Paris-Saclay Conference in Analysis and PDE

In Honor of Maciej Zworski's 60th Birthday

Inverse Problem for Nonlinear Equations

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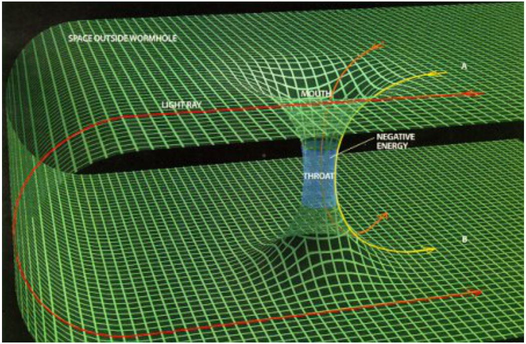
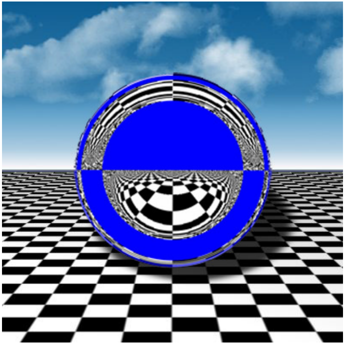
Orsay, May 27, 2024

Goal: To Determine the Topology and Metric of Space-Time





How can we determine the topology and metric of complicated structures in space-time with a radar-like device?

Figures: Anderson institute and Greenleaf-Kurylev-Lassas-U.

Non-linearity Helps

We will consider inverse problems for non-linear wave equations, e.g.

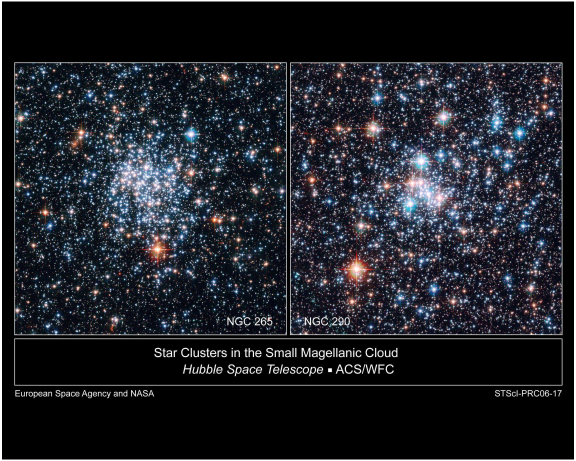
$$\frac{\partial^2}{\partial t^2}u(t,y) - c(t,y)^2\Delta u(t,y) + a(t,y)u(t,y)^2 = f(t,y).$$

We will show that:

-Non-linearity helps to solve the inverse problem,

-"Scattering" from the interacting wave packets determines the structure of the spacetime.

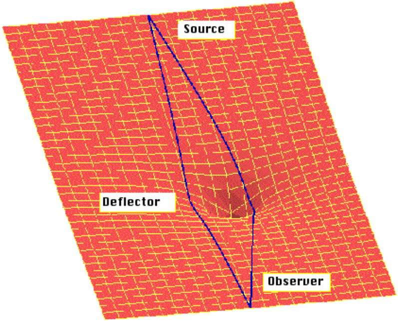
Inverse Problems in Space-Time: Passive Measurements

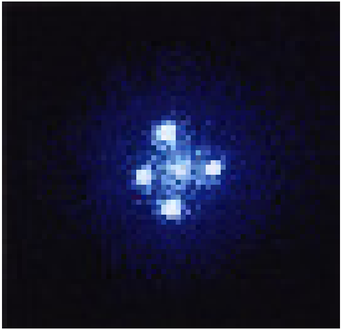


Can we determine the structure of space-time when we see light coming from many point sources varying in time? We can also observe gravitational waves.

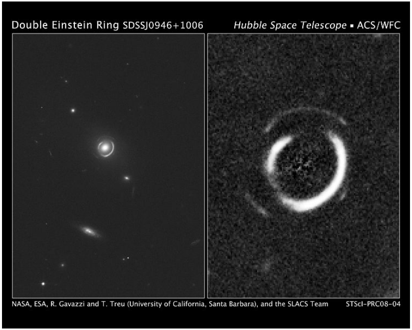
Gravitational Lensing

We consider e.g. light or X-ray observations or measurements of gravitational waves.

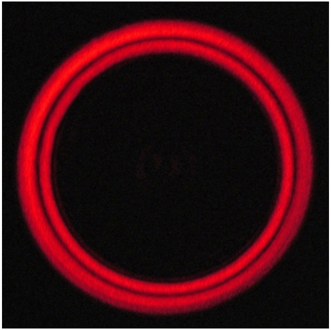




Gravitational Lensing



Double Einstein Ring

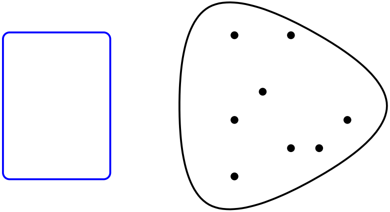


Conical Refraction

Passive Measurements: Gravitational Waves

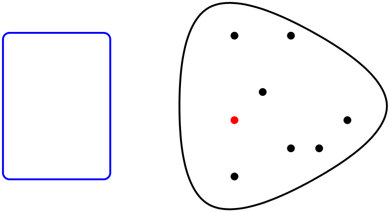
NSF Announcement, Feb 11, 2015

Inverse Problem for Passive Measurements



Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



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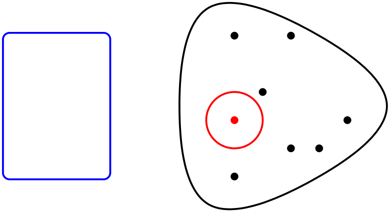

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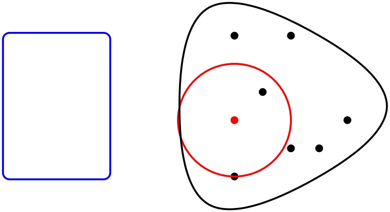

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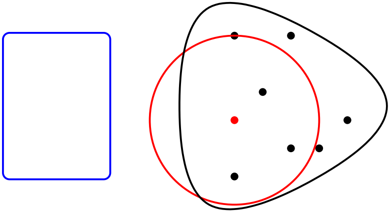

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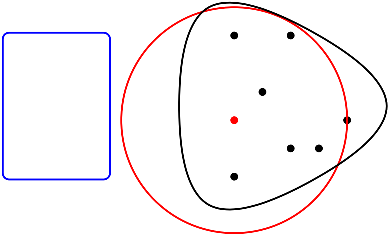

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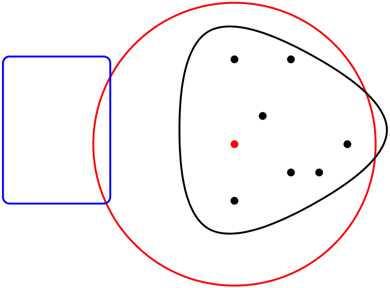

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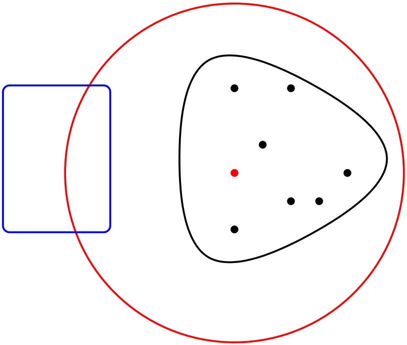

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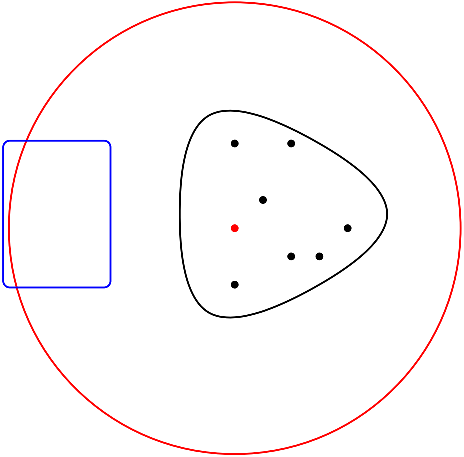

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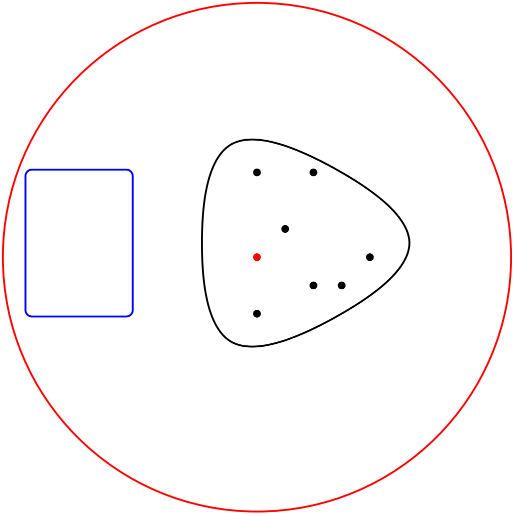

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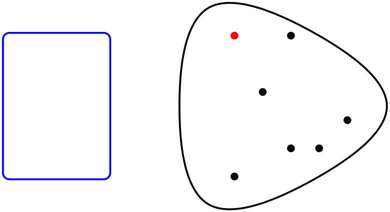

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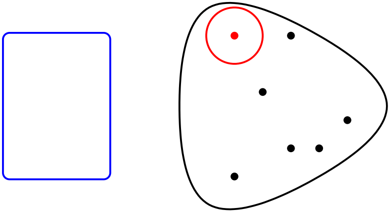

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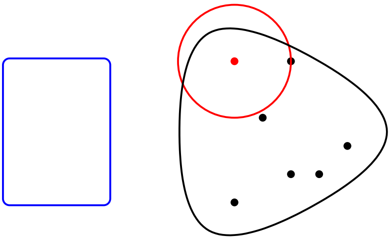

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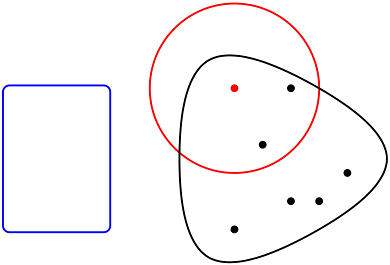

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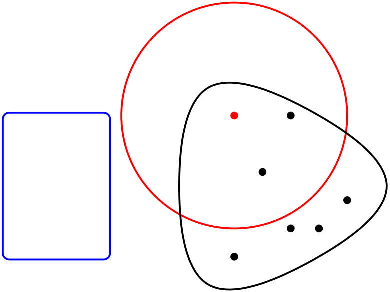

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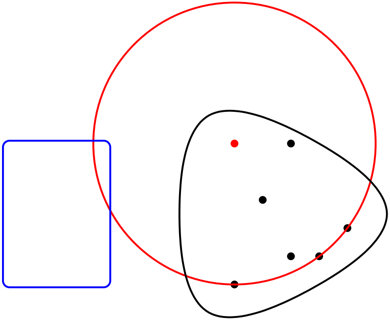

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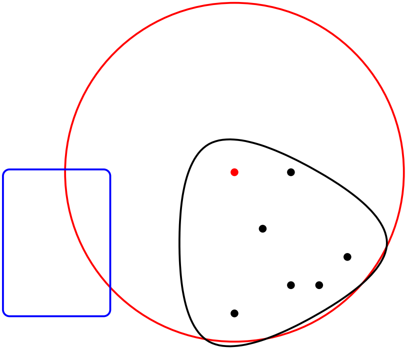

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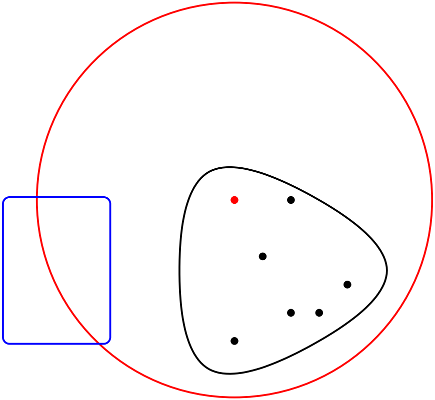


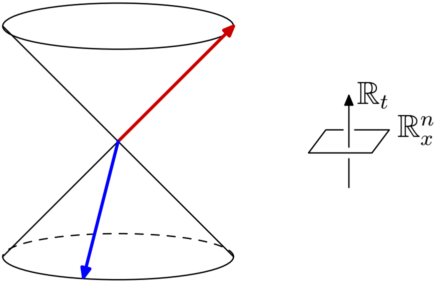
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Lorentzian Geometry

$(n+1)$-dimensional Minkowski space: (M, g)

$$M = \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x, \quad \text{metric: } g = -dt^2 + dx^2.$$

Null/lightlike vectors: $V \in T_qM$ with $g(V,V) = 0$.



$L^{\pm}_q M$: future/past null vectors

Lorentzian Geometry

In general:

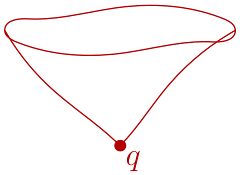
$$M = (n+1)\text{-dimensional manifold},\ g \text{ Lorentzian } (-,+,\dots,+).$$

Assume: existence of time orientation.

$$T_qM \cong (\mathbb{R}^{1+n}, \text{Minkowski metric}).$$

Null-geodesics: $\gamma(s) = \exp_q(sV)$, $V \in T_qM$ null.

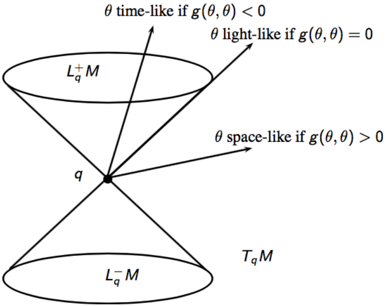
Future light cone: $\mathcal{L}_q^+ = \{\exp_q(V) \colon V \text{ future null}\}$



Lorentzian Manifolds

Let (M, g) be a $1+3$ dimensional time oriented Lorentzian manifold.
The signature of g is $(-,+,+,+)$.
Example: Minkowski space-time $(\mathbb{R}^4, g_m)$, $g_m = -dt^2 + dx^2 + dy^2 + dz^2$.

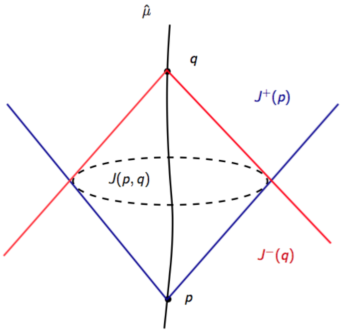
- $L_q^{\pm} M$ is the set of future (past) pointing light like vectors at q.
- Casual vectors are the collection of time-like and light-like vectors.
- A curve γ is time-like (light-like, causal) if the tangent vectors are time-like (light-like, causal).



Causal Relations

Let $\widehat{\mu}$ be a time-like geodesic, which corresponds to the world-line of an observer in general relativity. For $p, q \in M$, $p \ll q$ means p, q can be joined by future pointing time-like curves, and $p < q$ means p, q can be joined by future pointing causal curves.

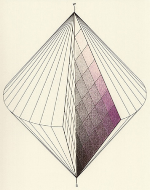
- The chronological future of $p \in M$ is $I^+(p) = \{q \in M : p \ll q\}$.
- The causal future of $p \in M$ is $J^+(p) = \{q \in M : q < p\}$.
- $J(p, q) = J^+(p) \cap J^-(q)$, $I(p, q) = I^+(p) \cap I^-(q)$.



Global Hyperbolicity

A Lorentzian manifold (M, g) is globally hyperbolic if

- there is no closed causal paths in M;
- for any $p, q \in M$
  and $p < q$, the set $J(p, q)$ is compact.



Then hyperbolic equations are well-posed on (M, g)
Also, (M, g) is isometric to the product manifold

$$\mathbb{R} \times N \text{ with } g = -\beta(t, y)dt^2 + \kappa(t, y).$$

Here $\beta : \mathbb{R} \times N \to \mathbb{R}_+$ is smooth, N is a 3 dimensional manifold and κ is a Riemannian metric on N and smooth in t.
We shall use $x = (t, y) = (x_0, x_1, x_2, x_3)$ as the local coordinates on M.

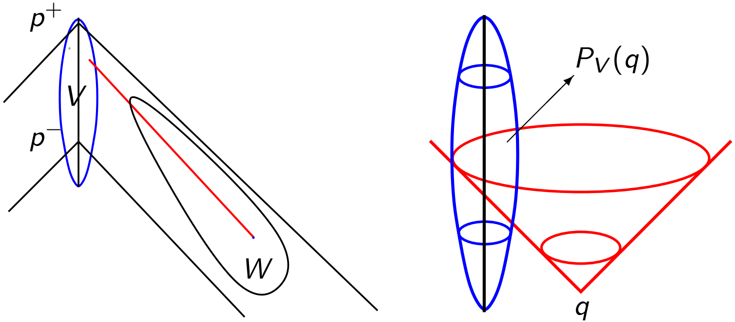
Light Observation Set

Let $\mu = \mu([-1,1]) \subset M$ be time-like geodesics containing p^- and p^+. We consider observations in a neighborhood $V \subset M$ of μ.

Let $W \subset I^-(p^+) \setminus J^-(p^-)$ be relatively compact and open set.

The light observation set for $q \in W$ is

$$P_V(q) := \{\gamma_{q,\xi}(r) \in V;\ r \geq 0,\ \xi \in L_q^+ M\}.$$



Inverse Problems with Passive Measurements

The earliest light observation set of $q \in M$ in V is

$$\mathcal{E}_V(q) = \{x \in \mathcal{P}_V(q) : \text{ there is no } y \in \mathcal{P}_V(q) \text{ and future pointing time like path } \alpha \text{ such that } \alpha(0) = y \text{ and } \alpha(1) = x\} \subset V.$$

In the physics literature the light observation sets are called light-cone cuts (Engelhardt-Horowitz, arXiv 2016)

Theorem (Kurylev-Lassas-U 2018, arXiv 2014)

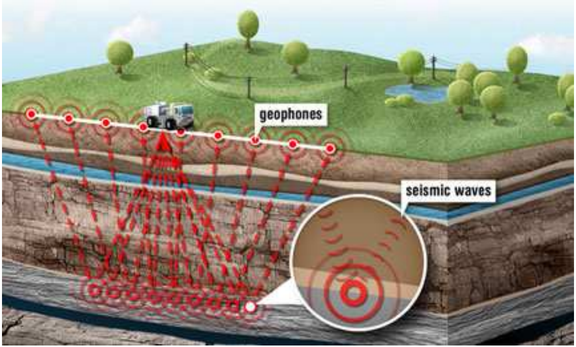
Let (M, g) be an open smooth globally hyperbolic Lorentzian manifold of dimension $n \geq 3$ and let $p^+, p^- \in M$ be the points of a time-like geodesic $\widehat{\mu}([-1, 1]) \subset M$, $p^\pm = \widehat{\mu}(s_\pm)$. Let $V \subset M$ be a neighborhood of $\widehat{\mu}([-1, 1])$ and $W \subset M$ be a relatively compact set. Assume that we know

$$\mathcal{E}_V(W).$$

Then we can determine the topological structure, the differential structure, and the conformal structure of W, up to diffeomorphism.

Inverse Problems for Linear Hyperbolic Equations

- Rakesh-Symes 1987: Inverse problem for $\partial_t^2 - \Delta + q$.
- Belishev-Kurylev 1992 and Tataru 1995: Reconstruction of a Riemannian manifold with time-independent metric.
- Unique continuation needed for Belishev-Kurylev-Tataru results fail for time-depending wave speed.



Active Measurements

Wave equation: Let $g = [g_{jk}(y)]_{j,k=1}^n$ and $u = u^f(y,t)$ be the solution of

$$
\begin{aligned}
&(\partial_t^2 u - \Delta_g)u = 0 \quad \text{on } N \times \mathbb{R}_+,\\
&u|_{\partial N \times \mathbb{R}_+} = f,\\
&u|_{t=0} = 0, \quad u_t|_{t=0} = 0.
\end{aligned}
$$

Here N is a Riemannian manifold, ν is the unit normal of ∂N,

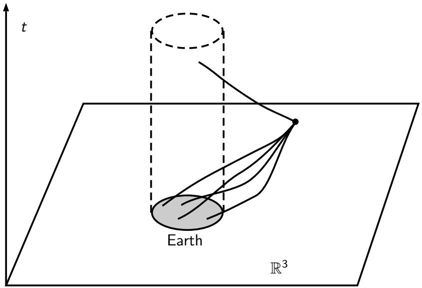
$$\Delta_g u = \sum_{j,k=1}^n |g|^{-1/2} \frac{\partial}{\partial y^j}(|g|^{1/2} g^{jk} \frac{\partial}{\partial y^k} u),$$

where $|g| = \det(g_{ij})$ and $[g_{ij}] = [g^{jk}]^{-1}$. Let

$$\Lambda f = \partial_\nu u^f|_{\partial N \times \mathbb{R}_+}.$$

We are given boundary data $(\partial N, \Lambda)$.

Interaction of Nonlinear Waves



Inverse Problem for a Non-linear Wave Equation

Consider the non-linear wave equation

$$\square_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } M^0 = (-\infty, T) \times N,$$
$$\text{supp } (u) \subset J_g^+(\text{supp } (f)),$$

where $\text{supp}(f) \subset V$, $V \subset M$ is open,

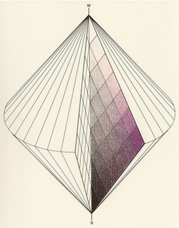
$$\square_g u = -\sum_{p,q=1}^{4} (-\det(g(x)))^{-1/2} \frac{\partial}{\partial x^p} \left((-\det(g(x)))^{1/2} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right),$$

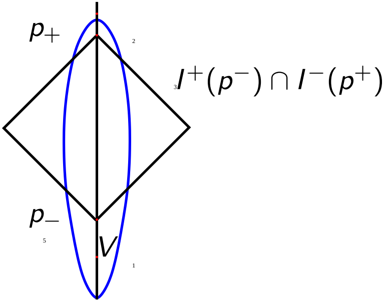
$\det(g) = \det((g_{pq}(x))_{p,q=1}^4)$, $f \in C_0^6(V)$ is a controllable source, and $a(x)$ is a non-vanishing C^∞-smooth function.
In a neighborhood $\mathcal{W} \subset C_0^2(V)$ of the zero-function, define the measurement operator by

$$L_V : f \mapsto u|_V, \quad f \in C_0^6(V).$$

Theorem (Kurylev-Lassas-U, 2018)

Let (M, g) be a globally hyperbolic Lorentzian manifold of dimension $(1+3)$. Let μ be a time-like path containing p^- and p^+, $V \subset M$ be a neighborhood of μ, and $a : M \to \mathbb{R}$ be a non-vanishing function. Then $(V, g|_V)$ and the measurement operator L_V determines the set $I^+(p^-) \cap I^-(p^+) \subset M$ and the conformal class of the metric g, up to a change of coordinates, in $I^+(p^-) \cap I^-(p^+)$.

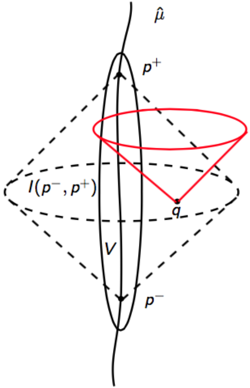




Idea of the Proof in the Case of Quadratic Nonlinearity: Interaction of Singularities

We construct the earliest light observation set by producing artificial point sources in $I(p_-, p_+)$. The key is the singularities generated from nonlinear interaction of linear waves.

- We construct sources f so that the solution u has new singularities.
- We characterize the type of the singularities.
- We determine the order of the singularities and find the principal symbols.



Non-linear Geometrical Optics

Let $u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$ satisfy

$$\square_g u + au^2 = f, \quad \text{in } M^0 = (-\infty, T) \times N,$$
$$u|_{(-\infty,0)\times N} = 0$$

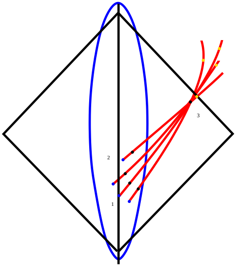
with $f = \varepsilon f_1$. When $Q = \square_g^{-1}$, we have

$$\begin{aligned} w_1 &= Qf, \\ w_2 &= -Q(a\, w_1\, w_1), \\ w_3 &= 2Q(a\, w_1\, Q(a\, w_1\, w_1)), \\ w_4 &= -Q(a\, Q(a\, w_1\, w_1)\, Q(a\, w_1\, w_1)) \\ &\quad -4Q(a\, w_1\, Q(a\, w_1\, Q(a\, w_1\, w_1))), \\ \|E_\varepsilon\| &\leq C\varepsilon^5. \end{aligned}$$

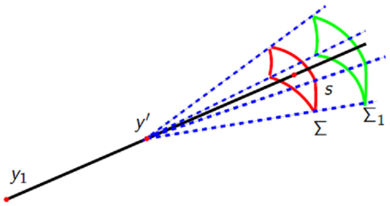
Non-linear Geometrical Optics

The product has, in a suitable microlocal sense, a principal symbol.

There is a lot of technology availale for the interaction analysis of conormal waves: intersecting pairs of conormal distributions (Melrose-U, 1979, Guillemin-U, 1981, Greenleaf-U, 1991).



Pieces of spherical waves



Consider solutions of $\square_g u_1 = f_1$, where f_1 is a conormal distribution that is singular on $\{t_0\} \times \Sigma$. The solution u_1 is a distribution associated to two intersecting Lagrangian manifolds. We can control the width s of the waves.

From $\Box_g u_1 = f_1$ we have

$$u_1 = \Box_g^{-1} f_1.$$

Thus,

$$\mathrm{WF} u_1 \subset \mathrm{WF} f_1 \cup \Lambda_p(\mathrm{WF} f_1)$$

where

$$\Lambda_p(\mathrm{WF} f_1) = \text{forward flow out by } H_p \text{ starting at } \mathrm{WF} f_1 \text{ intersected with } \{p = 0\}.$$

Here $p = \tau^2 - \sum g^{ij}(y)\xi_i\xi_j$.

H_p is the Hamiltonian vector field.

Notice that $\{p = 0\}$ is the light cone.

Interaction of Waves in Minkowski Space $\mathbb{R}^4$

Let x^j, $j = 1, 2, 3, 4$ be coordinates such that $\{x^j = 0\}$ are light-like. We consider waves

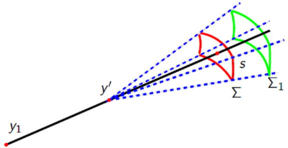
$$\begin{aligned} u_j(x) &= v \cdot (x^j)_+^m, \quad (s)_+^m = |s|^m H(s), \quad v \in \mathbb{R}, j = 1, 2, 3, 4. \\ x^j &= t - x \cdot \omega_j, \quad |\omega_j| = 1 \end{aligned}$$

Waves u_j are conormal distributions, $u_j \in I^{m+1}(K_j)$, where

$$K_j = \{x^j = 0\}, \quad j = 1, 2, 3, 4.$$

The interaction of the waves $u_j(x)$ produce new sources on

$$\begin{aligned} K_{12} &= K_1 \cap K_2, \\ K_{123} &= K_1 \cap K_2 \cap K_3 = \text{line}, \\ K_{1234} &= K_1 \cap K_2 \cap K_3 \cap K_4 = \{q\} = \text{one point}. \end{aligned}$$



Interaction of Two Waves (Second order linearization)

If we consider sources $f_{\vec{\varepsilon}}(x) = \varepsilon_1 f_{(1)}(x) + \varepsilon_2 f_{(2)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have

$$\begin{aligned} W_2(x) &= \frac{\partial}{\partial \varepsilon_1} \frac{\partial}{\partial \varepsilon_2} u_{\vec{\varepsilon}}(x)|_{\vec{\varepsilon}=0} \\ &= Q(a\, u_{(1)} \cdot u_{(2)}), \end{aligned}$$

where $Q = \square_g^{-1}$ and

$$u_{(j)} = Qf_{(j)}.$$

Recall that $K_{12} = K_1 \cap K_2 = \{x^1 = x^2 = 0\}$. Since the normal bundle N^*K_{12} contain only light-like directions $N^*K_1 \cup N^*K_2$,

$$\text{singsupp}(W_2) \subset K_1 \cup K_2.$$

Thus no new interesting singularities are produced by the interaction of two waves (Greenleaf-U, 1991).

Three plane waves interact and produce a conic wave. (Bony, 1986, Melrose-Ritter, 1987, Rauch-Reed, 1982)

Interaction of Three Waves (Third order linearization)

If we consider sources $f_{\vec\varepsilon}(x) = \sum_{j=1}^{3} \varepsilon_j f_{(j)}(x)$, $\vec\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, and the corresponding solution $u_{\vec\varepsilon}$, we have

$$\begin{aligned} W_3 &= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec\varepsilon}|_{\vec\varepsilon=0} \\ &= \quad 4Q(a\, u_{(1)}\ Q(a\, u_{(2)}\ u_{(3)})) \\ &\quad +4Q(a\, u_{(2)}\ Q(a\, u_{(1)}\ u_{(3)})) \\ &\quad +4Q(a\, u_{(3)}\ Q(a\, u_{(1)}\ u_{(2)})), \end{aligned}$$

where $Q = \Box_g^{-1}$. The interaction of the three waves happens on the line $K_{123} = K_1 \cap K_2 \cap K_3$.

The normal bundle N^*K_{123} contains light-like directions that are not in $N^*K_1 \cup N^*K_2 \cup N^*K_3$ and hence new singularities are produced.

Interaction of Four Waves (Fourth order linearization)

If we consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{4} \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have following. Consider

$$W_4 = \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}.$$

Since $K_{1234} = \{q\}$ we have $N^*K_{1234} = T_q^*M$. Hence new singularities are produced and

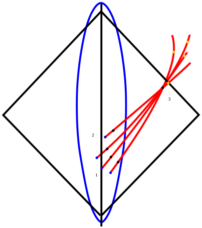
$$\text{singsupp}(W_4) \subset (\cup_{j=1}^{4} K_j) \cup \Sigma \cup \mathcal{L}_q^+ M,$$

where Σ is the union of conic waves produced by sources on K_{123}, K_{134}, K_{124}, and K_{234}. Moreover, $\mathcal{L}_q^+ M$ is the union of future going light-like geodesics starting from the point q.

Interaction of Four Waves

The 3-interaction produces conic waves (only one is shown below).

The 4-interaction produces a spherical wave from the point q that determines the light observation set $P_V(q)$.



Active and Passive Measurements

(M, g) $(2+1)$-dimensional, $\Box_g u = u^3 + f$.

Idea (Kurylev-Lassas-U 2018, arXiv 2014): Using nonlinearity to create point sources in $I(p_-, p_+)$.

$$f = \sum_{i=1}^{3} \epsilon_i f_i, \quad u_i := \Box_g^{-1} f_i.$$

Take $f_i =$ conormal distribution, e.g.

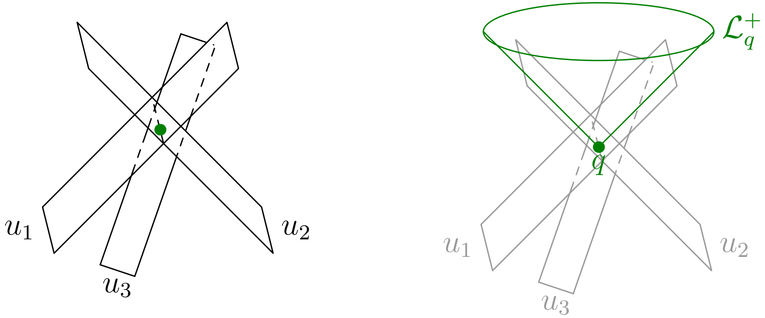
$$f_1(t,x) = (t - x_1)_+^{11} \chi(t,x), \quad \chi \in C_c^\infty(\mathbb{R}^{1+2}).$$

Then

$$u \approx \sum \epsilon_i u_i + 6\epsilon_1\epsilon_2\epsilon_3 \Box_g^{-1}(u_1 u_2 u_3).$$

Generating Point Sources

non-linear interaction of conormal waves $u_i = \Box_g^{-1} f_i$: $\Box_g^{-1}(u_1 u_2 u_3)$



$$q = \bigcap_{i=1}^{3} \mathit{sing\ supp}\ u_i, \quad \mathcal{L}_q^+ = \mathit{sing\ supp}\ \Box_g^{-1}(u_1 u_2 u_3)$$

$\Rightarrow$ singularities of $\partial^3_{\epsilon_1 \epsilon_2 \epsilon_3} u$ give light observation sets $\mathcal{L}_q^+$

Further Developments

1. Einstein's equations coupled with scalar fields (Kurylev-Lassas-U, 2013; Kurylev-Lassas-Oksanen-U, 2022)

2. Einstein-Maxwell's equations in vacuum (Lassas-U-Wang, 2017)

3. Einstein's equations (U-Wang, 2020)

4. Non-linear elasticity (de Hoop-U-Wang, 2020; U-Zhai, 2021)

5. Yang-Mills (Chen-Lassas-Oksanen-Paternain, 2021, 2022)

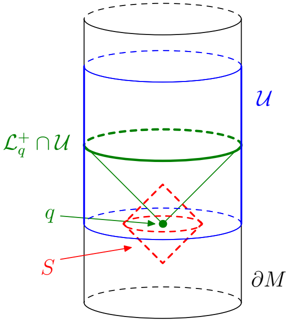
6. Inverse Scattering (Sa Barreto-U-Wang, 2022)

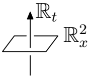
7. Semilinear equations (Kurylev-Lassas-U, 2018; Wang-U, 2018; Wang-Zhou, 2019; Hintz-U-Zhai, 2022; Stefanov-Sa Barreto, 2021; U-Zhang 2021; Hintz-U-Zhai, 2022)

8. Non-linear Acoustics (Acosta-U-Zhai, 2023; U-Zhang, 2023)

Boundary Light Observation Set

$M = \{(t, x) \colon |x| < 1\} \subset \mathbb{R}^{1+2}$.





Set of sources $S \subset M^\circ$.

Observations in $\mathcal{U} \subset \partial M$.

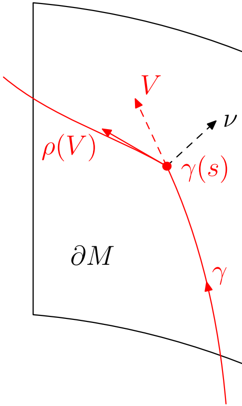
Data: $\mathscr{S} = \{\mathcal{L}_q^+ \cap \mathcal{U} \colon q \in S\}$

Theorem

The collection $\mathscr{S}$ determines the topological, differentiable, and conformal structure $[g|_S] = \{fg|_S \colon f > 0\}$ of S.

Reflection at the Boundary

γ null-geodesic until $\gamma(s) \in \partial M$.



$\rho(V) =$ reflection of V across ∂M. (Snell's law.)

$\rightarrow$ continuation of γ as broken null-geodesic

Null-convexity

Simplest case:

$$\text{All null-geodesics starting in } M^\circ \text{ hit } \partial M \text{ transversally.} \qquad (1)$$

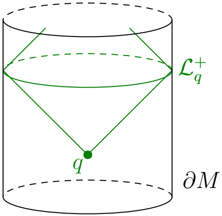
Proposition

(1) *is equivalent to null-convexity of* ∂M*:*

$$II(W,W) = g(\nabla_W \nu, W) \geq 0, \quad W \in T\partial M \text{ null.}$$

Stronger notion: strict null-convexity. ($II(W,W) > 0$, $W \neq 0$.)

Define light cones $\mathcal{L}_q^+$ using broken null-geodesics.



Main Result

Setup:

- (M, g) Lorentzian, dim ≥ 2, strictly null-convex boundary
- existence of $t\colon M \to \mathbb{R}$ proper, timelike
- sources: $S \subset M^\circ$ with $\bar{S}$ compact
- observations in $\mathcal{U} \subset \partial M$ open

Assumptions:

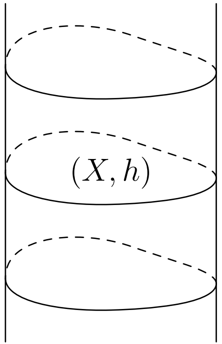
1. $\mathcal{L}_{q_1}^+ \cap \mathcal{U} \neq \mathcal{L}_{q_2}^+ \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$
2. points in S and $\mathcal{U}$ are not (null-)conjugate

Theorem (Hintz–U, 2019)

The smooth manifold $\mathcal{U}$ and the unlabelled collection $\mathscr{S} = \{\mathcal{L}_q^+ \cap \mathcal{U} \colon q \in S\} \subset 2^{\mathcal{U}}$ uniquely determine $(S, [g|_S])$ (topologically, differentiably, and conformally).

Example for (M, g)

(X, h) compact Riemannian manifold with boundary.

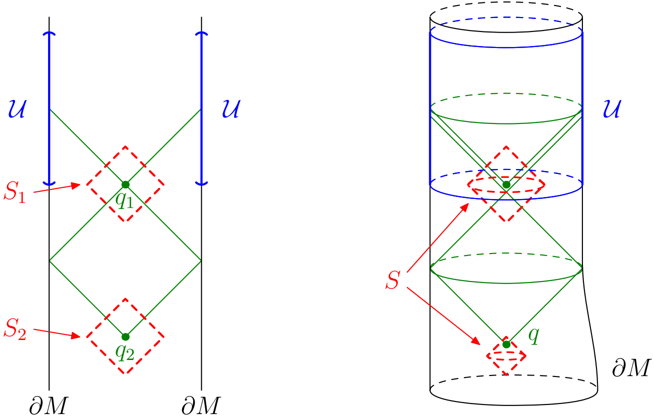


$$M = \mathbb{R}_t \times X, \quad g = -dt^2 + h.$$

(Strict) null-convexity of $\partial M \iff$ (strict) convexity of ∂X

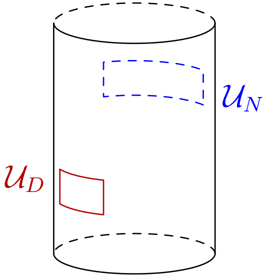
‘Counterexamples’

Necessity of assumption 1. ($\mathcal{L}^+_{q_1} \cap \mathcal{U} \neq \mathcal{L}^+_{q_2} \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$)



S_1 and $S_1 \cup S_2$ are indistinguishable from $\mathcal{U}$.

Active Measurements for Boundary Value Problems



(Special case: $\mathcal{U}_N = \mathcal{U}_D$.)

Propagation of singularities: (strict) null-convexity assumption simplifies structure of null-geodesic flow. (Melrose 1975, Taylor 1975, Melrose–Sjöstrand 1978.)

Inverse Boundary Value Problem

Assume $M = \mathbb{R} \times N$ is a Lorentzian manifold of dimension $(1+3)$ with time-like boundary.

$$\begin{aligned} \Box_g u(x) + a(x)u(x)^4 &= 0, && \text{on } M, \\ u(x) &= f(x), && \text{on } \partial M, \\ u(t,y) &= 0, && t < 0, \end{aligned}$$

Inverse Problem: determine the metric g and the coefficient a from the Dirichlet-to-Neumann map.

The Main Result

Theorem (Hintz-U-Zhai, 2022)

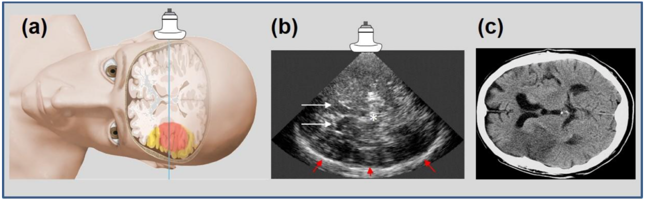
Consider the semilinear wave equations

$$\square_{g^{(j)}} u(x) + a^{(j)} u(x)^4 = 0, \qquad j = 1, 2,$$

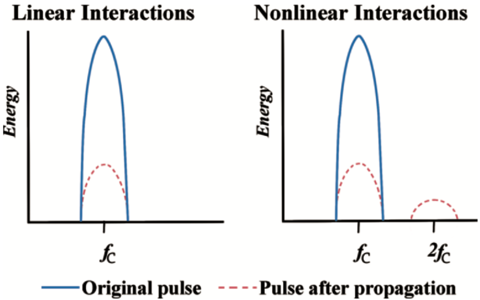
on Lorentzian manifold $M^{(j)}$ *with the same boundary* $\mathbb{R} \times \partial N$*. If the Dirichlet-to-Neumann maps* $\Lambda^{(j)}$ *acting on* $\mathcal{C}^5([0, T] \times \partial N)$ *are equal,* $\Lambda^{(1)} = \Lambda^{(2)}$*, then there exist a diffeomorphism* $\Psi \colon U_{g^{(1)}} \to U_{g^{(2)}}$ *with* $\Psi|_{(0,T)\times\partial N} = \mathcal{I}d$ *and a smooth function* $\beta \in C^\infty(M^{(1)})$, $\beta|_{(0,T)\times\partial N} = \partial_\nu \beta|_{(0,T)\times\partial N} = 0$*, so that, in* $U_{g^{(1)}}$*,*

$$\Psi^* g^{(2)} = e^{-2\beta} g^{(1)}, \quad \Psi^* a^{(2)} = e^{-\beta} a^{(1)}, \quad \square_g e^{-\beta} = 0.$$

Ultrasound Imaging



Nonlinear interaction: waves at frequency f_C generate waves at frequency $2f_C$:



Inverse Boundary Value Problem

The acoustic waves are modeled by the Westervelt-type equation

$$\frac{1}{c^2(x)}\partial_t^2 p(t,x) - \beta(x)\partial_t^2 p^2(t,x) = \Delta p(t,x), \quad \text{in } (0,T)\times\Omega,$$

$$p(t,x) = f, \quad \text{on } (0,T)\times\partial\Omega,$$

$$p = \frac{\partial p}{\partial t} = 0, \quad \text{on } \{t=0\},$$

- c: wavespeed
- β: nonlinear parameter

Inverse problem: recover β from the Dirichlet-to-Neumann map Λ.

Second Order Linearization

Second order linearization and the resulted integral identity:

$$\int_0^T \int_{\partial\Omega} \frac{\partial^2}{\partial\epsilon_1 \partial\epsilon_2} \Lambda(\epsilon_1 f_1 + \epsilon_2 f_2)\Big|_{\epsilon_1=\epsilon_2=0} f_0 \, dS \, dt$$
$$= 2 \int_0^T \int_{\Omega} \beta(x) \partial_t(u_1 u_2) \partial_t u_0 \, dx \, dt.$$

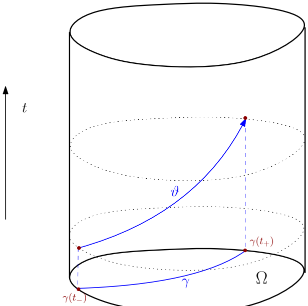
where u_j, $j = 1, 2$ are solutions to the linear wave equation

$$\frac{1}{c^2} \partial_t^2 u_i(t,x) - \Delta u_j(t,x) = 0$$

with $u_j|_{(0,T)\times\partial\Omega} = f_j$, and u_0 is the solution to the backward wave equation with $u_0|_{(0,T)\times\partial\Omega} = f_0$

Reduction to a Weighted Ray Transform

Construct Gaussian beam solutions u_0, u_1, u_2 traveling along the same null-geodesic $\vartheta(t) = (t, \gamma(t))$, where $\gamma(t), t \in (t_-, t_+)$ is the geodesic in (Ω, g) joining two boundary points $\gamma(t_-), \gamma(t_+) \in \partial\Omega$.



Insert into the integral identity, one can extract the Jacobi-weighted ray transform of $f = \beta c^{3/2}$ $\Rightarrow$ invert this weighted ray transform (Paternain-Salo-U-Zhou, 2019; Feizmohammadi-Oksanen, 2020)

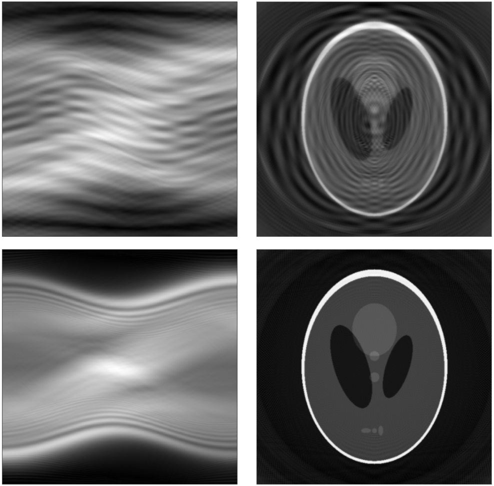


Figure: $L/\lambda = 10$ (top row) and $L/\lambda = 100$ (bottom row) where L is the size of the image and λ is the wavelength.

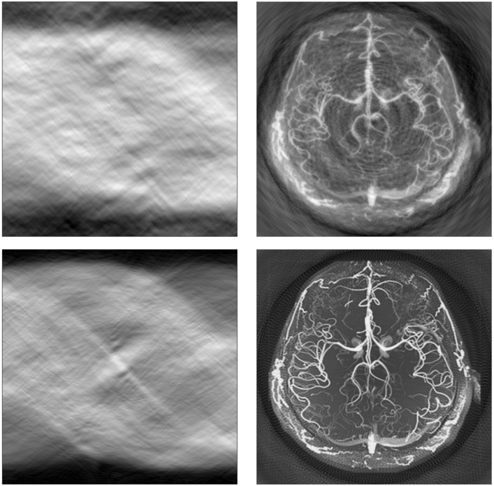


Figure: $L/\lambda = 10$ (top row) and $L/\lambda = 100$ (bottom row) where L is the size of the image and λ is the wavelength.

Happy Birthday, Maciej!