^{*,r}} + \|B_3 Pu\|_{H^{*,r+1}} + \|u\|_{H^{*,-N}})$$

as illustrated.



At the source/sink:

- $B_2 u$ can be dropped if $r > -1/2$ (a threshold value);
- if $r < -1/2$ then one can propagate estimates in this form from a punctured neighborhood of the radial set to the radial set itself.



Issue: being Fredholm needs estimates for both P and P^* on *dual* spaces, so

- for both we need high regularity (as measured by decay) at either the source or sink,
- which means low regularity at the same place for the dual,
- so we need $r > -1/2$, say, at source, $r < -1/2$ at the sink,
- so the decay order needs to be variable.

There are such variable order (or anisotropic) Sobolev spaces (going back to Unterberger, Duistermaat...), and indeed can be defined via variable order A, essentially $\mathrm{Op}(\langle\zeta\rangle^s\langle z\rangle^r)$, but $r = r(z,\zeta)$ is a homogeneous degree 0 function in z: $u \in H^{s,r}$ if for such elliptic variable order A, $Au \in L^2$. (Other uses of variable order spaces: Anosov dynamical systems: Faure, Sjöstrand, Dyatlov, Zworski, Guillarmou...)

The results then extend to:

Theorem (V, 2017)

$$P(\sigma) : \{u \in H^{s,r} : \; P(\sigma)u \in H^{s-2,r+1}\} \to H^{s-2,r+1}$$

is invertible (in particular Fredholm), provided r is monotone along the H_p-flow in the characteristic set, $< -1/2$ at one of the source/sink, $> -1/2$ at the other.

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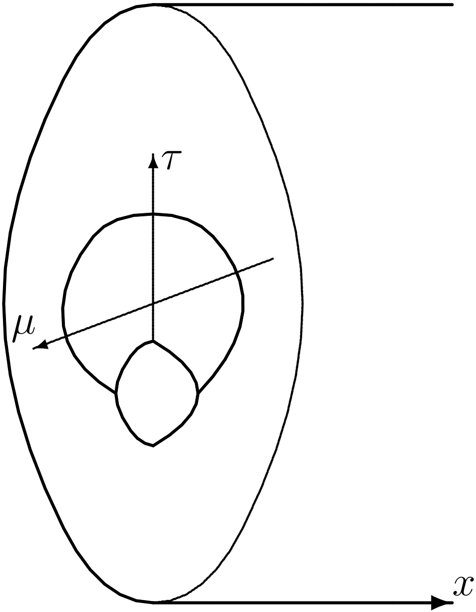
is invertible (in particular Fredholm), provided r is monotone along the H_p-flow in the characteristic set, $< -1/2$ at one of the source/sink, $> -1/2$ at the other.

We do *not* need to make sense of the limiting absorption principle resolvent as a limit; it is an honest Fredholm problem, thus sharp!

Here we can make r high everywhere except in a small neighborhood of the sink, say.

But shouldn't we be able to make it high everywhere but *at* the sink? Here comes 2-microlocalization.

Informally, 2-microlocalization blows up (resolves) the phase space, and a version goes back to Bony in the 80s.



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Informally, 2-microlocalization blows up (resolves) the phase space.

- Here we blow up the outgoing radial set, which creates a new boundary hypersurface.
- Symbolic orders, as well as Sobolev space orders, arise from order of vanishing at the boundary hypersurfaces.
- Thus, we have three orders now: differentiability, general decay (call it sc-decay) and outgoing decay (call it b-decay): $H^{s,r,l}$
- We can have $r > -1/2$, $l < -1/2$ constant.

Theorem (V., 2019)

$$P(\sigma) : \{u \in H^{s,r,l} : \ P(\sigma)u \in H^{s-2,r+1,l+1}\} \to H^{s-2,r+1,l+1}$$

is invertible.

More precisely, what are these spaces?

One can conjugate

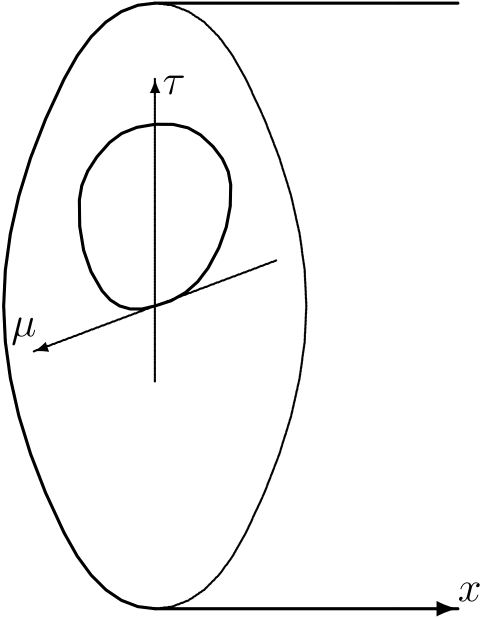
$$\hat{P}(\sigma) = e^{-i\sigma/x} P(\sigma) e^{i\sigma/x}.$$

When applied to 'nice' functions (e.g. Schwartz), the expected behavior of $P(\sigma)^{-1}$ is $e^{i\sigma/x}$ times an expansion, namely conormality, which for $\hat{P}(\sigma)^{-1}$ means just conormality!

The effect of the conjugation is, at the phase space level,

- replacing τ by $\tau - \sigma$ (leaving x, y, μ unchanged),
- thus moving the outgoing radial set, with $\tau = -\sigma$, to the zero section,
- so the new principal symbol is

$$(\tau - \sigma)^2 + |\mu|^2 - \sigma^2 = \tau^2 + |\mu|^2 - 2\tau\sigma.$$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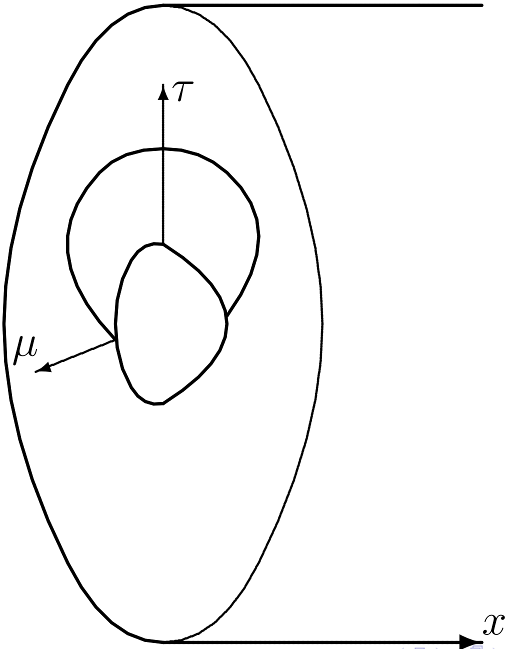
The blow up of the zero section $\tau = 0$, $\mu = 0$, at $x = 0$, in projective coordinates is

$$\tau_{\mathrm{b}} = \frac{\tau}{x},\ \mu_{\mathrm{b}} = \frac{\mu}{x}.$$

But

- τ is the principal symbol of $x^2 D_x = -D_\rho$, μ of $xD_y = \frac{1}{\rho} D_y$,
- so τ_{b} corresponds to $xD_x = -\rho D_\rho$, μ_{b} to D_y,
- regularity with respect to $\tau_{\mathrm{b}}, \mu_{\mathrm{b}}$ is conormality, or b-regularity (regularity with respect to vector fields tangent to the boundary $x = 0$).

With this insight, we can think of phase space in a different way, starting with the b-phase space and blowing up the corner.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With this insight, we can think of phase space in a different way, starting with the b-phase space and blowing up the corner.
Advantage:

- Symbolic regularity is unaffected by the blow-up of a corner, so we get a non-classical version of Melrose's b-ps.d.o. algebra (we can simply generalize orders as well),
- which gives a good symbol calculus,
- except at finite $\tau_{\mathrm{b}}, \mu_{\mathrm{b}}$.
- But for that one can proceed as in other b-situations, considering the normal operator, i.e. a model at the boundary!

Explicitly:

$$K_{\mathrm{Op}_{\mathrm{b}}(a)} = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\left(\frac{x-x'}{x}\tau_{\mathrm{b}} + (y-y')\mu_{\mathrm{b}}\right)} \phi\Big(\frac{x-x'}{x}\Big) a(x, y, \tau_{\mathrm{b}}, \mu_{\mathrm{b}})\, d\tau_{\mathrm{b}}\, d\mu_{\mathrm{b}},$$

and one needs to add smooth Schwartz kernels $K_R(x, \frac{x'}{x}, y, y')$, rapidly vanishing at 0 and infinite in the second variable.

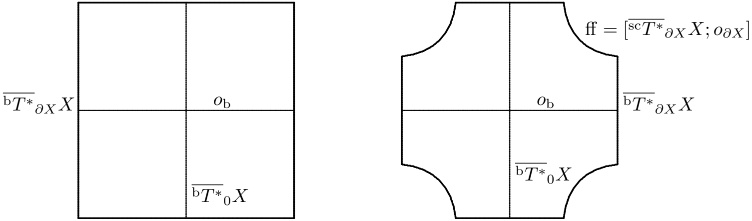

Figure: The second microlocal space, on the right, obtained by blowing up the corner of $\overline{{}^{b}T^*X}$, shown on the left.

Another perspective: if Schwartz in (τ, μ) at infinity, i.e. order $-\infty$ in the differential sense as a scattering ps.d.o.,

$$\int_{\mathbb{R}^n} e^{i\left(\frac{x-x'}{x^2}\tau+\frac{y-y'}{x}\mu\right)}\phi\left(\frac{x-x'}{x}\right)a(x,y,\tau/x,\mu/x,\tau,\mu)\,d\tau\,d\mu$$

$$\sim\int_{\mathbb{R}^{2n}} e^{i\left(\frac{1-\frac{x'}{x}-\bar{s}}{x}\tau+\frac{y-y'-\bar{y}}{x}\mu\right)}\phi\left(\frac{x-x'}{x}\right)\check{a}(x,y,\bar{s},\bar{y},\tau,\mu)\,d\bar{s}\,d\bar{y}\,d\tau\,d\mu,$$

$$\int_{\mathbb{R}^{2n}} e^{i\left(\frac{1-\frac{x'}{x}-\bar{s}}{x}\tau+\frac{y-y'-\bar{y}}{x}\mu\right)}\phi\Big(\frac{x-x'}{x}\Big)\check{a}(x,y,\bar{s},\bar{y},\tau,\mu)\,d\bar{s}\,d\bar{y}\,d\tau\,d\mu$$

a symbolic in the third and fourth variables, and Schwartz in (τ,μ) (overall decay included in x, relative decay in $(\tau_{\mathrm{b}},\mu_{\mathrm{b}})$), so $\check{a}$ is conormal to 0 in $(\bar{s},\bar{y})$. Intersecting Legendrians: Melrose-Zworski, Hassell-V after Melrose-Uhlmann and Guillemin-Uhlmann.

Since $\hat{P}(\sigma)$ is like

$$(x^2D_x)^2+(xD_y)^2-2\sigma\Big(x^2D_x+i\frac{n-1}{2}x\Big)$$

and the first two terms are $O(x^2)$ as b-differential operators, the normal operator is

$$-2\sigma\Big(x^2D_x+i\frac{n-1}{2}x\Big).$$

The normal operator of $\hat{P}(\sigma)$ is

$$-2\sigma\Big(x^2 D_x + i\frac{n-1}{2}x\Big).$$

Factor out $-2x$:

$$\sigma\Big(xD_x + i\frac{n-1}{2}\Big).$$

Mellin transform

$$\sigma\Big(\tau_{\mathrm{b}} + i\frac{n-1}{2}\Big),$$

or really

$$\sigma\Big(\tau_{\mathrm{b}} - i\frac{1}{2}\Big)$$

after weights are taken into account, so invertible as long as not borderline, which means $l \neq -1/2$.

The propagation estimates still hold, so one has a Fredholm theory, proving the main theorem!

The high energy version, i.e. $|\sigma| \to \infty$ holds via 'standard' semiclassical rescaling.

The low energy version, $\sigma \to 0$, is more delicate as can be seen by the vanishing of the normal operator (but *not* the whole operator) at $\sigma = 0$.

This low energy analysis requires yet another blow up at $\sigma = 0, x = 0$, but then it gives uniform results on the resolved spaces H_{res}. If $n \geq 3$, $P(0)$ invertible on the approrpiate spaces, it *implies* a uniform estimate

$$\|(x+|\sigma|)^\alpha u\|_{H^{s,s+l-1,l}} \leq C\|(x+|\sigma|)^\alpha \hat{P}(\sigma)u\|_{H^{s-2,s+l,l+2}},$$

where $\alpha \in (l+1-\frac{n-2}{2}, l+1+\frac{n-2}{2})$, $l < -1/2$, $r > -1/2$, $s = r - l + 1$.

In combination, this gives the analytic framework for linearized Kerr stability, as in the recent work with Häfner and Hintz.

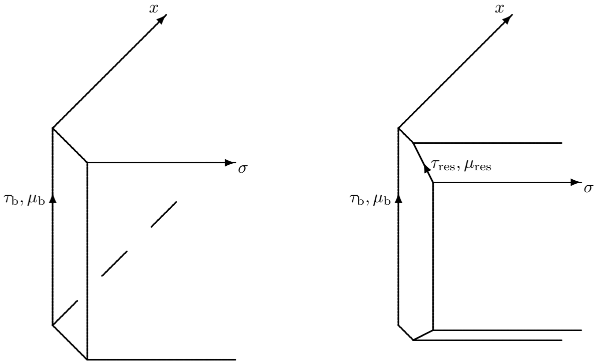

Figure: The resolved b-cotangent bundle on the left, and its scattering-b resolution on the right obtained by blowing up the corner $x/\sigma = 0$ at fiber infinity (nearest horizontal edges) of the resolved b-cotangent bundle. At the pseudodifferential operator level the symbolic calculus works at resolved b-fiber infinity which is the top (as well as bottom!) face on both pictures, as well as new face on the right picture, which corresponds to rescaled sc-decay.

Theorem (V., 2019)

Let $|l'+1| < \frac{n-2}{2}$*, and suppose that* $P(0) : H^{\infty,\infty,l'} \to H^{\infty,\infty,l'+2}$ *has trivial nullspace, an assumption independent of* l' *in this range. Suppose also that either* $r > -1/2$, $l < -1/2$, *or* $r < -1/2$, $l > -1/2$.
There exists $\sigma_0 > 0$ *such that*

$$\hat{P}(\sigma) : \{u \in H^{s,r,l}_{\mathrm{res}} : \ \hat{P}(\sigma)u \in H^{s-2,r+1,l+1}_{\mathrm{res}}\} \to H^{s-2,r+1,l+1}_{\mathrm{res}}$$

is invertible for $0 < |\sigma| \le \sigma_0$, $\operatorname{Im}\sigma \ge 0$, *and we have the estimate*

$$\|(x+|\sigma|)^\alpha u\|_{H^{s,r,l}_{\mathrm{res}}} \le C\|(x+|\sigma|)^{\alpha-1}\hat{P}(\sigma)u\|_{H^{s-2,r+1,l+1}_{\mathrm{res}}}$$

for

$$\alpha \in \left(l+1-\frac{n-2}{2}, l+1+\frac{n-2}{2}\right).$$

Happy Birthday, Maciej!