The small data global well-posedness conjectures for cubic dispersive flows

Daniel Tataru

University of California, Berkeley

Orsay, May 2024

This is joint work with Mihaela Ifrim

Nonlinear dispersive problems:

$$i\partial_t u - A(D_x)u = N(u), \qquad u(0) = u_0$$

Characteristic set:

$$\Sigma = \{\tau + a(\xi) = 0\}$$

Group velocity:

$$v_\xi = \nabla_\xi a(\xi)$$

Dispersive models:

$$\nabla_\xi^2 a(\xi) \neq 0$$

Smooth nonlinearity:

$$N(u) = N(u, \bar{u})$$

Symmetries:

- Translation invariant
- (1D) phase rotation, $u \to ue^{i\theta}$.

Resonant/nonresonant interactions

Question: Are there global dispersive solutions for small initial data ?

Examples of dispersion relations

- NLS: $a(\xi) = \xi^2$
- Klein-Gordon: $a(\xi) = (1 + \xi^2)^{\frac{1}{2}}$
- KdV: $a(\xi) = \xi^3$
- Deep gravity waves $a(\xi) = |\xi|^{\frac{1}{2}}$
- Capillary waves $a(\xi) = |\xi|^{\frac{3}{2}}$
- Shallow gravity waves $a(\xi) = \sqrt{|\xi| \tanh |\xi|}$
- Shallow capillary waves: $a(\xi) = \sqrt{|\xi|^3 \tanh |\xi|}$

The nonlinearity

a) Classified by strength:

- semilinear (e.g. NLS3, KdV), Lipschitz dependence on data
- quasilinear (e.g. water waves), continuous dependence on data

b) Classified by leading homogeneity:

- quadratic,

$$N(u) = Q_1(u,u) + Q_2(u,\bar{u}) + Q_3(\bar{u},\bar{u})$$

- cubic, e.g.

$$N(u) = C(u,\bar{u},u)$$

- higher order

c) Classified by leading order nonlinear effect (cubic case):

- defocusing
- focusing

Long-time/global dynamics

Linear effects: dispersive decay

vs.

Nonlinear effects: ode growth /oscillation

Key concept: Nonlinear wave interactions

- resonant
- nonresonant

What is linear dispersion ?

1. Fundamental solution:

$$K(t,x) \approx \frac{1}{t^{\frac{n}{2}} |\det \nabla^2 a(\xi_v)|^{\frac{1}{2}}} e^{it\phi(v)}, \qquad v = x/t$$

$$a'(\xi_v) = v, \qquad \phi'(v) = \xi_v \qquad \text{(Legendre)}$$

A1: $t^{-\frac{n}{2}}$ decay (for localized or L^1 data)

2. Translation invariant bounds:

$$\|e^{itA} u_0\|_S \lesssim \|u_0\|_{L^2} \qquad \text{(Strichartz)}$$

$$\swarrow \downarrow \searrow$$

$$L^\infty_t L^2_x \qquad L^6 (L^{\frac{2(n+2)}{n}}) \quad L^4 L^\infty (L^2 L^{\frac{2n}{n-2}})$$

$$\|u_A u_B\|_{L^2} \lesssim |v_A - v_B|^{-\frac{1}{2}} \|u_{A0}\|_{L^2} \|u_{B0}\|_{L^2} \qquad \text{(bilinear } L^2\text{, 1D)}$$

A2: Strichartz + transversal L^2 bounds (for L^2 data)

Why cubic nonlinearity ?

Quick answer: higher order $\Rightarrow$ more decay from dispersion

1. Quadratic case:
   - three wave interactions
   - Resonant vs. nonresonant or null interactions
   - Algebraically,
   $$\pm\Sigma \pm \Sigma \to \Sigma$$
   - Nonresonant/null $\Rightarrow$ normal form reduction to cubic

2. Cubic case: (with phase rotation symmetry)
   - four wave interactions
   $$(\xi_1, \xi_2, \xi_3) \to \xi_4 = \xi_1 - \xi_2 + \xi_3, \qquad 0 = \Delta^4 \xi := \xi_1 - \xi_2 + \xi_3 - \xi_4$$
   - Resonance: same for time frequencies,
   $$(a(\xi_1), a(\xi_2), a(\xi_3)) \to a(\xi_4), \qquad 0 = \Delta^4 a(\xi)$$
   - Resonant interactions:
   $$(1D): \qquad \Delta^4 \xi = 0, \quad \Delta^4 a(\xi) = 0 \Rightarrow \{\xi_1, \xi_3\} = \{\xi_2, \xi_4\}$$

   Many resonant interactions in higher dimensions.

Trilinear wave packet interactions

$$u \to C(u, \bar{u}, u)$$

Equal frequencies:

$$(\xi, \xi, \xi) \to \xi$$

Amplitude equation:

$$i\dot{A} = c(\xi, \xi, \xi) A |A|^2,$$

always nonperturbative on large time scales, at least in 1D.
Here $c(\xi, \xi, \xi) \in \mathbb{R}$ prevents blow-up (exponential growth).

Two assumptions on the symbol of C:

1. Conservative: $c(\xi, \xi, \xi), \nabla c(\xi, \xi, \xi) \in \mathbb{R}$
   $\to$ Wave packet interactions do not increase energy

2. Focusing vs. defocusing:
   $\to$ determined by the sign of $c(\xi, \xi, \xi)$

Semilinear vs quasilinear

Semilinear example:

$$i\partial_t u + \Delta u = \pm u|u|^2$$

- Can directly use dispersive decay (Strichartz)
- Nonlinearity is perturbative
- Lipschitz dependence of solutions on data
- sign choice corresponds to focusing/defocusing

Quasilinear example:

$$i\partial_t u + g^{jk}(u)\partial_j\partial_k u = 0, \qquad g(u) = I_n + O(|u|^2)$$

- No access to dispersive decay (Strichartz)
- Nonlinearity is nonperturbative
- Continuous dependence of solutions on data

A brief history of global solutions

1. Classical:

$$\text{Conserved energy} + \text{LWP} \Rightarrow \text{GWP}$$

   - no dispersive decay information

2. Modern (semilinear):

$$\text{Strichartz} \Rightarrow \text{GWP} + \text{scattering} \qquad \text{(small data)}$$

$$S \subset \overset{4}{L^6} \xrightarrow{N} \overset{4/3}{L^{\frac{6}{5}}} \subset S'$$

   - requires quintic or higher nonlinearity in 1D ,
     cubic and higher nonlinearity in 2D

3. Contemporary:

$$\text{nD Small and localized data} \Rightarrow \text{GWP with } t^{-\frac{n}{2}} \text{ decay}$$

   - conservative cubic nonlinearity (1D)
   - vector field methods
   - 1D expository notes Ifrim-T. '22

Our set-up for the global problem

- Small data
  - dispersion has time to kick in
- Nonlocalized data
  - nonlinear interactions at every location
- Rough data
  - nonlinear interactions at every scale
- Cubic nonlinearity
  - stronger than dispersion in 1D (semilinear, quasilinear)
  - balances dispersion in 2D (quasilinear)

The non-localized data defocusing global well-posedness conjecture in 1D:

Assume:

- 1D dispersive problem
- cubic nonlinearity which is conservative and defocusing

Then:

$$\text{Small data} \Longrightarrow \text{global dispersive solutions.}$$

The non-localized data (focusing) long time well-posedness conjecture 1D:

Assume:

- 1D dispersive problem
- cubic nonlinearity which is conservative

Then:

$$\epsilon\text{-small data} \Longrightarrow \text{long time } \epsilon^{-8} \text{ dispersive solutions.}$$

The non-localized data global well-posedness conjecture in 2D:

Assume:

- Dispersive quasilinear problem
- cubic nonlinearity

Then:

$$\text{Small data} \Longrightarrow \text{global scattering solutions.}$$

Scattering

Classical formulation: Given a nonlinear solution u_{nonlin} there exists a linear solution $\tilde{u}_{lin}$ so that

$$\lim_{t\to\infty} \|u_{nonlin}(t) - \tilde{u}_{lin}(t)\|_{H^s} = 0$$

1D cubic problem

- No classical scattering can hold
- Dispersive decay:
  - L^6 Strichartz estimates, with loss of derivatives.
  - bilinear L^2 estimates, without loss of derivatives

2D cubic problem

- Classical scattering should hold
- Dispersive decay:
  - L^4 Strichartz estimates, with loss of derivatives.
  - bilinear L^2 estimates, without loss of derivatives

A semilinear result (1D defocusing)

Theorem (Ifrim-T. '22)

$$i\partial_t u + \Delta u = C(u, \bar{u}, u), \qquad u(0) = u_0$$

Suppose the nonlinearity C is cubic, conservative and defocusing. Then for small initial data

$$\|u_0\|_{L^2} \leq \epsilon \ll 1$$

there exists a unique global solution u so that

$$\|u\|_{L^\infty L^2} \lesssim \epsilon \qquad \textit{(Energy)}$$

$$\|u(t)\|_{L^6} \lesssim \epsilon^{\frac{2}{3}} \qquad \textit{(Strichartz)}$$

$$\|P_A u P_B u\|_{L^2} \lesssim d(v_A, v_B)^{-\frac{1}{2}} \epsilon^2 \qquad \textit{(bilinear } L^2\textit{)}$$

- First result of this type
- no energy conservation is assumed
- global dispersive bounds are obtained
- work in progress: general dispersion relations

A special case: defocusing $NLS^3(\mathbb{R})$

$$i\partial_t + \Delta u = u|u|^2$$

- Globally well-posed in L^2.
- Completely integrable $\Rightarrow$ Conserved energies

Theorem

L^2 solutions satisfy the Strichartz bound

$$\|u\|_{L^6} \lesssim \|u_0\|_{L^2}$$

and the bilinear L^2 bound

$$\|\partial_x |u|^2\|_{cL^2 + \dot{H}^{-\frac{1}{2}}} \lesssim \|u_0\|_{L^2}^2, \qquad c = \|u_0\|_{L^2}$$

Earlier dispersive bounds for H^1 solutions by Planchon-Vega.

A semilinear result (focusing case)

Theorem (Ifrim-T. '22)

$$i\partial_t u + \Delta u = C(u, \bar{u}, u), \qquad u(0) = u_0$$

Suppose the nonlinearity C is cubic and conservative. Then for small initial data

$$\|u_0\|_{L^2} \leq \epsilon \ll 1$$

there exists a solution u in $[0, \epsilon^{-8}]$ so that

$$\|u\|_{L^\infty[0,\epsilon^{-8};L^2]} \lesssim \epsilon \qquad \textit{(Energy)}$$

and also on ϵ^{-6} time intervals we have:

$$\|u(t)\|_{L^6} \lesssim \epsilon^{\frac{2}{3}} \qquad \textit{(Strichartz)}$$

$$\|P_A u P_B u\|_{L^2} \lesssim d(v_A, v_B)^{-\frac{1}{2}} \epsilon^2 \qquad \textit{(bilinear } L^2\textit{)}$$

- Sharp result, because of the existence of small solitons.

A quasilinear Schrödinger model

$$\begin{cases} iu_t + g^{jk}(u)\partial_j\partial_k u = N(u, \partial_x u), \quad u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C} \\ \\ u(0, x) = u_0(x) \end{cases} \tag{QNLS}$$

- $g = g(u, \bar{u})$ smooth, real valued, $g(0) = 1$.
- $N = N(u, \bar{u}, \partial u, \partial \bar{u})$ is smooth, complex valued, at most quadratic in ∂u.

$$\begin{cases} iu_t + g^{jk}(u, \partial_x u)\partial_j\partial_k u = N(u, \partial_x u), \quad u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C} \\ \\ u(0, x) = u_0(x) \end{cases} \tag{DQNLS}$$

Structural assumptions

1. Cubic nonlinearity:
   - $g - I$ is at least quadratic
   - N is at least cubic
2. Phase rotation symmetry:
   - $u \to ue^{i\theta}$.

$$i\partial_t u + \Delta u = C(u, \bar{u}, u) + \text{higher order}$$

3. Conservative nonlinearity:
   - $c(\xi, \xi, \xi), \nabla c(\xi, \xi, \xi) \in \mathbb{R}$.
4. Defocusing:
   - $c(\xi, \xi, \xi) \gtrsim \langle \xi \rangle^2$

Sharp local well-posedness 1D

Theorem (Ifrim-T. '23)

a) The 1D cubic (QNLS) *is locally well-posed for small data in H^s for $s > 1$, and the solutions satisfy*

1. *Uniform H^s bounds*
2. *Loss-less Strichartz estimates*
3. *Transversal bilinear L^2 bounds.*

b) The same result holds for the cubic (DQNLS) *for $s > 2$.*

- Scaling index $s_c = \frac{1}{2}$ (QNLS) (resp. $s_c = \frac{3}{2}$ (DQNLS))
- Regular sols with localized data Kenig-Ponce-Vega '04
- Rough sols with $s > 2$ (resp $s > 3$) Marzuola-Metcalfe-Tataru '14
- Should be generically ill-posed below H^1 (resp. H^2):
  - comparison with NLS^3 below L^2.

Other remarks:

- difference between quadratic and cubic problems (Mizohata, Doi)
- difference between small and large data [KPV], [MMT]

Sharp local well-posedness 2D+

Theorem (Ifrim-T. '24)

a) The nD cubic (QNLS) *is locally well-posed for small data in* H^s *for* $s > \frac{n+1}{2}$*, and the solutions satisfy*

1. *Uniform* H^s *bounds*
2. L^4 *Strichartz estimates with* $1/6$ *(no) derivative loss (2D) (3D+).*
3. *Transversal bilinear* L^2 *bounds.*

b) The same result holds for the cubic (DQNLS) *for* $s > \frac{n+3}{2}$.

- Scaling index $s_c = \frac{n}{2}$ (resp. $s_c = \frac{n+2}{2}$)
- Regular sols with localized data Kenig-Ponce-Vega '04
- Rough sols $s > \frac{n+3}{2}$ (resp $s > \frac{n+5}{2}$) Marzuola-Metcalfe-Tataru '14
- Should be generically ill-posed below $H^{\frac{n+1}{2}}$ (resp. $H^{\frac{n+3}{2}}$):
  - failure of nontrapping requirement

Other remarks:

- difference between quadratic and cubic problems (Mizohata, Doi)
- difference between small and large data, nontrapping [KPV], [MMT]

Quasilinear local well-posedness

[Enhanced] Hadamard local well-posedness in Sobolev spaces

$$u(0) \in H^s$$

- existence of solutions u in the class $C(0, T; H^s)$
- uniqueness of solutions, either directly for regular solutions, or as unique limits of smooth solutions
- continuous dependence in H^s, i.e. continuity of the data to solution map
$$H^s \ni u(0) \to u \in C(0, T; H^s)$$
- weak Lipschitz dependence, i.e. for two H^s solutions u and v we have the difference bound
$$\|u - v\|_{C(0,T;L^2)} \lesssim \|u(0) - v(0)\|_{L^2}$$
- higher regularity, $u_0 \in H^N \Rightarrow u \in C(H^N)$

Defocusing global well-posedness 1D

Theorem (Ifrim-T. '23)

a) Consider the cubic (QNLS) *with phase rotation symmetry, conservative and defocusing. Let $s > 1$. Then for small initial data*

$$\|u_0\|_{H^s} \leq \epsilon \ll 1$$

there exists a unique global solution u which satisfies

1. *Uniform H^s bounds*
2. *L^6 Strichartz estimates with $1/6$ derivative loss.*
3. *Transversal bilinear L^2 bounds (loss-less).*

- First proof of the defocusing GWP conjecture in a quasilinear setting.
- Sharp result in terms of regularity
- Global in time integrated decay bounds ("scattering")

Focusing long time well-posedness

Theorem (Ifrim-T. '23)

a) Consider the cubic (QNLS) *with phase rotation symmetry, and conservative. Let $s > 1$. Then for small initial data*

$$\|u_0\|_{H^s} \leq \epsilon \ll 1$$

there exists a unique global solution u in $[0, \epsilon^{-8}]$ which satisfies

1. *Uniform H^s bounds*
2. *Strichartz estimates with $1/6$ derivative loss on ϵ^{-6} time scale*
3. *Transversal bilinear L^2 bounds (loss-less) on ϵ^{-6} timescale.*

- First quasilinear proof of the focusing long time WP conjecture.
- Sharp result in terms of regularity.
- Sharp result in terms of time scales (small solitons).

Global well-posedness 2D+

Theorem (Ifrim-T. '24)

a) Consider the 2+D cubic (QNLS) *Let* $s > \frac{n+1}{2}$ *(*$n \geq 3$*) or* $s \geq \frac{7}{4}$ *(*$n = 2$*). Then for small initial data*

$$\|u_0\|_{H^s} \leq \epsilon \ll 1$$

there exists a unique global solution u *which satisfies*

1. *Uniform* H^s *bounds*
2. L^4 *Strichartz estimates with* $1/2$*(no) derivative loss (2D) (3D+).*
3. *Transversal bilinear* L^2 *bounds (loss-less).*
4. *Scattering in* H^s*.*

- First proof of the 2D GWP conjecture in a quasilinear setting.
- Sharp result in terms of regularity, $n \geq 3$
- Global in time integrated decay bounds ("scattering")

Global well-posedness 2D, Take 2

Theorem (Ifrim-T. '24)

a) Consider the 2D cubic (QNLS), *conservative. Let* $s > \frac{n+1}{2} = \frac{3}{2}$. *Then for small initial data*

$$\|u_0\|_{H^s} \leq \epsilon \ll 1$$

there exists a unique global solution u which satisfies

1. *Uniform* H^s *bounds*
2. L^4 *Strichartz estimates with* $1/2$ *derivative loss.*
3. *Transversal bilinear* L^2 *bounds (loss-less).*
4. *Scattering in* H^s.

- First proof of the 2D GWP conjecture at sharp Sobolev regularity.
- Sharp result in terms of regularity
- Global in time integrated decay bounds

Five key ideas

1. Bootstrap argument via frequency envelopes
   - associated to a dyadic frequency decomposition
2. Energy estimates via density flux identities.
   - carried out in a nonlocal setting, where both the densities and the fluxes involve translation invariant multilinear forms.
3. Modified energies, akin to the I-method.
   - we implement this at the level of density-flux identities, rather than for energy functionals
4. Interaction Morawetz bounds.
   - extended to the setting and language of nonlocal multilinear forms.
5. Strichartz estimates.
   - 1D: via wave packet parametrices, peeling off "perturbative" errors
   - nD global: by comparison with flat metric, with derivative loss

The Littlewood-Paley decomposition

Dichotomy for multilinear forms:

- parallel interactions $\longrightarrow$ rely on L^6 Strichartz (L^4 if $n \geq 2$)
- transverse interactions $\longrightarrow$ rely on bilinear L^2

Dyadic frequency decomposition:

$$u = \sum_{\lambda \in 2^{\mathbb{N}}} u_\lambda,$$

size of LP regions dictated by the Hamilton flow.

Goal:

- estimate each u_λ separately
- estimate bilinear interactions

A collection of related equations

Full equation:

$$iu_t + g^{jk}(u)\partial_j\partial_k u = N(u, \partial_x u). \tag{QNLS}$$

Linearized equation:

$$iv_t + g^{jk}(u)\partial_j\partial_k v = N^{lin}(u)v. \tag{QNLS-lin}$$

Paradifferential equation:

$$iw_{\lambda t} + \partial_j g^{jk}(u_{<\lambda})\partial_k w_\lambda = f_\lambda \tag{QNLS-para}$$

Full equation in paradifferential form,

$$iu_{\lambda t} + \partial_j g^{jk}(u_{<\lambda})\partial_k u_\lambda = N_\lambda\ (u, \partial_x u) \tag{QNLS}$$

Linearized equation in paradifferential form

$$iv_{\lambda t} + \partial_j g^{jk}(u_{<\lambda})\partial_k v_\lambda = N_\lambda^{lin}(u)v. \tag{QNLS-lin}$$

A collection of related equations

Full equation:

$$iu_t + g^{jk}(u)\partial_j\partial_k u = N(u, \partial_x u). \tag{QNLS}$$

Linearized equation:

$$iv_t + g^{jk}(u)\partial_j\partial_k v = N^{lin}(u)v. \tag{QNLS-lin}$$

Paradifferential equation:

$$iw_{\lambda t} + \partial_j g^{jk}(u_{<\lambda})\partial_k w_\lambda = f_\lambda \tag{QNLS-para}$$

Full equation in paradifferential form, long time analysis

$$iu_{\lambda t} + \partial_j g^{jk}(u_{<\lambda})\partial_k u_\lambda = N^{nr}_\lambda(u, \partial_x u) + C_\lambda(u, \bar{u}, u) \tag{QNLS}$$

Linearized equation in paradifferential form

$$iv_{\lambda t} + \partial_j g^{jk}(u_{<\lambda})\partial_k v_\lambda = N^{lin}_\lambda(u)v. \tag{QNLS-lin}$$

Frequency envelopes

-introduced by Tao to track the time evolution of dyadic energies

- Start with frequency envelope $\{c_\lambda\} \in \ell^2$ for the initial data
$$\|u_{0\lambda}\|_{H^s} \lesssim \epsilon c_\lambda$$
- Show that similar bounds carry over to solutions
- Key assumption on c: *slowly varying*, to control nonlinear leakage.
$$\frac{c_\lambda}{c_\mu} \leq \left(\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right)^\delta .$$

Bootstrap hypothesis 1D:

(energy) $$\|u_\lambda\|_{L^\infty L^2} \lesssim C\epsilon c_\lambda \lambda^{-s}$$

(unbalanced bilinear) $$\|u_\lambda \bar{u}^h_\mu\|_{L^2} \lesssim C^2\epsilon^2 c_\lambda c_\mu \lambda^{-s-\frac{1}{2}}\mu^{-s} \qquad \mu \ll \lambda$$

(balanced bilinear) $$\|\partial_x(u_\lambda \bar{u}^h_\mu)\|_{L^2} \lesssim C^2\epsilon^2 c_\lambda c_\mu \lambda^{-s+\frac{1}{2}}\mu^{-s}(1+\lambda h), \qquad \lambda \approx \mu$$

(Strichartz) $$\|u_\lambda(t)\|_{L^6} \lesssim C\epsilon c_\lambda \lambda^{-s+\frac{1}{6}}$$

- bootstraping both Strichartz and bilinear: Ifrim-T., Benjamin-Ono

Frequency envelopes

-introduced by Tao to track the time evolution of dyadic energies

- Start with frequency envelope $\{c_\lambda\} \in \ell^2$ for the initial data
$$\|u_{0\lambda}\|_{H^s} \lesssim \epsilon c_\lambda$$
- Show that similar bounds carry over to solutions
- Key assumption on c: *slowly varying*, to control nonlinear leakage.
$$\frac{c_\lambda}{c_\mu} \leq \left(\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right)^\delta .$$

Bootstrap hypothesis nD:

(energy) $$\|u_\lambda\|_{L^\infty L^2} \lesssim C\epsilon c_\lambda \lambda^{-s}$$

(unbalanced bilinear) $$\|u_\lambda \bar{u}^h_\mu\|_{L^2} \lesssim C^2\epsilon^2 c_\lambda c_\mu \lambda^{-s-\frac{1}{2}} \mu^{-s+\frac{n-1}{2}} \qquad \mu \ll \lambda$$

(balanced bi) $$\||D_x|^{\frac{3-n}{2}}(u_\lambda \bar{u}^h_\mu)\|_{L^2} \lesssim C^2\epsilon^2 c_\lambda c_\mu \lambda^{-s+\frac{1}{2}} \mu^{-s}(1+\lambda h), \qquad \mu \approx \lambda$$

(Strichartz) $$\|u_\lambda(t)\|_{L^4} \lesssim C\epsilon c_\lambda \lambda^{-s+\frac{n-2}{2}}$$

- bootstraping both Strichartz and bilinear: Ifrim-T., Benjamin-Ono

Conservation laws in density flux form

- Integral laws in linear/nonlinear case:

$$\mathbf{M} = \int |u|^2 \, dx, \qquad \frac{d}{dt}\mathbf{M} = \int C_m^4(u, \bar{u}, u, \bar{u}) \, dx$$

- Well chosen mass/momentum densities

$$\mathbf{M} = \int M(u, \bar{u}) \, dx, \qquad \mathbf{P} = \int P(u, \bar{u}) \, dx$$

- Density flux identities in linear/nonlinear case:

$$\partial_t M(u, \bar{u}) = \partial_x [g P(u, \bar{u})] + C_m^4(u, \bar{u}, u, \bar{u})$$

$$\partial_t P(u, \bar{u}) = \partial_x [g E(u, \bar{u})] + C_p^4(u, \bar{u}, u, \bar{u})$$

- Frequency localized density-flux identities:

$$\partial_t M_\lambda(u, \bar{u}) = \partial_x [g_{<\lambda} P_\lambda(u, \bar{u})] + C_{m,\lambda}^4(u, \bar{u}, u, \bar{u})$$

$$\partial_t P_\lambda(u, \bar{u}) = \partial_x [g_{<\lambda} E_\lambda(u, \bar{u})] + C_{p,\lambda}^4(u, \bar{u}, u, \bar{u})$$

Energy corrections for long time results 1D, 2D

♣ second generation I-method: correct energies for better conservation (I-team:=Colliander-Keel-Stafillani-Takaoka-Tao)

♡ better strategy: correct densities and fluxes

- Quartic energy correction

$$M^\sharp_\lambda(u,\bar u) = M_\lambda(u,\bar u) + B^4_{\lambda,m}(u,\bar u,u,\bar u),$$

$$P^\sharp_\lambda(u,\bar u) = P_\lambda(u,\bar u) + B^4_{\lambda,p}(u,\bar u,u,\bar u),$$

- Density-flux identities:

$$\partial_t M^\sharp_\lambda = \partial_x(P_\lambda + R^4_{\lambda,m}) + F^{4,nr}_{\lambda,m} + R^6_{\lambda,m}$$

$$\partial_t P^\sharp_\lambda = \partial_x(E_\lambda + R^4_{\lambda,p}) + F^{4,nr}_{\lambda,p} + R^6_{\lambda,p}$$

  - This requires solving a nontrivial division problem,

$$c^4 = \Delta^4\xi^2 \cdot b^4 + \Delta^4\xi \cdot r^4 + (\xi_{odd} - \xi_{even})^2 q^{4,nr}$$

  - Energy bounds follow by direct integration

Bilinear L^2 estimates

- cannot use linear theory, as (i) problem is quasilinear and (ii) nonlinearity is nonperturbative
- Nonlinear idea: Interaction Morawetz
  - introduced by I-team '03 for 3D NLS
  - one dimensional version by Planchon-Vega

Baby version: $u, v \geq 0$ densities

$$\partial_t u = \partial_x f, \qquad \text{moves to the left } f > 0$$

$$\partial_t v = \partial_x g, \qquad \text{moves to the right } g < 0$$

Interaction functional:

$$I(u, v) = \int_{x<y} u(x)v(y)\, dxdt$$

$$\frac{dI}{dt} = \int_{\mathbb{R}} fv - ug\, dx > 0 \qquad \text{(transversality bound)}$$

Dispersive Interaction Morawetz 1D

"momentum is moving to the right faster than the mass"

1. Interaction Morawetz functional, diagonal case:

$$I(u_\lambda, u_\lambda) = \int_{x<y} M^\sharp_\lambda(x) P^\sharp_\lambda(y) - M^\sharp_\lambda(y) P^\sharp_\lambda(x)\, dx dy$$

Time differentiation:

$$\frac{d}{dt} I(u_\lambda, u_\lambda) \approx \|\partial_x(u_\lambda \bar{u}_\lambda)\|^2_{L^2} + \|u_\lambda\|^6_{L^6} + \text{Errors (6,8,10)}$$

- used to prove the L^6 Strichartz and diagonal bilinear L^2.

2. Transversal Interaction Morawetz functional:

$$I(u_\lambda, u_\mu) = \int_{x<y} M^\sharp_\lambda(x) P^\sharp_\mu(y) - M^\sharp_\mu(y) P^\sharp_\lambda(x)\, dx dy$$

Time differentiation:

$$\frac{d}{dt} I(u_\lambda, u_\mu) \approx \|\partial_x(u_\lambda \bar{u}_\mu)\|^2_{L^2} + \text{Errors (6,8,10)}$$

- used to prove the off-diagonal bilinear L^2 bound.

Dispersive Interaction Morawetz nD

1. Interaction Morawetz functional, diagonal case:

$$I(u_\lambda, u_\lambda) = \int a_j(x-y)(M_\lambda^\sharp(x)P_\lambda^{\sharp,j}(y) - M_\lambda^\sharp(y)P_\lambda^{\sharp,j}(x))\,dxdy$$

$$a_j(x) = \partial_j a(x), \qquad a(x) = |x|.$$

Time differentiation:

$$\frac{d}{dt}I(u_\lambda, u_\lambda) \approx \int a_{jk}(x-y)F^j\bar{F}^k dxdy + \text{Errors (6,8,10)}$$

$$F^j(x,y) = \partial^j u_\lambda(x)\bar{u}_\lambda(y) + u_\lambda(x)\partial^j\bar{u}_\lambda(y)$$

- used to prove the balanced bilinear L^2.

2. Transversal Interaction Morawetz functional:

$$I(u_\lambda, u_\mu) = \int a_j(x-y)(M_\lambda^\sharp(x)P_\mu^{\sharp,j}(y) - M_\mu^\sharp(y)P_\lambda^{\sharp,j}(x))\,dxdy$$

- Time differentiation as above.
- used to prove the unbalanced bilinear L^2 bound.

Lossless Strichartz estimates 1D

Established at the level of the paradifferential equation:

$$iw_{\lambda t} + \partial_x g(u_{<\lambda})\partial_x w_\lambda = f_\lambda, \qquad w(0) = w_0 \qquad \text{(QNLS-para)}$$

Main challenge: variable coefficient problem

- SE with derivative loss: from sharp SE on semiclassical time scales (Staffilani-Tataru '02, Burq-Gerard-Tzvetkov '06, etc.)
- SE without loss on asymptotically flat spaces (Robbiano-Zuily '06, Hassell-Tao-Wunsch '06, Tataru 07)

All the above require at least C^2 coefficients. Here, $g - 1 \in L^\infty H^{1+}$!

Key ideas:

- flatten metric with change of coordinates
- use equation for u
- allow for a large class of source terms f_λ
- use bilinear L^2 estimates
- use wave packet parametrix (Marzuola-Metcalfe-Tataru)

Summary

1. new global well-posedness conjectures for
   - 1D cubic defocusing problems with nonlocalized data
   - 2D+ cubic quasilinear problems with nonlocalized data
2. first 1D global well-posedness results
   - semilinear NLS
   - quasilinear NLS
3. first 2D quasilinear global well-posedness result
   - quasilinear NLS
4. new, sharp local well-posedness results in all dimensions
5. global well-posedness holds at optimal regularity (same as in the local result)

Happy birthday Maciej !