Paris-Saclay conference in Analysis and PDE, 27-31 May 2024
on the occasion of Maciej Zworski's 60th birthday

Quantum transport, exponential sums and lattice point statistics

Jens Marklof

School of Mathematics, University of Bristol
http://www.maths.bristol.ac.uk

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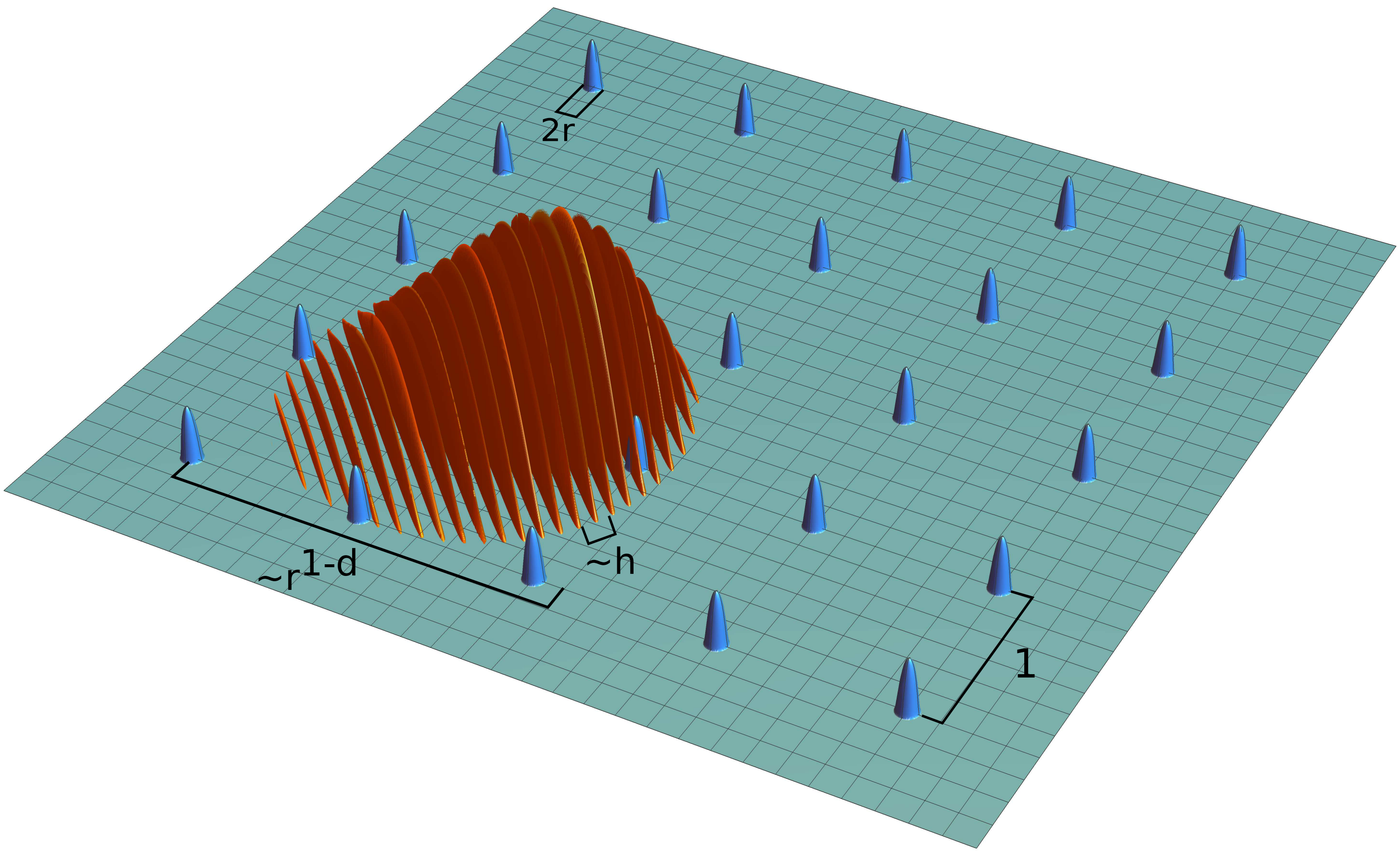
The quantum Lorentz gas



The quantum Lorentz gas

- Schrödinger equation

$$\mathrm{i}\frac{h}{2\pi}\,\partial_t f(t,\boldsymbol{x}) = H_{h,\lambda} f(t,\boldsymbol{x}), \qquad f(0,\boldsymbol{x}) = f_0(\boldsymbol{x})$$

- quantum Hamiltonian

$$H_{h,\lambda} = -\frac{h^2}{8\pi^2}\,\Delta + \lambda V(\boldsymbol{x})$$

- potential

$$V(\boldsymbol{x}) = V_r(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\mathcal{P}} W(r^{-1}(\boldsymbol{x}+\boldsymbol{m})), \qquad W \in \mathcal{S}(\mathbb{R}^d)$$

  with $\mathcal{P}$ point set describing location of scatterers (e.g. $\mathcal{P} = \mathbb{Z}^d$ or random point set)
- solution

$$f(t,\boldsymbol{x}) = U_{h,\lambda}(t) f_0(\boldsymbol{x}), \qquad U_{h,\lambda}(t) = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{h} H_{h,\lambda}\, t}$$

- note: classical mean free path length in this setting $\sim r^{1-d}$

Observables

- time evolution of linear operators $A(t)$ ("quantum observables") given by Heisenberg evolution $A(t) = U_{h,\lambda}(t)\, A\, U_{h,\lambda}(t)^{-1}$.
- L^2 inner product on classical phase space

$$\langle a, b\rangle = \int_{\mathbb{R}^d\times\mathbb{R}^d} a(\boldsymbol{x},\boldsymbol{y})\,\overline{b(\boldsymbol{x},\boldsymbol{y})}\,d\boldsymbol{x}d\boldsymbol{y},$$

- Hilbert-Schmidt inner product $\langle A, B\rangle_{\mathsf{HS}} = \mathsf{Tr}\, AB^\dagger$.
- semiclassical Boltzmann-Grad scaling

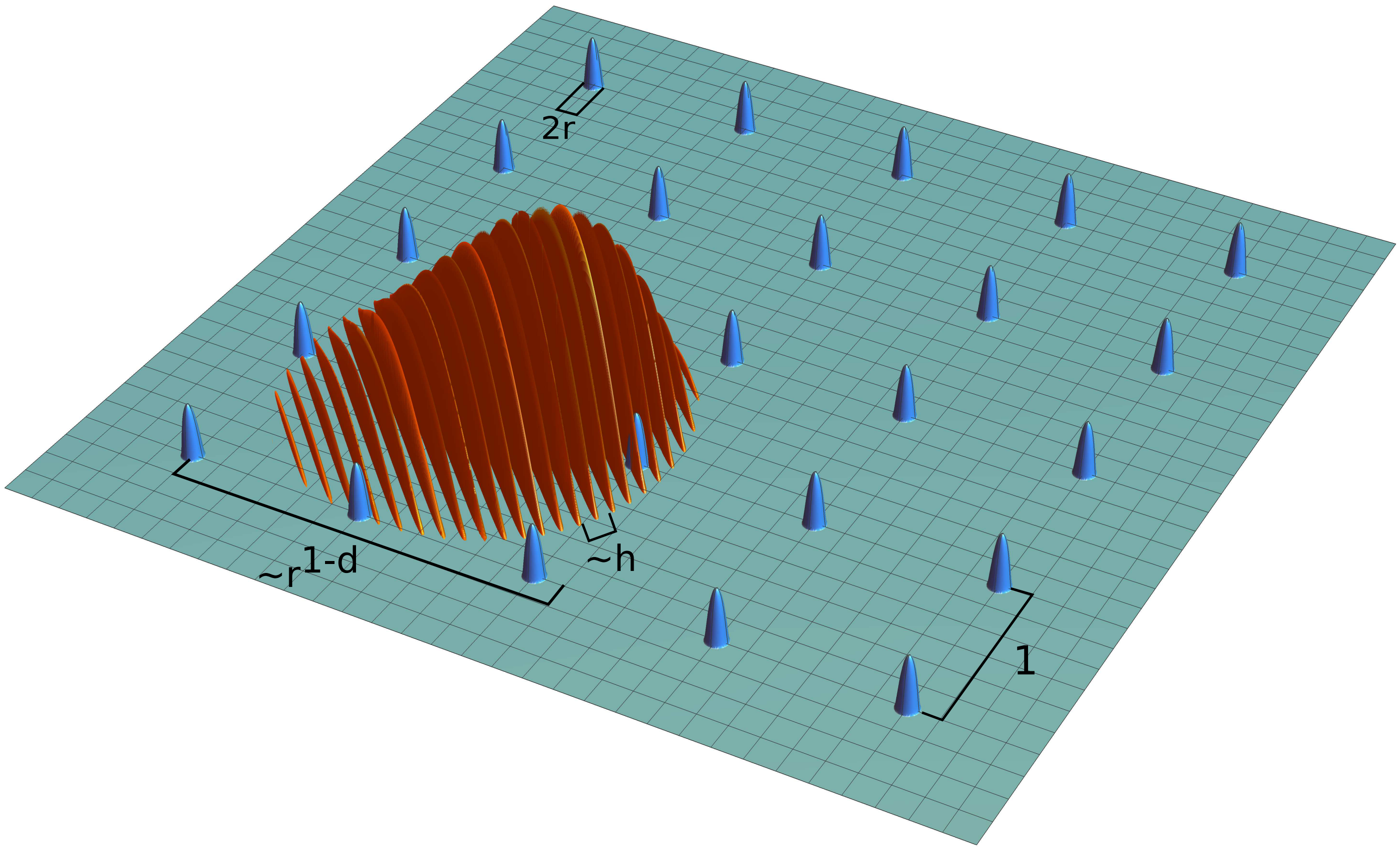
$$D_{r,h}a(\boldsymbol{x},\boldsymbol{y}) = r^{d(d-1)/2}h^{d/2}\, a(r^{d-1}\boldsymbol{x}, h\boldsymbol{y}),$$

- standard Weyl quantisation of $a \in \mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)$,

$$\mathsf{Op}(a)f(\boldsymbol{x}) = \int_{\mathbb{R}^d\times\mathbb{R}^d} a(\tfrac{1}{2}(\boldsymbol{x}+\boldsymbol{x}'),\boldsymbol{y})\,\mathsf{e}((\boldsymbol{x}-\boldsymbol{x}')\cdot\boldsymbol{y})\,f(\boldsymbol{x}')\,d\boldsymbol{x}'d\boldsymbol{y}$$

- set $\mathsf{Op}_{r,h} = \mathsf{Op}\circ D_{r,h}$ and $\mathsf{Op}_h = \mathsf{Op}_{1,h}$.

In the following assume $h = r$:



$>>>$**Semiclassical propagation with quantum scattering**$<<<$

A limiting transport process?

Pick your favourite scatterer configuration $\mathcal{P}$ (random or deterministic).

Recall $h = r$.

Questions:

(i) Does there exist a family of operators $L(t) : \mathsf{L}^1(\mathbb{R}^d \times \mathbb{R}^d) \to \mathsf{L}^1(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all $a, b \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, $A = \mathrm{Op}_{r,h}(a)$, $B = \mathrm{Op}_{r,h}(b)$,

$$\lim_{r \to 0} \langle A(tr^{-(d-1)}), B \rangle_{\mathrm{HS}} = \langle L(t)a, b \rangle \quad ?$$

(ii) Is $f_t(\boldsymbol{x}, \boldsymbol{y}) = L(t)a(\boldsymbol{x}, \boldsymbol{y})$ a solution of the linear Boltzmann equation?

Random scatterer configurations

For random scatterer configurations Eng and Erdös (Rev Math Phys 2005)* have proved convergence (in the annealed case) to a limit $f_t = L(t)a$, which in fact is a solution to the linear Boltzmann equation

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla_{\boldsymbol{Q}}\right] f_t(\boldsymbol{Q},\boldsymbol{V}) = \int_{\mathrm{S}_1^{d-1}} \left[f_t(\boldsymbol{Q},\boldsymbol{V}') - f_t(\boldsymbol{Q},\boldsymbol{V})\right] \Sigma(\boldsymbol{V},\boldsymbol{V}')\, d\boldsymbol{V}'$$

with the standard quantum mechanical collision kernel

$$\Sigma(\boldsymbol{y},\boldsymbol{y}') = 8\pi^2\, \delta(\|\boldsymbol{y}\|^2 - \|\boldsymbol{y}'\|^2)\, |T(\boldsymbol{y},\boldsymbol{y}')|^2 .$$

Here $T(\boldsymbol{y},\boldsymbol{y}')$ is the (single scatterer) T-matrix.

$>>>$**Semiclassical propagation with quantum scattering**$<<<$

*for uniformly distributed scatterers in a large torus, building on work by Erdös and Yau (Comm Pure Appl Math 2000) for the weak-coupling limit; see also Mikkelsen (preprint 2023, arXiv:2303.05176) for a proof for Poisson scatterer configuration in $\mathbb{R}^3$ and general semiclassical Wigner measures

Periodic scatterer configurations

Consider the periodic scatterer configuration $\mathcal{P} = \mathbb{Z}^d$ (or any other lattice in $\mathbb{R}^d$ of full rank).

Theorem (Griffin & JM, J Stat Phys 2021).
For $d \geq 3$, and conditional on a generalised Berry-Tabor conjecture:
(i) There exists a family of operators $L(t) : \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d) \to \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all $a, b \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, $A = \mathrm{Op}_{r,h}(a)$, $B = \mathrm{Op}_{r,h}(b)$, $t > 0$ and $0 < \lambda \leq \lambda_0$ (λ_0 sufficently small)

$$\lim_{r \to 0} \langle A(tr^{-(d-1)}), B \rangle_{\mathrm{HS}} = \langle L(t)a, b \rangle$$

(ii) $f(t, \boldsymbol{x}, \boldsymbol{y}) = L(t)a(\boldsymbol{x}, \boldsymbol{y})$ is **NOT** a solution of the linear Boltzmann equation.

Castella & Plagne (Indiana Math J 2001) showed that the low-density limit diverges for zero Bloch vector (=small scatterer limit on a unit torus).

Collision series for linear Boltzmann

Total scattering cross section $\Sigma_{\mathrm{tot}}(\boldsymbol{y}) = \int_{\mathbb{R}^d} \Sigma(\boldsymbol{y}', \boldsymbol{y})\, d\boldsymbol{y}'$

Collision series for solution of the linear Boltzmann equation

$$f_{\mathrm{LB}}(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{\infty} f_{\mathrm{LB}}^{(k)}(t, \boldsymbol{x}, \boldsymbol{y})$$

with the zero-collision term

$$f_{\mathrm{LB}}^{(1)}(t, \boldsymbol{x}, \boldsymbol{y}) = a(\boldsymbol{x} - t\boldsymbol{y}, \boldsymbol{y})\, \mathrm{e}^{-t\Sigma_{\mathrm{tot}}(\boldsymbol{y})},$$

and the $(k-1)$-collision term ...

Collision series for linear Boltzmann

...and the $(k-1)$-collision term

$$f_{\mathsf{LB}}^{(k)}(t,\boldsymbol{x},\boldsymbol{y}) = \int_{(\mathbb{R}^d)^k} \int_{\mathbb{R}^k_{\geq 0}} \delta(\boldsymbol{y}-\boldsymbol{y}_1)\, a\Bigg(\boldsymbol{x}-\sum_{j=1}^{k} u_j \boldsymbol{y}_j, \boldsymbol{y}_k\Bigg) \\ \times r_{\mathsf{LB}}^{(k)}(\boldsymbol{u},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)\, \delta\Bigg(t-\sum_{j=1}^{k} u_j\Bigg)\, d\boldsymbol{u}\, d\boldsymbol{y}_1 \cdots d\boldsymbol{y}_k$$

with

$$r_{\mathsf{LB}}^{(k)}(\boldsymbol{u},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k) = \prod_{i=1}^{k} \mathrm{e}^{-u_i \Sigma_{\mathsf{tot}}(\boldsymbol{y}_i)} \prod_{j=1}^{k-1} \Sigma(\boldsymbol{y}_j,\boldsymbol{y}_{j+1}).$$

The product form of the density $r_{\mathsf{LB}}^{(k)}$ shows that the corresponding random flight process is Markovian, and describes a particle moving along a random piecewise linear curve with momenta $\boldsymbol{y}_i$ and exponentially distributed flight times u_i.

Collision series for our limit process

Collision series

$$f(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{k=1}^{\infty} f^{(k)}(t,\boldsymbol{x},\boldsymbol{y})$$

with the zero-collision term (as for LB)

$$f^{(1)}(t,\boldsymbol{x},\boldsymbol{y}) = f^{(1)}_{\mathsf{LB}}(t,\boldsymbol{x},\boldsymbol{y}) = a(\boldsymbol{x}-t\boldsymbol{y},\boldsymbol{y})\,\mathrm{e}^{-t\Sigma_{\mathsf{tot}}(\boldsymbol{y})},$$

and the $(k-1)$-collision term ...

$$f^{(k)}(t,\boldsymbol{x},\boldsymbol{y}) = \frac{1}{k!}\sum_{\ell,m=1}^{k}\int_{(\mathbb{R}^d)^k}\int_{\mathbb{R}^k_{\geq 0}} \delta(\boldsymbol{y}-\boldsymbol{y}_\ell)\,a\Bigg(\boldsymbol{x}-\sum_{j=1}^{k}u_j\boldsymbol{y}_j,\boldsymbol{y}_m\Bigg)$$
$$\times\, r^{(k)}_{\ell m}(\boldsymbol{u},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)\,\delta\Bigg(t-\sum_{j=1}^{k}u_j\Bigg)\,d\boldsymbol{u}\,d\boldsymbol{y}_1\cdots d\boldsymbol{y}_k,$$

with the collision densities ...

Collision series for our limit process

...with the collision densities (**positive!**)

$$r_{\ell m}^{(k)}(\boldsymbol{u},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)=\left|g_{\ell m}^{(k)}(\boldsymbol{u},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)\right|^2\omega_k(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)\,\prod_{i=1}^{k}\mathrm{e}^{-u_i\Sigma_{\mathrm{tot}}(\boldsymbol{y}_i)}.$$

Here

$$\omega_k(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)=\prod_{j=1}^{k-1}\delta\Big(\tfrac{1}{2}\|\boldsymbol{y}_j\|^2-\tfrac{1}{2}\|\boldsymbol{y}_{j+1}\|^2\Big)$$

and $g_{\ell m}^{(k)}$ are the coefficients of the matrix valued function

$$\mathbb{G}^{(k)}(\boldsymbol{u},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)=\frac{1}{(2\pi\mathrm{i})^k}\oint\cdots\oint\Big(\mathbb{D}(\boldsymbol{z})-\mathbb{W}\Big)^{-1}\exp(\boldsymbol{u}\cdot\boldsymbol{z})\,dz_1\cdots dz_k,$$

where $\mathbb{D}(\boldsymbol{z})=\mathrm{diag}(z_1,\ldots,z_k)$ and $\mathbb{W}=\mathbb{W}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)$ with entries

$$w_{ij}=\begin{cases}0 & (i=j)\\ -2\pi\mathrm{i}T(\boldsymbol{y}_i,\boldsymbol{y}_j) & (i\neq j).\end{cases}$$

$>>>$**Strong correlation with past momenta**$<<<$

Collision series for our limit process

Explicitly, for the one collision terms

$$r_{11}^{(2)}(\boldsymbol{u},\boldsymbol{y}_1,\boldsymbol{y}_2) = r_{\mathsf{LB}}^{(2)}(\boldsymbol{u},\boldsymbol{y}_1,\boldsymbol{y}_2)\left|\frac{u_1 T(\boldsymbol{y}_2,\boldsymbol{y}_1)}{u_2 T(\boldsymbol{y}_1,\boldsymbol{y}_2)}\right| \\ \times \left|J_1\Big(4\pi[u_1u_2T(\boldsymbol{y}_1,\boldsymbol{y}_2)T(\boldsymbol{y}_2,\boldsymbol{y}_1)]^{1/2}\Big)\right|^2.$$

and

$$r_{12}^{(2)}(\boldsymbol{u},\boldsymbol{y}_1,\boldsymbol{y}_2) = r_{\mathsf{LB}}^{(2)}(\boldsymbol{u},\boldsymbol{y}_1,\boldsymbol{y}_2)\left|J_0\Big(4\pi[u_1u_2T(\boldsymbol{y}_1,\boldsymbol{y}_2)T(\boldsymbol{y}_2,\boldsymbol{y}_1)]^{1/2}\Big)\right|^2$$

with J_k the standard Bessel functions.

The remaining matrix elements can be computed via the identities

$$r_{22}^{(2)}(u_1,u_2,\boldsymbol{y}_1,\boldsymbol{y}_2) = r_{11}^{(2)}(u_2,u_1,\boldsymbol{y}_2,\boldsymbol{y}_1),$$

$$r_{21}^{(2)}(u_1,u_2,\boldsymbol{y}_1,\boldsymbol{y}_2) = r_{12}^{(2)}(u_2,u_1,\boldsymbol{y}_2,\boldsymbol{y}_1).$$

Above formulas strikingly similar to those for two-point spectral statistics in diffractive systems (Bogomolny and Giraud, Nonlinearity 2002)

Key steps in proof

- Use Floquet-Bloch decomposition to reduce problem to L^2 subspaces of functions

$$\psi(\boldsymbol{x}+\boldsymbol{k}) = \mathsf{e}(\boldsymbol{k}\cdot\boldsymbol{\alpha})\psi(\boldsymbol{x}), \quad \forall \boldsymbol{k}\in\mathbb{Z}^d$$

with $\boldsymbol{\alpha}\in[0,1)^d$

- Consider each $\boldsymbol{\alpha}$-subspace separately
- Use iterated application of Duhamel formula for quantum propagator,

$$U_{\lambda,h}(t) = U_{0,h}(t) - 2\pi\mathrm{i}\lambda\int_0^t U_{\lambda,h}(t-s)\,\mathrm{Op}(V)U_{0,h}(s)ds,$$

to produce perturbation expansion

- The eigenphases of $U_{0,h}(t)$ restricted to $\boldsymbol{\alpha}$-subspace are of the form

$$\pi\, t\,\|\boldsymbol{m}+\boldsymbol{\alpha}\|^2, \quad \boldsymbol{m}\in\mathbb{Z}^d$$

Key steps in proof

- Set $\mathcal{P}_{\boldsymbol{\alpha}} = \mathbb{Z}^d + \boldsymbol{\alpha}$
- The $(n-1)$ collision term can be expressed in the form

$$r^d \sum_{\substack{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n=\boldsymbol{p}_0\in\mathcal{P}_{\boldsymbol{\alpha}} \\ \text{non-consec}}} H_{t,\ell,n}\Big(r^{2-d}(\tfrac{1}{2}\|\boldsymbol{p}_0\|^2,\ldots,\tfrac{1}{2}\|\boldsymbol{p}_n\|^2), r\boldsymbol{p}_0,\ldots,r\boldsymbol{p}_n\Big)$$

  form some (not so well behaved) function $H_{t,\ell,n}$, which has translation invariance in the first coordinates so that it only depends on the differences between the $\|\boldsymbol{p}_j\|^2$
- The above expression is thus the n-point correlation density of $\mathcal{P}$ tested against $H_{t,\ell,n}$ — measured on the scale of their mean separation
- **Our key assumption in this work is that we can replace $\mathcal{P}_{\boldsymbol{\alpha}}$, for typical (or on average) $\boldsymbol{\alpha}$ by a Poisson point process in $\mathbb{R}^d$ of intensity one**

$\Longrightarrow$ **Berry-Tabor conjecture, quantitative Oppenheim conjecture**

Key steps in proof

- Set $\mathcal{P}_\alpha$ =
- The $(n$ –

r^d $\ldots, r\boldsymbol{p}_n)$

$\boldsymbol{p}_1$

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- The ab of $\mathcal{P}$ tested
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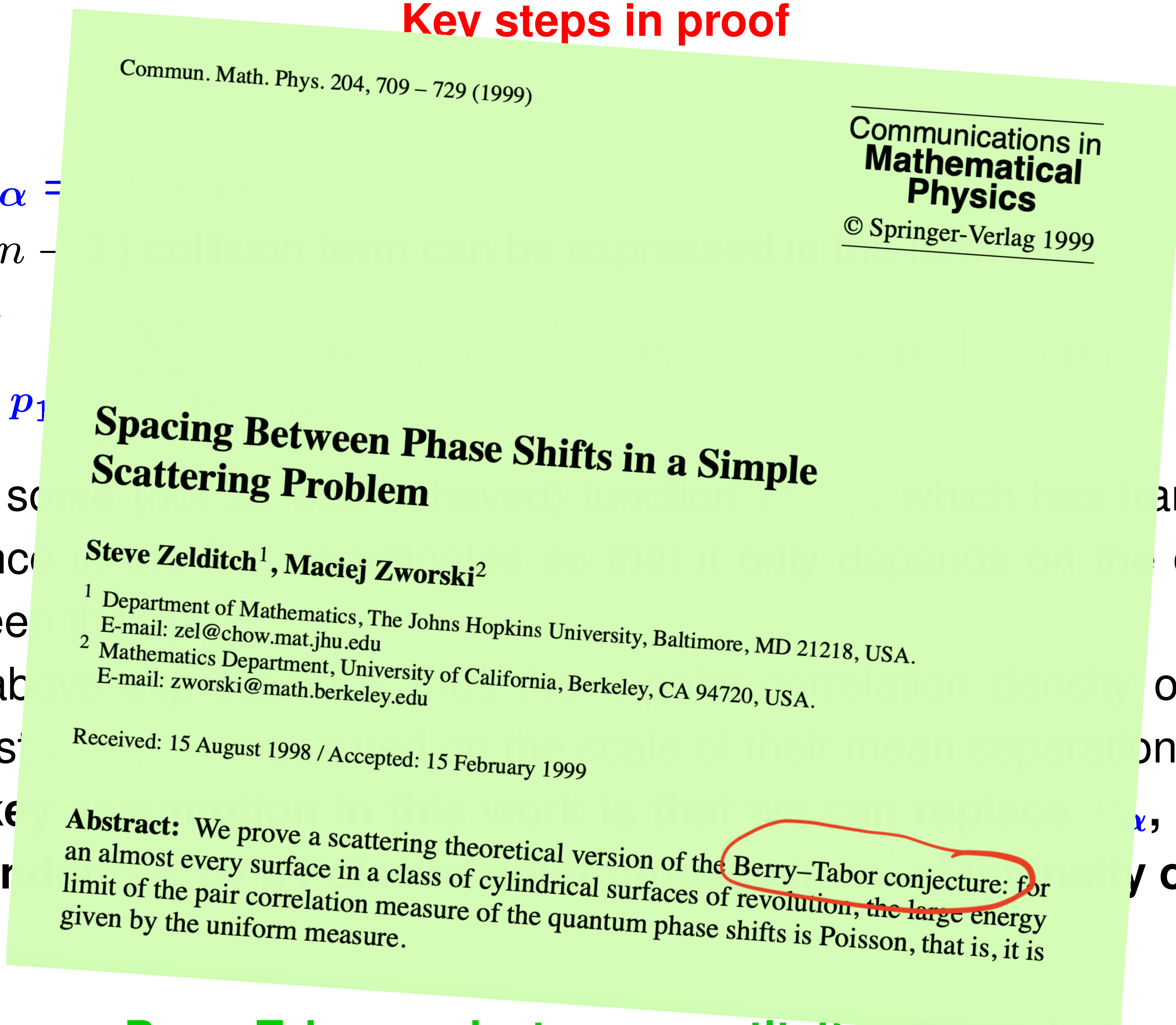
Commun. Math. Phys. 204, 709 – 729 (1999)

Communications in Mathematical Physics

Spacing Between Phase Shifts in a Simple Scattering Problem

Steve Zelditch[1], Maciej Zworski[2]

[1] Department of Mathematics, The Johns Hopkins University, Baltimore, MD 21218, USA.
E-mail: zel@chow.mat.jhu.edu
[2] Mathematics Department, University of California, Berkeley, CA 94720, USA.
E-mail: zworski@math.berkeley.edu

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Abstract: We prove a scattering theoretical version of the Berry–Tabor conjecture: for an almost every surface in a class of cylindrical surfaces of revolution, the large energy limit of the pair correlation measure of the quantum phase shifts is Poisson, that is, it is given by the uniform measure.

$\Longrightarrow$ **Berry-Tabor conjecture, quantitative Oppenheim conjecture**

Moments of exponential sums

- The key task is to understand the distribution of exponential sums (quadratic Weyl sums) of the form

$$\Theta(t+\mathrm{i}r^2) = r^{d/2} \sum_{j=1}^{\infty} \psi(r\boldsymbol{p}_j) \mathrm{e}^{2\pi \mathrm{i} \|\boldsymbol{p}_j\|^2 t}$$

  in the limit $r \to 0$.
- In particular, second-order correlations follow from the asymptotics of the second moment

$$\int_a^b \left| \Theta(r^{2-d} t + \mathrm{i}r^2) \right|^2 dt$$

  which holds in the case of Diophantine $\boldsymbol{\alpha} \in \mathbb{R}^d$ (JM, Annals Math 2003, Duke Math J 2002). This yields a proof of the Poisson nature of second-order correlations. See also Eskin, Margulis & Mozes (Annals 2005) and Margulis & Mohammadi (Duke Math J 2011) for general homogeneous/inhomogeneous forms in 2D
- Gives rigorous expansion of quantum transport up to order λ^2 (Griffin & JM, Pure & Applied Analysis 2019).

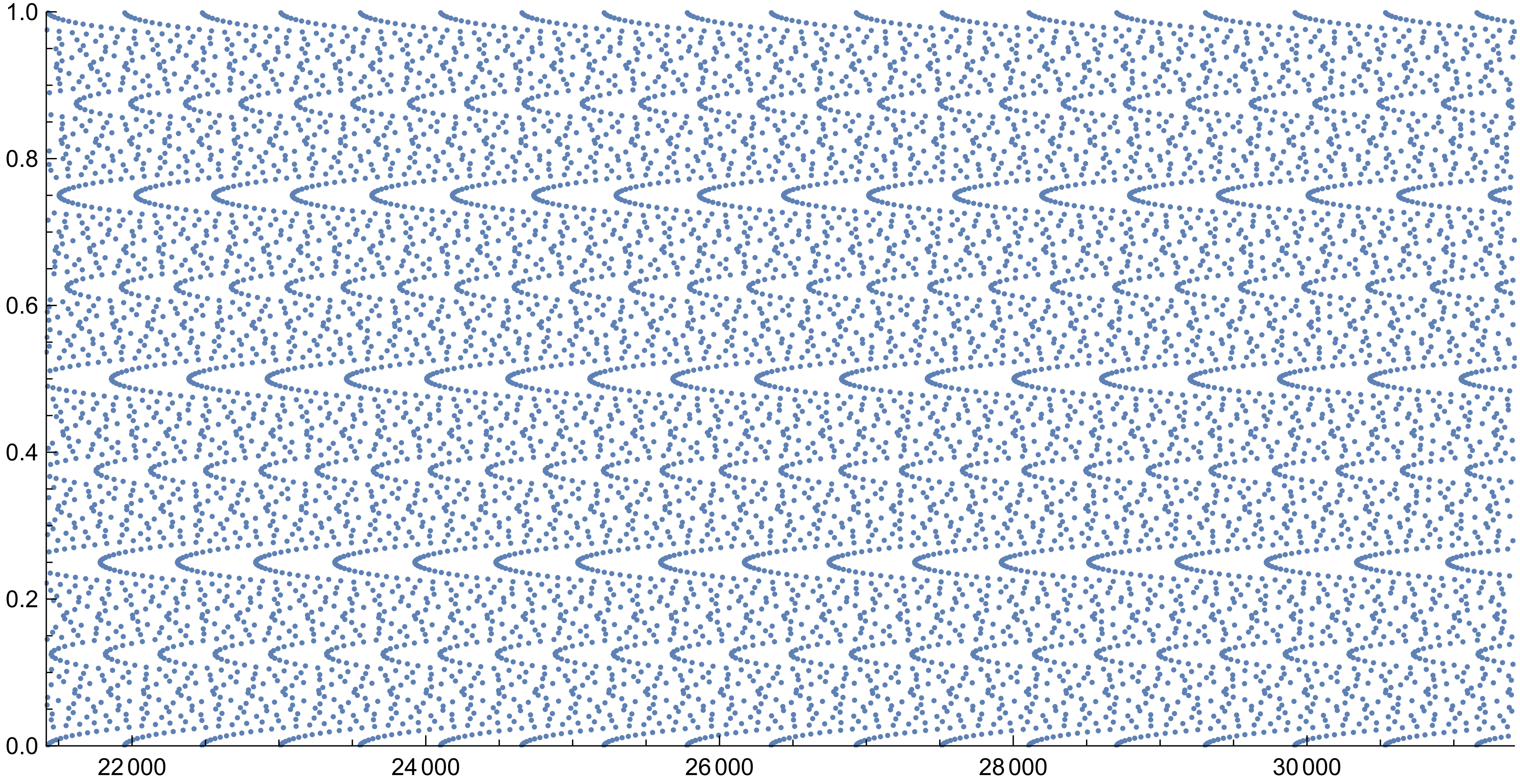
How random is $\mathcal{P}_{\boldsymbol{\alpha}} = \mathbb{Z}^d + \boldsymbol{\alpha}$?

Illustrative example for $d = 2$:

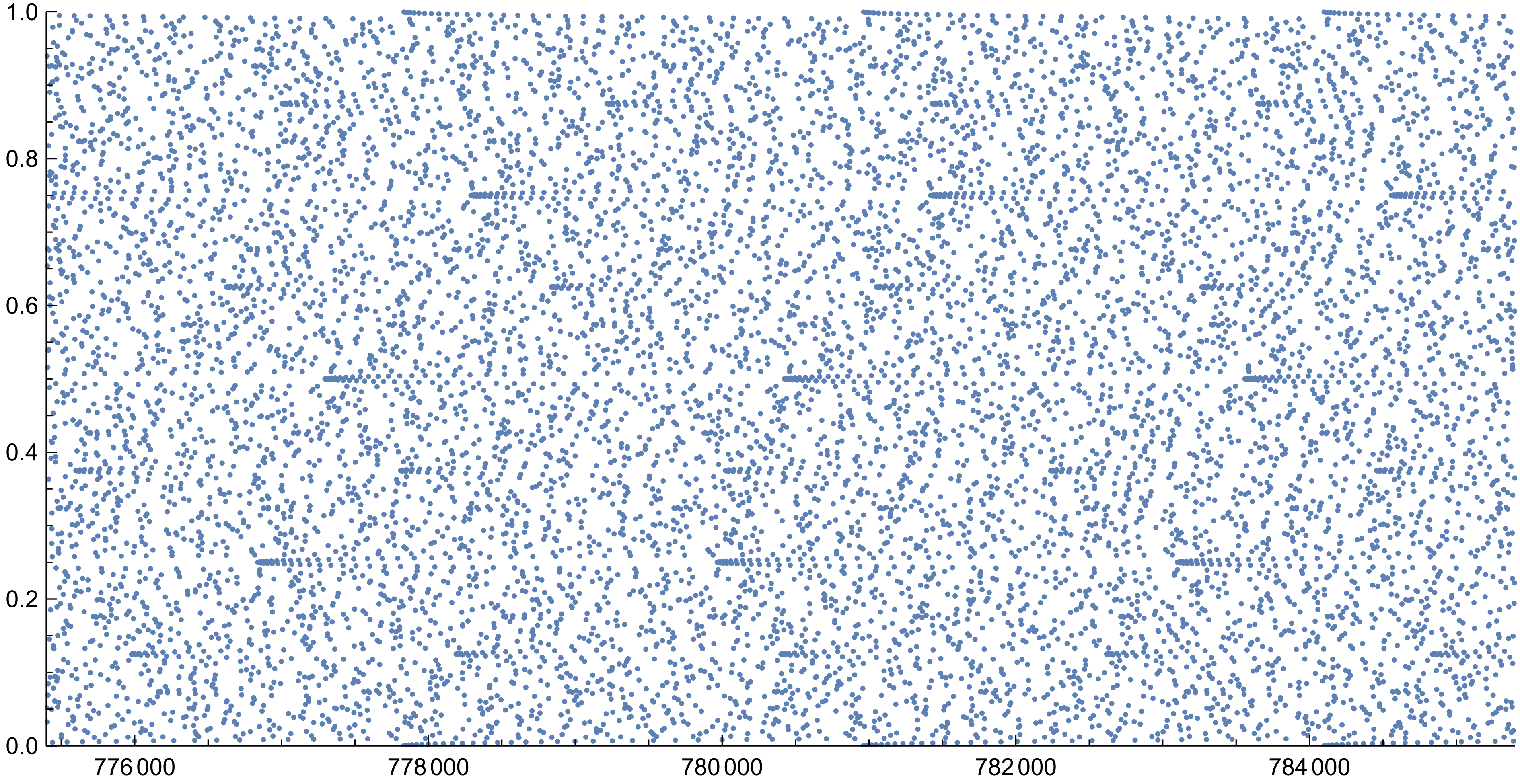
- Fix $\boldsymbol{\alpha} = (\sqrt{2}, \sqrt{3}) \longleftarrow$ not even generic

- Consider the sequence $(\lambda_i, \theta_i)_{i\in\mathbb{N}}$ of elements of the set
$$\left\{ \left(\pi \|\boldsymbol{n}+\boldsymbol{\alpha}\|^2, \frac{1}{2\pi}\arg(\boldsymbol{n}+\boldsymbol{\alpha}) \right) \in \mathbb{R}_{\geq 0} \times [0,1) \;\middle|\; \boldsymbol{n} \in \mathbb{Z}^2 \right\}$$
arranged in increasing order according to the first component

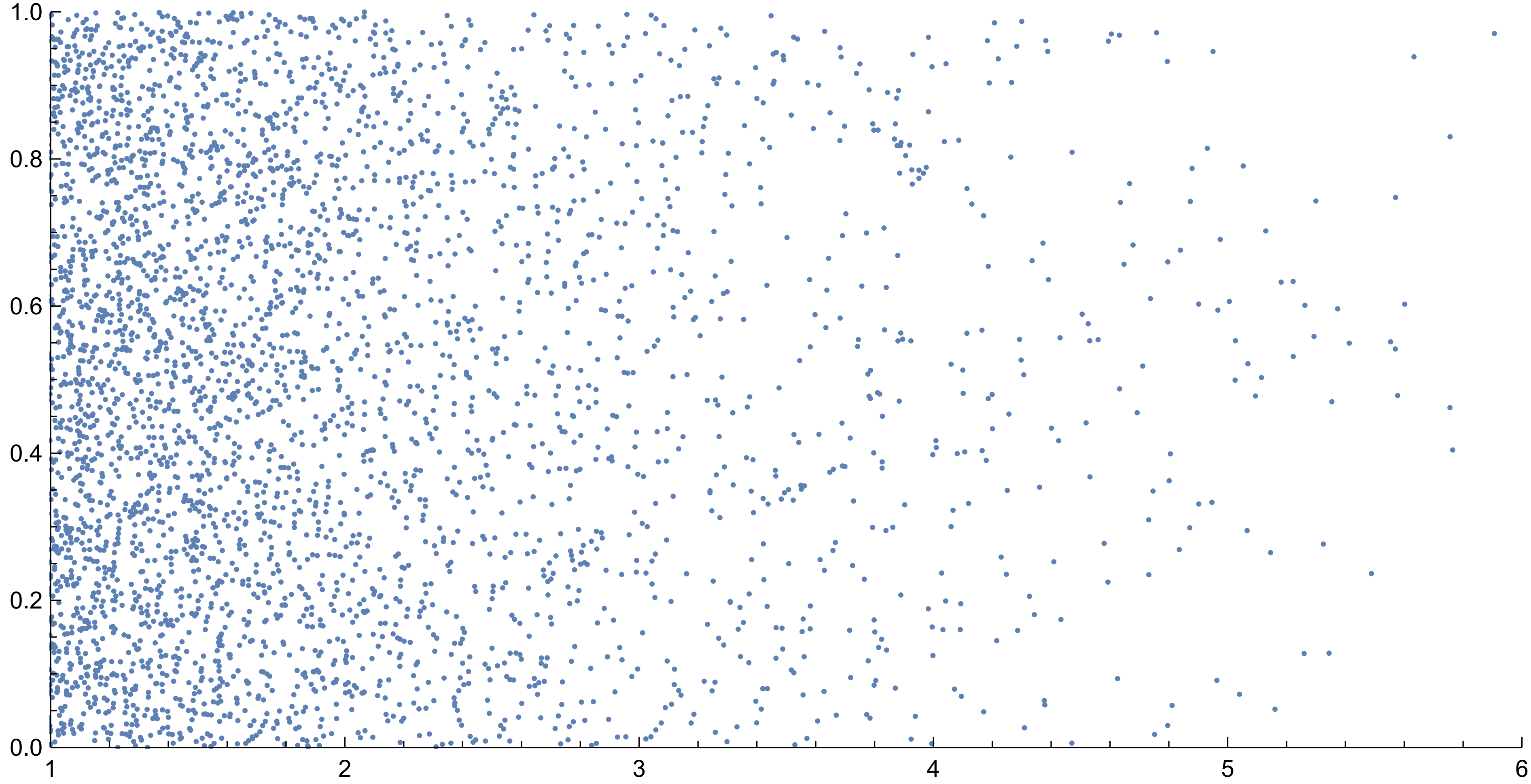
- Our assumption is concerned with the distribution of points (λ_i, θ_i) restricted to a strip $[R - \Delta R, R) \times [0,1)$ for $\Delta R > 0$ fixed and $R \to \infty$



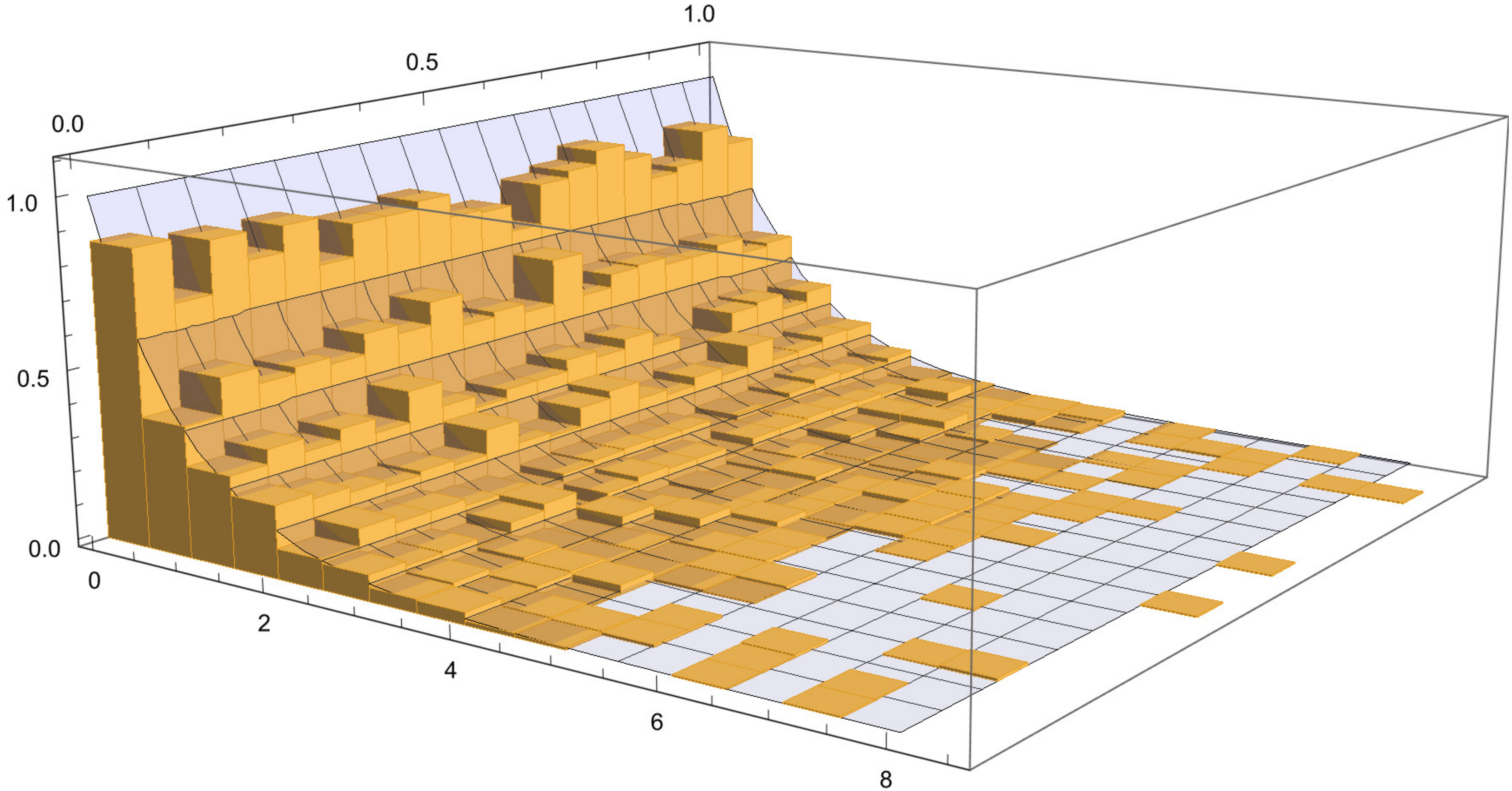
Scatter plots of (λ_i, θ_i) in the strip $[R - \Delta R, R) \times [0, 1)$ for $R = \pi \times 100^2$, with $\Delta R = 10^4$. For large R we expect the point set to be modelled by a Poisson point process.



Scatter plots of (λ_i, θ_i) in the strip $[R - \Delta R, R) \times [0, 1)$ for $R = \pi \times 500^2$, with $\Delta R = 10^4$. For large R we expect the point set to be modelled by a Poisson point process.



Scatter plot for the sequence $(\lambda_{i+1} - \lambda_i, \theta_i)$ for $R = \pi \times 500^2$ and $\Delta R = 10^4$



Histogram for the sequence $(\lambda_{i+1} - \lambda_i, \theta_i)$ for $R = \pi \times 500^2$ and $\Delta R = 10^4$

How random is $\mathcal{P}_{\boldsymbol{\alpha}} = \mathbb{Z}^d + \boldsymbol{\alpha}$?

- Previously we considered the distribution of points

  $$(\lambda_i, \theta_i)_{i\in\mathbb{N}} = \left\{ \left(\pi \|\boldsymbol{n}+\boldsymbol{\alpha}\|^2, \frac{1}{2\pi} \arg(\boldsymbol{n}+\boldsymbol{\alpha}) \right) \in \mathbb{R}_{\geq 0} \times [0,1) \,\middle|\, \boldsymbol{n} \in \mathbb{Z}^2 \right\}$$

  restricted to a strip $[R-\Delta R, R) \times [0,1)$ for $\Delta R > 0$ fixed and $R \to \infty$

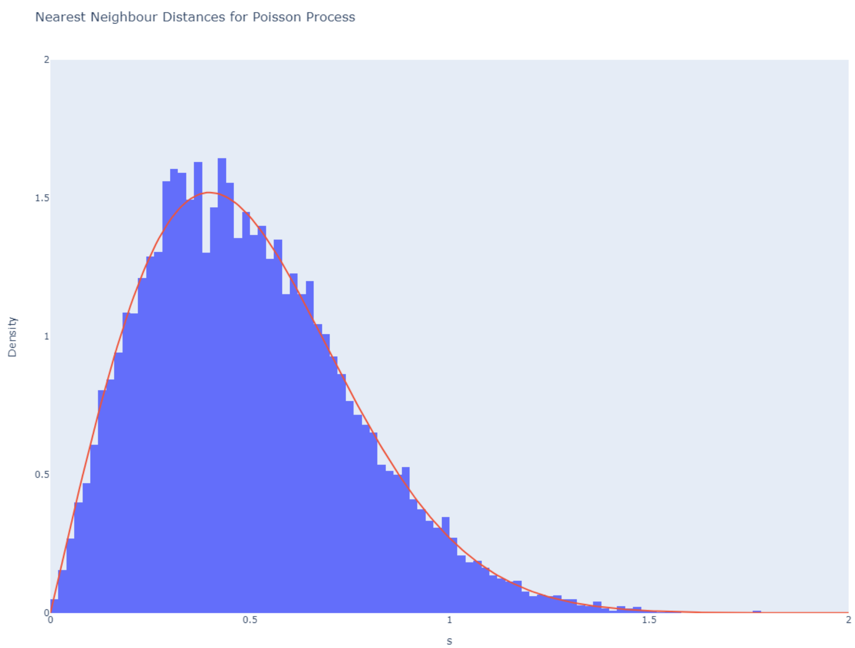
- Let us now look at the fine-scale statistics of the angles $\theta_i \in [0,1)$ only, with $\lambda_i \in [0,R)$

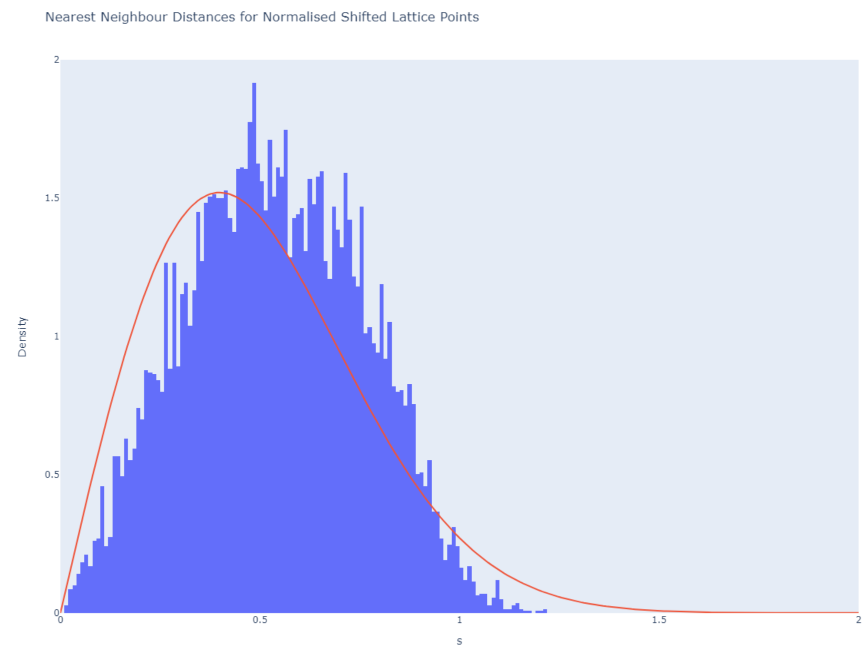
- In higher dimensions, this leads to the statistics of unit vectors $\boldsymbol{v}_i$ given by

  $$\frac{\boldsymbol{n}+\boldsymbol{\alpha}}{\|\boldsymbol{n}+\boldsymbol{\alpha}\|} \in \mathsf{S}_1^{d-1}$$

  where the $\boldsymbol{v}_i$ are corresponding to the ordering of λ_i, i.e., $\|\boldsymbol{n}+\boldsymbol{\alpha}\|^2$

Nearest-neighbour distributions for directions on S_1^2 vs. $2\pi s e^{-\pi s^2}$





Fixed realisation of a Poisson point process in $\mathbb{R}^3$
(JM & Strömbergsson, Mem. AMS 2024)

Affice lattice $\mathbb{Z}^3 + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (2^{1/4}, 3^{1/4}, 5^{1/4})$
(JM & Strömbergsson, Annals 2010)

Numerics by Jory Griffin

Pair correlation (Ripley's K-function)

- Particularly popular fine-scale statistics

$$R_N^2(s) = \frac{\#\left\{(i,j) \in \mathbb{N}^2 \,:\, 0 < i \neq j \leq N,\ c_d N^{\frac{1}{d-1}} \operatorname{dist}_{\mathbb{S}^{d-1}}\left(\boldsymbol{v}_i, \boldsymbol{v}_j\right) \leq s\right\}}{N},$$

$$c_d = \operatorname{vol}_{\mathbb{S}^{d-1}}(\mathbb{S}^{d-1})^{-\frac{1}{d-1}}, \quad \operatorname{dist}_{\mathbb{S}^{d-1}} = \text{arc length}$$

Theorem A.
Let $\mathcal{P}$ be a realization of a Poisson point process Ξ of unit intensity. Then almost surely we have that, for all $s > 0$,

$$\lim_{T\to\infty} R_T^2(s) = \frac{\pi^{\frac{d-1}{2}} s^{d-1}}{\Gamma(\frac{d+1}{2})}$$

Theorem B. (Wooyeon Kim & JM, ETDS 2024)

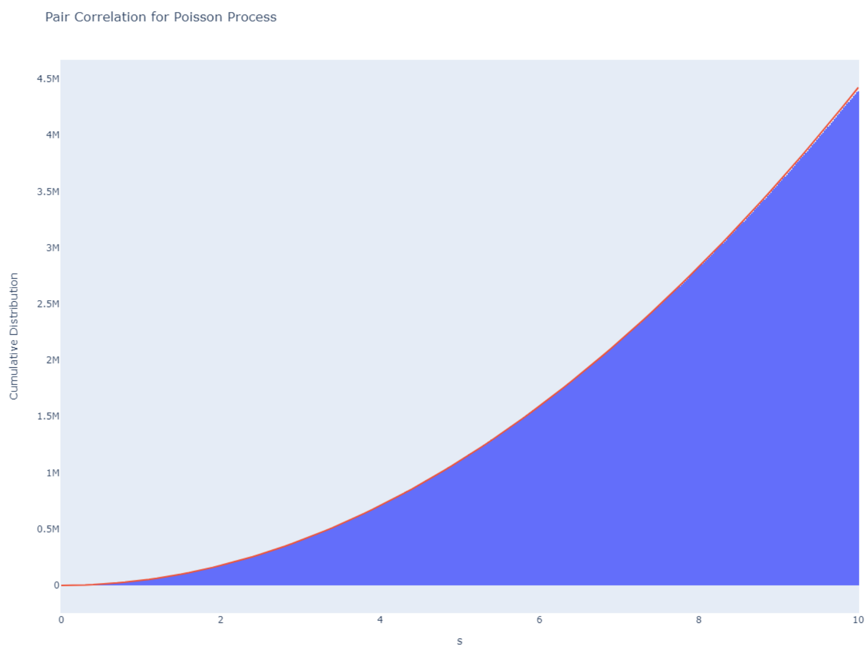
Let $\mathcal{P} = \mathbb{Z}^d + \boldsymbol{\alpha}$, with $\boldsymbol{\alpha} \notin \mathbb{Q}^d$ (plus being (0,0,2)-vaguely Diophantine* if $d = 2$). Then, for all $s > 0$,

$$\lim_{T\to\infty} R_T^2(s) = \frac{\pi^{\frac{d-1}{2}} s^{d-1}}{\Gamma(\frac{d+1}{2})}$$

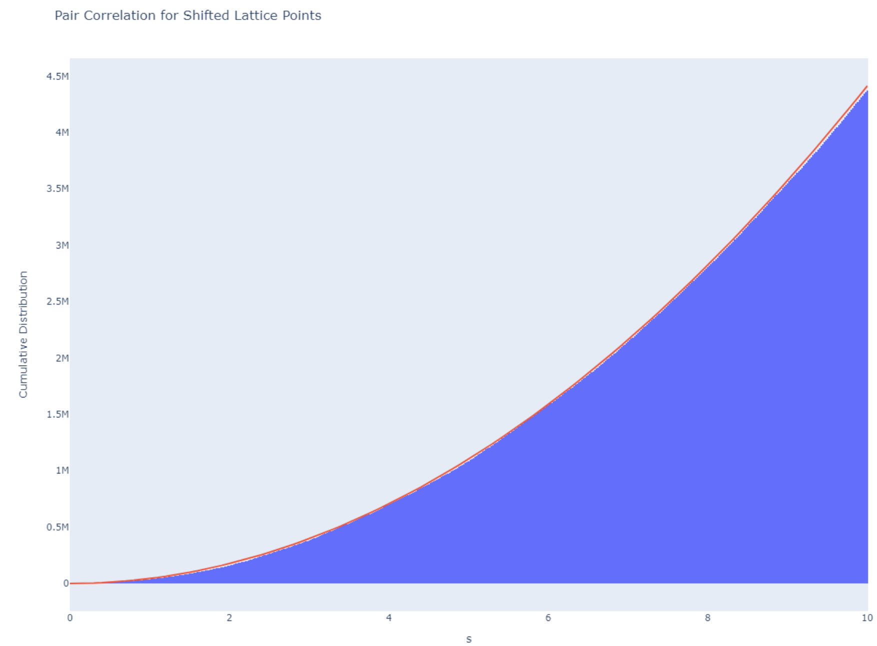
- Previously known for $d = 2$ (El Baz, Vinogradov & JM, IMRN 2015)
- The key points in the proof are: (1) escape of mass estimates for embedded $\mathrm{SL}(d,\mathbb{R})$-horospheres in $\mathrm{ASL}(d,\mathbb{Z})\backslash \mathrm{ASL}(d,\mathbb{R})$, and (2) a Rogers type volume formula that shows that the limit variance is Poissonian.

Pair correlation in 3d

Pair correlation numerics vs. $\dfrac{\pi^{\frac{d-1}{2}} s^{d-1}}{\Gamma(\frac{d+1}{2})}$:



Fixed realisation of a Poisson point process in $\mathbb{R}^3$



Affice lattice $\mathbb{Z}^3 + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (2^{1/4}, 3^{1/4}, 5^{1/4})$, Wooyeon Kim & JM, ETDS 2024

Numerics by Jory Griffin

Fact sheet on Diophantine condition

For $\kappa \geq d$, we say that $\boldsymbol{\alpha} \in \mathbb{R}^d$ is Diophantine of type κ if there exists $C_\kappa > 0$ such that

$$|\boldsymbol{\alpha} \cdot \boldsymbol{m}|_{\mathbb{Z}} > C_\kappa |\boldsymbol{m}|^{-\kappa}$$

for any $\boldsymbol{m} \in \mathbb{Z}^d \setminus \{0\}$, where $|\cdot|$ denotes the supremum norm of $\mathbb{R}^d$, and $|\cdot|_{\mathbb{Z}}$ denotes the supremum distance from $0 \in \mathbb{T}^d$. We will in fact only require a milder Diophantine condition. Define the function $\zeta : \mathbb{R}^d \times \mathbb{R}_{>0} \to \mathbb{N}$ by

$$\zeta(\boldsymbol{\alpha}, T) := \min \left\{ N \in \mathbb{N} : \min_{\substack{\boldsymbol{m} \in \mathbb{Z}^d \setminus \{0\} \\ 0 < |\boldsymbol{m}| \leq N}} |\boldsymbol{\alpha} \cdot \boldsymbol{m}|_{\mathbb{Z}} \leq \frac{1}{T} \right\}.$$

In view of Dirichlet's pigeon hole principle, we have that $\zeta(\boldsymbol{\alpha}, T) \leq T^{1/d}$ and, if $\boldsymbol{\alpha}$ is of Diophantine type $\kappa \geq d$, then $\zeta(\boldsymbol{\alpha}, T) > (C_\kappa T)^{\frac{1}{\kappa}}$.

We say $\boldsymbol{\alpha} \in \mathbb{R}^d$ is (r, μ, ν)-*vaguely* Diophantine, if $\displaystyle\sum_{l=1}^{\infty} l^r \, 2^\mu \zeta(\boldsymbol{\alpha}, 2^{l-1})^{-\nu} < \infty$.

Thus, if $\boldsymbol{\alpha}$ is Diophantine type κ, then it is also (r, μ, ν)-vaguely Diophantine for $\kappa\mu < \nu$.

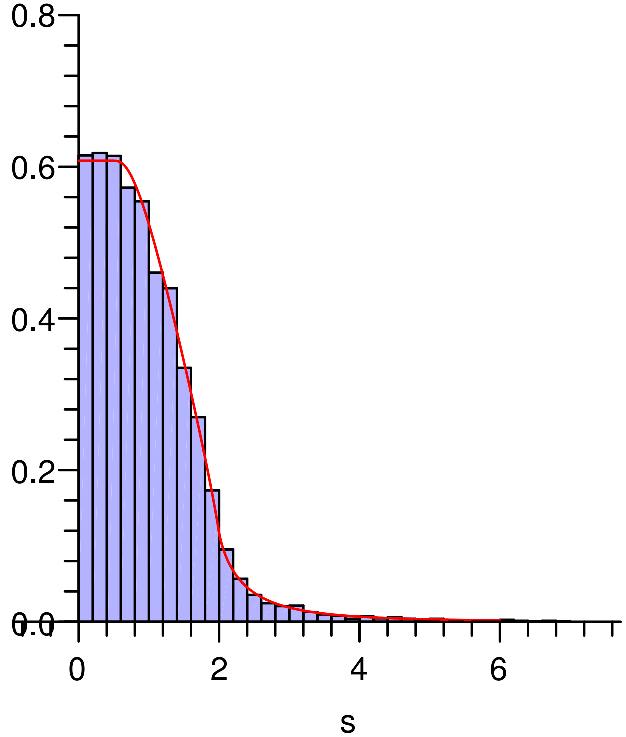
If $\boldsymbol{\alpha}$ satisfies the generalised s-Brjuno Diophantine condition[a] $\displaystyle\sum_{n=0}^{\infty} 2^{-\frac{n}{s}} \max_{\substack{\boldsymbol{m} \in \mathbb{Z}^d \setminus \{0\} \\ 0 < |\boldsymbol{m}| \leq 2^n}} \log \frac{1}{|\boldsymbol{\alpha} \cdot \boldsymbol{m}|_{\mathbb{Z}}} < \infty$

then it is $(r, 0, \nu)$-vaguely Diophantine for $s > \frac{r+1}{\nu}$.

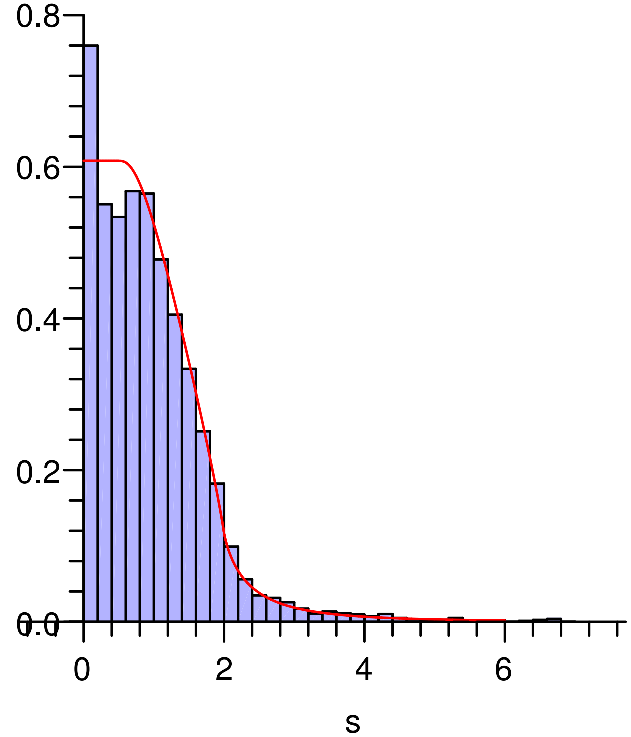
[a]Bounemoura & Féjoz, Ann Sc Norm Super Pisa Cl Sci (2019)
Lopes Dias & Gaivão, J Diff Equ 267 (2019)

Gap distribution for directions in a 2d affine lattice

Gap distribution of directions in 2d affine lattice vs. Elkies-McMullen distribution:



Affice lattice $\mathbb{Z}^2 + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (\sqrt{2}, 0)$
(JM & Strömbergsson, Annals 2010)



$\sqrt{n} \bmod 1, n = 1, \ldots, 7765$
Elkies & McMullen Duke 2005

Both proofs use Ratner's measure classification theorem on the same space for same test function – but for different unipotent flows! Tail is $\sim \frac{3}{\pi^2} s^{-3}$. Note that for $\boldsymbol{\alpha} = 0$ we would (taking only the visible lattice points) recover the classical Hall distribution (Hall, J LMS 1970) for the gaps between Farey points. Hall density has tail $\sim \frac{36}{\pi^4} s^{-3}$.

Conclusion

- Complete understanding of the Boltzmann-Grad limit for the quantum Lorentz gas remains major challenge; currently only annealed limit for random scatterer configuration fully understood

- Periodic setting leads to subtle lattice point problems and requires a major hypothesis (Berry-Tabor conjecture)

- Is the long-time limit of the macroscopic process (super-) diffusive? Other scalling limits: $h \ll r$ or $r \ll h$? Extension to quasicrystals or other scatterer configurations with long-range correlations?

Happy Birthday, Maciej!