Analytic hypoellipticity and classically forbidden regions for twisted bilayer graphene

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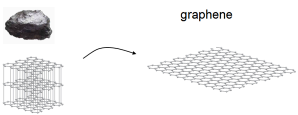
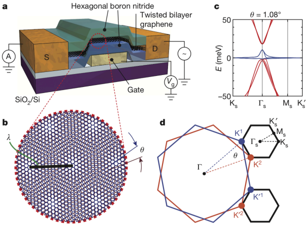
In honor of MACIEJ ZWORSKI

Joint work with M. Zworski

Motivation : twisted bilayer graphene

In this talk, we shall be concerned with some aspects of semiclassical analysis for a class of non-self-adjoint operators coming from condensed matter physics of 2D materials.





Cao et al (2018), Yankovitz et al (2019) : superconductivity at $\theta \simeq 1.08°$.
Predicted by R. Bistritzer – A. MacDonald (2011).

The chiral model of twisted bilayer graphene

G. Tarnopolsky – A. Kruchkov – A. Vishwanath (2019) :

$$H(\alpha) := \begin{pmatrix} 0 & D(\alpha)^* \\ D(\alpha) & 0 \end{pmatrix}, \quad D(\alpha) := \begin{pmatrix} 2D_{\bar{z}} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar{z}} \end{pmatrix},$$

$$z = x_1 + ix_2 \in \mathbb{C}, \quad D_{\bar{z}} = \tfrac{1}{i}\partial_{\bar{z}} = \tfrac{1}{2i}(\partial_{x_1} + i\partial_{x_2}),$$

acting on $L^2(\mathbb{C};\mathbb{C}^4)$. Here $U(z)$ is the Bistritzer–MacDonald potential,

$$U(z) = -iK \sum_{\ell=0}^{2} \omega^\ell e^{i\langle z, \omega^\ell K\rangle}, \quad K = \frac{4\pi}{3}, \;\; \omega = e^{2\pi i/3}, \;\; \langle z, w\rangle = \mathrm{Re}\,(z\overline{w}).$$

The Hamiltonian $H(\alpha)$ is derived from the full Bistritzer–MacDonald Hamiltonian by removing certain tunneling interactions between the two sheets of graphene. The dimensionless coupling constant α is such that the angle of twisting $\asymp 1/\alpha$.

Mathematical derivation :

Cancès-Garrigue-Gontier (2023), Watson-Kong-MacDonald-Luskin (2023).

Let

$$\Lambda := \omega\mathbb{Z} \oplus \mathbb{Z}, \quad \Lambda^* = \frac{4\pi i}{\sqrt{3}}\Lambda.$$

Here

$$\Lambda^* = \{k \in \mathbb{R}^2; \langle k, \gamma\rangle \in 2\pi\mathbb{Z} \text{ for every } \gamma \in \Lambda\}$$

is the dual lattice.

Symmetries of the potential U :

$$U(z+\gamma) = e^{i\langle\gamma,K\rangle}U(z), \quad \gamma \in \Lambda, \quad U(\omega z) = \omega U(z), \quad \overline{U(\bar{z})} = -U(-z),$$

$\Longrightarrow$ U is periodic with respect to $\Gamma = 3\Lambda$.

Flat bands

Performing a Floquet reduction of $H(\alpha)$, we are led to consider the family

$$H_k(\alpha) := e^{i\langle z,k\rangle} H(\alpha) e^{-i\langle z,k\rangle} = \begin{pmatrix} 0 & D(\alpha)^* - \overline{k} \\ D(\alpha) - k & 0 \end{pmatrix}, \quad k \in \mathbb{C}/\Gamma^*,$$

acting on $L^2(\mathbb{C}/\Gamma; \mathbb{C}^4)$, with the domain $H^1(\mathbb{C}/\Gamma; \mathbb{C}^4)$. Here Γ^* is the dual lattice of Γ. A flat band at zero energy for $H(\alpha)$ occurs when

$$0 \in \mathrm{Spec}_{L^2(\mathbb{C}/\Gamma)}(H_k(\alpha))$$

for all $k \in \mathbb{C}$, or equivalently, when

$$\mathrm{Spec}_{L^2(\mathbb{C}/\Gamma)}(D(\alpha)) = \mathbb{C}.$$

We have

$$D(\alpha) : H^1(\mathbb{C}/\Gamma; \mathbb{C}^2) \to L^2(\mathbb{C}/\Gamma; \mathbb{C}^2), \quad \alpha \in \mathbb{C},$$

is Fredholm of index 0 such that

$$\mathrm{Spec}_{L^2(\mathbb{C}/\Gamma)}(D(\alpha)) = \mathrm{Spec}_{L^2(\mathbb{C}/\Gamma)}(D(\alpha)) + k, \quad k \in \Gamma^*.$$

The spectrum of $D(\alpha)$ and magic angles

Theorem (S. Becker, M. Embree, J. Wittsten, and M. Zworski (2022))

There exists a discrete set $\mathcal{A} \subset \mathbb{C}$ such that

$$\mathrm{Spec}_{L^2(\mathbb{C}/\Gamma)} D(\alpha) = \begin{cases} \Gamma^*, & \alpha \notin \mathcal{A}, \\ \mathbb{C}, & \alpha \in \mathcal{A}. \end{cases}$$

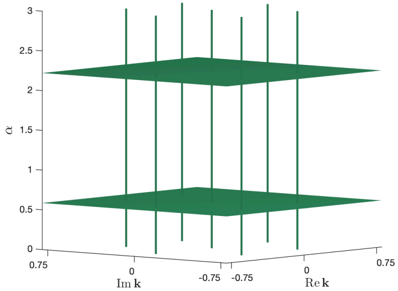


FIGURE – Spectrum of $D(\alpha)$ as α varies. Magic angles : $1/\alpha$, $\alpha \in \mathcal{A}$.

Crucial component of the proof : symmetry protected eigenstates at 0,

$$\mathrm{Ker}_{L^2_{\rho_1,0}(\mathbb{C}/\Gamma)} D(\alpha) \neq \{0\}, \quad \alpha \in \mathbb{C}.$$

J. Galkowski – M. Zworski (2023) : an abstract formulation of the flat band condition.

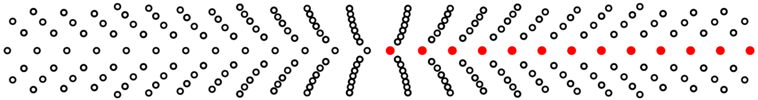


FIGURE – Reciprocals of magic angles for the Bistritzer-MacDonald potential (Becker–Embree–Wittsten–Zworski (2022)).

S. Becker – T. Humbert – M. Zworski (2023) : the set $\mathcal{A}$ is infinite.

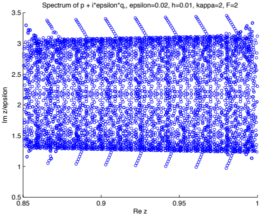
M. Luskin – A. Watson (2021), S. Becker – T. Humbert – M. Zworski (2023) : existence of the first real positive magic α.

Quantization condition for magic angles ?

Numerical observation by Tarnopolsky – Kruchkov – Vishwanath (2019), Becker – Embree – Wittsten – Zworski (2022) : if $\alpha_1 < \alpha_2 < \cdots \alpha_j < \cdots$ is the sequence of all real α's in $\mathcal{A}$, then

$$\alpha_{j+1} - \alpha_j \simeq 1.515, \quad j \leq 13.$$

A. Melin – J. Sjöstrand (2002), J. Sjöstrand – M.H. (2004 – 2018) : quantization rules for eigenvalues of semi-classical non-self-adjoint analytic operators in dimension 2.



Can we apply the 2D non-self-adjoint machinery in this setting ?

Spectra of elliptic first order scalar operators on tori

A. Melin – J. Sjöstrand (2002) : let

$$P = a(z)2D_{\bar{z}} + b(z)$$

on $L^2(\mathbb{C}/\Gamma)$, with a, $b \in C^\infty(\mathbb{C}/\Gamma)$, a nowhere vanishing. We have :

$$\lambda \in \mathrm{Spec}_{L^2(\mathbb{C}/\Gamma)}(P) \Longleftrightarrow \mathcal{F}\left(\frac{b}{a}\right)(0) - \lambda \mathcal{F}\left(\frac{1}{a}\right)(0) \in \Gamma^*.$$

In particular, we get a lattice of eigenvalues precisely when

$$\mathcal{F}\left(\frac{1}{a}\right)(0) \neq 0,$$

whereas if $\mathcal{F}(1/a)(0) = 0$, we get

$$\mathrm{Spec}_{L^2(\mathbb{C}/\Gamma)} P = \begin{cases} \mathbb{C}, & \mathcal{F}(b/a)(0) \in \Gamma^*, \\ \emptyset, & \mathcal{F}(b/a)(0) \notin \Gamma^*. \end{cases}$$

R. Seeley (1986) : a similar example in 1D, $P(\alpha) = e^{ix} D_x + \alpha e^{ix}$, $x \in \mathbb{R}/2\pi\mathbb{Z} : \mathrm{Spec}(P(\alpha) = \mathbb{C}$, $\alpha \in \mathbb{Z}$.

Protected states in the semiclassical limit

This talk : Understand the structure of protected eigenstates at 0 of $D(\alpha)$ in the small angle limit $0 < \alpha \to \infty$ (within or without the magic set),

$$D(\alpha)u = 0, \quad u \in L^2_{\rho_1,0}(\mathbb{C}/\Gamma; \mathbb{C}^2).$$

Semiclassical formulation with $0 < h = \dfrac{1}{\alpha} \ll 1$,

$$p(x, hD_x)u = 0, \quad p(x, hD_x) = hD(\alpha) = \begin{pmatrix} 2hD_{\bar{z}} & U(z) \\ U(-z) & 2hD_{\bar{z}} \end{pmatrix},$$

$$p(x, \xi) = \begin{pmatrix} 2\bar{\zeta} & U(z) \\ U(-z) & 2\bar{\zeta} \end{pmatrix}, \quad z = x_1 + ix_2, \ \ \zeta = \tfrac{1}{2}(\xi_1 - i\xi_2).$$

Principally scalar reduction

Observe that

$$(hD(-\alpha))(hD(\alpha)) = q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x),$$

where

$$q(x,\xi) = \det p(x,\xi) = 4\bar{\zeta}^2 - U(z)U(-z),$$
$$q(x, hD_x) = (2hD_{\bar{z}})^2 - U(z)U(-z),$$

and

$$R(x) = \begin{pmatrix} 0 & 2D_{\bar{z}}U(z) \\ -D_{\bar{z}}U(-z) & 0 \end{pmatrix},$$

to get

$$(q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x))\, u = 0.$$

Classically forbidden regions

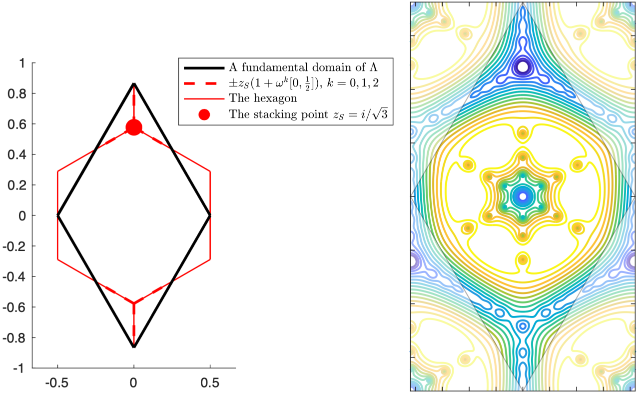


FIGURE – Left : the vertices of the hexagon in a fundamental domain of Λ are given by the *stacking points* $\pm z_S$, $z_S = i/\sqrt{3}$, i.e. points of high symmetry satisfying $\pm\omega z_S \equiv \pm z_S \mod \Lambda$. Right : plot of $\log|u(z,\alpha)|$ where u is the protected state in the kernel of $D(\alpha)$ on $H^1(\mathbb{C}/\Gamma)$ and $\alpha = 11.345$. Dark blue corresponds to $|u| \simeq 10^{-7}$ and yellow to $|u| \simeq 1$: we see exponential decay $|u(z,\alpha)| \leq e^{-c_0/h}$ near the hexagon and near its center.

The Poisson bracket $\{q, \overline{q}\}$

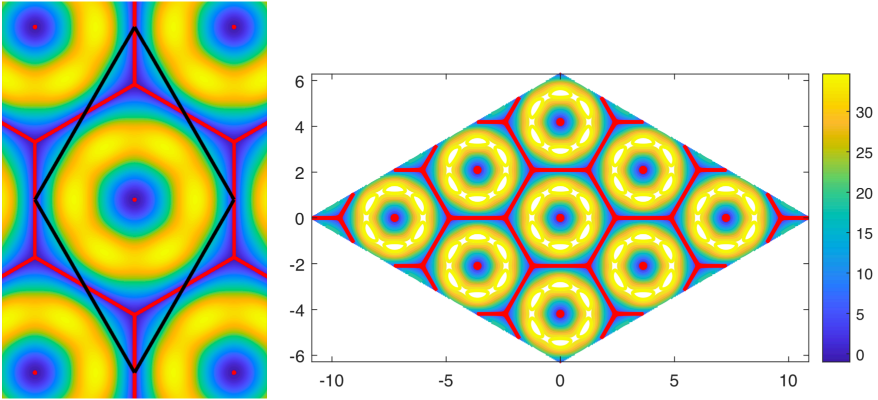


FIGURE – Left : the contour plot of $|\{q, \bar{q}\}|_{q^{-1}(0)}|$, for q given by the determinant of the semiclassical symbol of $hD(\alpha)$, $\alpha = 1/h$, $q(x, \xi) = (2\bar{\zeta})^2 - U(z)U(-z)$. Right : the contour plot of $|\{q, \bar{q}\}|_{q^{-1}(0)}|$ over a fundamental domain of $\Gamma = 3\Lambda$. The set where $\{q, \bar{q}\}|_{q^{-1}(0)} = 0$ is in red.

Classically forbidden regions for Schrödinger operators

Let

$$(-h^2\Delta + V(x) - E)u = 0, \quad x \in \mathbb{R}^n.$$

Exponential decay of eigenfunctions in the classically forbidden region $\mathcal{U} = \{x \in \mathbb{R}^n; V(x) > E\}$ is a consequence of ellipticity :

$$p(x_0,\xi) = \xi^2 + V(x_0) - E \neq 0, \quad x_0 \in \mathcal{U},\ \xi \in \mathbb{R}^n.$$

L. Lithner (1964), S. Agmon (1982), B. Simon (1984), B. Helffer – J. Sjöstrand (1984).

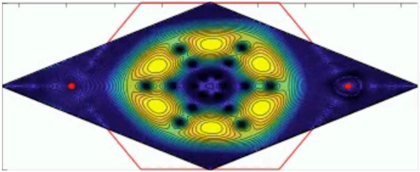
For $q(x,\xi) = (2\bar{\zeta})^2 - U(z)U(-z)$, there are no classically forbidden regions in the sense of ellipticity,

$$\forall\, x_0 \in \mathbb{R}^2 \ \ q^{-1}(0) \cap \pi^{-1}(x_0) = \{\xi \in \mathbb{R}^2; q(x_0,\xi) = 0\} \neq \emptyset.$$

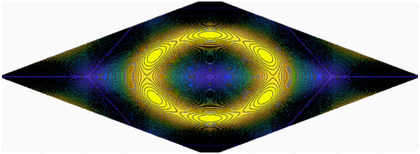
Here $\pi : T^*\mathbb{R}^2 \ni (x,\xi) \mapsto x \in \mathbb{R}^2$ is the natural projection.

Use (analytic) hypoellipticity as a replacement ?

Classically forbidden regions when $\{q, \bar{q}\} = 0$



$$(q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x))u = 0$$



$$|\{q, \bar{q}\}|_{q^{-1}(0)}$$

Theorem (M. Zworski – M.H. 2023)

Let $U \subset \mathbb{R}^2$ be open and let

$$P = P(x, hD_x; h) = Q \otimes 1_{\mathbb{C}^2} + h \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \quad x \in U,$$

be a principally scalar system of semiclassical differential operators with real analytic coefficients in U, such that $Q = q(x, hD_x)$ is classically elliptic of order 2, and $R_{k\ell} = R_{k\ell}(x, hD_x)$ are of order 1, for $1 \leq k, \ell \leq 2$. Assume that for $x_0 \in U$, we have

$$\{q, \bar{q}\}|_{q^{-1}(0) \cap \pi^{-1}(x_0)} = 0, \quad \{q, \{q, \bar{q}\}\}|_{q^{-1}(0) \cap \pi^{-1}(x_0)} \neq 0,$$

and $H_{\mathrm{Re}\, q}$ and $H_{\mathrm{Im}\, q}$ are linearly independent on $q^{-1}(0) \cap \pi^{-1}(x_0)$. If $Pu = 0$ in U and $\|u\|_{L^2(U)} \leq \mathcal{O}(1)$, then there exists an h-independent neighborhood Ω of x_0 and $C_0, c_0 > 0$ such that for all $0 < h \leq h_0$ we have,

$$|u(x; h)| \leq C_0\, e^{-c_0/h}, \quad x \in \Omega.$$

A simple example in 1D

Let

$$q(x,\xi) = \xi + ix^2, \quad (x,\xi) \in T^*\mathbb{R},$$

with $x_0 = 0$. Then

$$\{q,\bar{q}\}(0,0) = 0, \quad \{q,\{q,\bar{q}\}\} = -4i \neq 0,$$

so the bracket conditions hold.

If

$$0 = q(x, hD_x)u = \frac{h}{i}\left(\partial_x - \frac{x^2}{h}\right)u,$$

then

$$u(x;h) = u(0,h)e^{x^3/3h}.$$

For this to be uniformly bounded near 0, we need $u(0;h) = \mathcal{O}(1)e^{-c/h}$, $c > 0$, and hence $|u(x;h)| \leq \mathcal{O}(1)e^{-c/2h}$ for $|x|$ small.

A quick digression : tunneling for non-self-adjoint operators

J. Sjöstrand – M. Vogel (2023), J. Sjöstrand – M. Vogel – M.H. (work in progress, 2024).

Let $M = S^1 + iS^1$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and consider the conjugated $\overline{\partial}$–operator

$$P = e^{-\varphi/h} \circ h\overline{\partial} \circ e^{\varphi/h} = h\overline{\partial} + \overline{\partial}\varphi,$$

where $\varphi \in C^\infty(M;\mathbb{R})$, φ non-constant.

P has the symbol

$$p(x,y;\xi,\eta) = \frac{i}{2}(\xi + i\eta) + \overline{\partial}\varphi, \quad (x,y;\xi,\eta) \in T^*M,$$

and

$$\frac{1}{i}\{p,\overline{p}\} = \frac{1}{2}\Delta\varphi.$$

The operator P has singular values, i.e. eigenvalues of $(P^*P)^{1/2}$, that are $\mathcal{O}(e^{-1/Ch})$, for φ real analytic.

L. Hörmander (1960), M. Sato – T. Kawai – M. Kashiwara (1973), . . . M. Zworski (2001), N. Dencker – J. Sjöstrand – M. Zworski (2004).

Tunneling for non-self-adjoint operators

Model tunneling problem : establish Weyl asymptotics for exponentially small singular values of P.

J. Sjöstrand – M. Vogel – M.H. (work in progress, 2024) : preliminary results on upper and lower bounds on the number of singular values of P in $[0, e^{-\tau/h}]$, for $0 < \tau < \max\varphi - \min\varphi$, expressed in terms of suitable auxiliary upper and lower bound weights $\varphi - \tau \leq \psi_{\ell\mathrm{b}}, \psi_{\mathrm{ub}} \leq \varphi$,

$$\frac{1}{2\pi h}\iint_{\Lambda_\varphi} 1_{\{\psi_{\ell\mathrm{b}}=\varphi\}}(x)\, d\xi \wedge dx - \frac{o(1)}{h}$$

$$\leq \#\left(\mathrm{Spec}((P^*P)^{1/2}) \cap [0, e^{-\tau/h}]\right) \leq \frac{1}{2\pi h}\iint_{\Lambda_\varphi} 1_{\{\psi_{\mathrm{ub}}=\varphi\}}(x)\, d\xi \wedge dx + \frac{o(1)}{h},$$

Here

$$\Lambda_\varphi = \left\{ \left(z, \frac{2}{i}\frac{\partial\varphi}{\partial z}(z)\right); z \in M \right\}.$$

A basic case is when $\Delta\varphi$ changes sign along a curve $\subset M$.

Some words about the proof

Step I. Establish microlocal exponential decay of u.

Step II. From microlocal to local exponential decay.

When describing Step I, we need to recall the notion of the semiclassical analytic wave front set of an h–tempered family $h \mapsto u(h) \in \mathcal{D}'(U)$.

The key role is played by the FBI (Fourier-Bros-Iagolnitzer) – Bargmann transform,

$$T_h w(x) = \int_{\mathbb{R}^2} e^{\frac{i}{h}\varphi_0(x,y)} w(y)\, dy, \quad \varphi_0(x,y) = \frac{i}{2}(x-y)^2, \quad x \in \mathbb{C}^2.$$

Given $(y_0, \eta_0) \in T^*U$, we have $(y_0,\eta_0) \notin \mathrm{WF}_{a,h}(u)$ precisely when $\exists\, \delta > 0, C > 0,\ V = \mathrm{neigh}(y_0 - i\eta_0, \mathbb{C}^2)$ such that

$$|T_h(\chi u)(x)| \leq C\, e^{(\Phi_0(x)-\delta)/h}, \quad x \in V,\ 0 < h \leq h_0.$$

Here $\chi \in C_0^\infty(U)$, $\chi(y) = 1$ in a neighborhood of y_0, and

$$\Phi_0(x) := \frac{1}{2}\left|\mathrm{Im}\, x\right|^2.$$

FBI transforms are flexible objects, and in the proof we work with a local transform of the form

$$T_h u(x) = \int_{\mathbb{R}^2} e^{i\varphi(x,y)/h} a(x,y;h)\chi(y)\, u(y),\, dy, \quad x \in \mathrm{neigh}(x_0, \mathbb{C}^2).$$

Here $\chi \in C_0^\infty(U)$, $\chi = 1$ near y_0, and $\varphi \in \mathrm{Hol}\left(\mathrm{neigh}\left((x_0, y_0), \mathbb{C}^4\right)\right)$, for some $x_0 \in \mathbb{C}^2$, is such that

$$-\varphi'_y(x_0, y_0) = \eta_0, \quad \mathrm{Im}\,\varphi''_{yy}(x_0, y_0) > 0, \quad \det \varphi''_{xy}(x_0, y_0) \neq 0.$$

The amplitude $a(x, y; h)$ is an elliptic classical analytic symbol in a neighborhood of (x_0, y_0),

$$a(x, y; h) \sim \sum_{j=0}^{\infty} a_j(x, y) h^j,$$

where a_j are holomorphic, with $a_0 \neq 0$, and such that

$$|a_j(x, y)| \leq C^{j+1} j^j, \quad j = 0, 1, 2, \ldots.$$

J. Sjöstrand (1982), ..., D. Tataru (1999), A. Martinez (2002), L. Robbiano – C. Zuily (2002), ... , J. Galkowski – M. Zworski (2019).

The FBI transform $T_h u$ of u is holomorphic and satisfies for each $\varepsilon > 0$,

$$|T_h u(x)| \leq \mathcal{O}_\varepsilon(1) e^{(\Phi(x)+\varepsilon)/h}, \quad x \in \mathrm{neigh}(x_0, \mathbb{C}^2),$$

where the weight

$$\Phi(x) = \sup_{y \in \mathrm{neigh}(y_0, \mathbb{R}^2)} (-\mathrm{Im}\, \varphi(x,y))$$

is strictly plurisubharmonic.

The definition of $\mathrm{WF}_{a,h}(u)$ is independent of the choice of an FBI transform :

Theorem (J. Sjöstrand, 1982)

We have $(y_0, \eta_0) \notin \mathrm{WF}_{a,h}(u)$ if and only if there exist $\delta > 0$, $C > 0$, and $V = \mathrm{neigh}(x_0, \mathbb{C}^2)$ such that

$$|T_h u(x)| \leq C\, e^{(\Phi(x)-\delta)/h}, \quad x \in V,\ 0 < h \leq h_0.$$

Microlocal analytic hypoellipticity

Proposition

Let $(0, h_0] \ni h \mapsto u(h) \in \mathcal{D}'(U; \mathbb{C}^2)$ *be h–tempered. If at some point* $\rho = (y_0, \eta_0) \in q^{-1}(0)$ *we have*

$$\{q, \bar{q}\}(\rho) = 0, \quad \{q, \{q, \bar{q}\}\}(\rho) \neq 0, \quad H_q(\rho) \not\parallel H_{\bar{q}}(\rho), \quad \rho \notin \mathrm{WF}_{a,h}(Pu),$$

then $\rho \notin \mathrm{WF}_{a,h}(u)$.

This result is based on the work of M. Kashiwara and T. Kawai (1979), J.-M. Trépreau (1984), J. Sjöstrand (1982), A. Himonas (1986) in the setting of analytic hypoellipticity for classical PDO. An alternative proof, based on complex deformations of phase space, has been given recently by J. Sjöstrand (2024).

Reduction to hD_{x_1}

There exists a (vector-valued) FBI transform

$$T_h u(x) = \int e^{i\varphi(x,y)/h} a(x,y;h)\chi(y)\, u(y),\, dy, \quad x \in \mathrm{neigh}(0,\mathbb{C}^2),$$

such that for some $\delta > 0$,

$$hD_{x_1} T_h u(x) = T_h(Pu)(x) + \mathcal{O}(1)e^{(\Phi(x)-\delta)/h}, \quad x \in \mathrm{neigh}(0,\mathbb{C}^2).$$

Here φ satisfies the complex eikonal equation

$$\varphi'_{x_1}(x,y) = q(y, -\varphi'_y(x,y)), \quad (x,y) \in \mathrm{neigh}(0,\mathbb{C}^2) \times \mathrm{neigh}(y_0,\mathbb{C}^2),$$

$$-\varphi'_y(0,y_0) = \eta_0, \quad \mathrm{Im}\,\varphi''_{yy}(0,y_0) > 0, \quad \det \varphi''_{xy}(0,y_0) \neq 0.$$

This a well known consequence of analytic WKB in the scalar case (J. Sjöstrand (1982)), and it also works for principally scalar systems.

Subharmonic minorants

It follows that $U(x;h) = T_h u(x)$ is essentially independent of x_1,

$$hD_{x_1} U(x;h) = \mathcal{O}(1) e^{(\Phi(x)-\delta)/h}, \quad x \in \mathrm{neigh}(0, \mathbb{C}^2),$$

and therefore

$$|U(x;h)| \leq \mathcal{O}_\varepsilon(1) e^{(\Psi(x_2)+\varepsilon)/h}, \quad x \in \mathrm{neigh}(0, \mathbb{C}^2).$$

Here

$$\Psi(x_2) = \inf_{x_1} \Phi(x_1, x_2)$$

need no longer be subharmonic $\Longrightarrow$ we get

$$|U(x;h)| \leq \mathcal{O}_\varepsilon(1) e^{(\widetilde{\Psi}(x_2)+\varepsilon)/h}, \quad x \in \mathrm{neigh}(0, \mathbb{C}^2),$$

where $\widetilde{\Psi}$ is the largest subharmonic minorant of Ψ. If we can show that

$$\widetilde{\Psi}(0) < \Phi(0),$$

then $\rho \notin \mathrm{WF}_{a,h}(u)$. (Idea of Kashiwara (1979).)

Choosing the weight I

The complex eikonal equation has many solutions, so how do we choose the right one ?

Normal form : $q(y, \eta) = \eta_1 + i\eta_2 + iy_1^2$, $(y_0, \eta_0) = (0, 0)$.

Proposition (long tradition . . . A. Himonas 1986)

There exists a real analytic canonical transformation

$$\kappa : \mathrm{neigh}((y_0, \eta_0), T^*U) \to \mathrm{neigh}((0,0), T^*\mathbb{R}^2), \quad \kappa(y_0, \eta_0) = (0, 0),$$

and a real analytic function a defined in a neighborhood of $(0, 0)$, *with* $a(0, 0) \neq 0$, *such that*

$$q \circ \kappa^{-1} = a(y, \eta) q_0(y, \eta), \quad q_0(y, \eta) := \eta_1 + i\left(\eta_2 + y_1 g(y, \eta_2)\right),$$

where g is real valued real analytic satisfying $g(0) = 0$, $g'_{y_1}(0) \neq 0$, *and* $g'_{y_2}(0) = 0$.

Choosing the weight II

To work with the approximate model symbol in the proposition, we use a scaling argument (taking $a = 1$ for simplicity),

$$\widetilde{q}_0(y,\eta) = \frac{1}{\mu^2} q_0(\mu y, \mu^2 \eta), \quad 0 < \mu \ll 1.$$

We have, for $0 \neq c \in \mathbb{R}$,

$$\widetilde{q}_0(y,\eta) = \eta_1 + i\eta_2 + icy_1^2 + \mathcal{O}(\mu).$$

It follows that the complex eikonal equation

$$\begin{cases} \varphi'_{x_1}(x,y) = \widetilde{q}_0(y, -\varphi'_y(x,y)), \\ \varphi|_{x_1=0} = \frac{i}{2}(x_2 - y_2)^2 + iy_1^2 \end{cases}$$

has a unique solution in a small fixed neighborhood of $(0,0) \in \mathbb{C}^2_x \times \mathbb{C}^2_y$,

$$\varphi(x,y) = \frac{i}{2}(x_2 - y_2 + ix_1)^2 + i(y_1 - x_1)^2 + \frac{ic}{3}\left(y_1^3 - (y_1 - x_1)^3\right) + \mu\mathcal{O}((x,y)^3).$$

Choosing the weight III

The corresponding weight is given by

$$\Phi(x) = \frac{1}{2}(\mathrm{Im}\, x_2 + \mathrm{Re}\, x_1)^2 + (\mathrm{Im}\, x_1)^2 - \frac{1}{3}c(\mathrm{Re}\, x_1)^3 + \mathcal{O}(|x_1|^4) + \mathcal{O}(\mu)\,|x|^3\,,$$

and in particular,

$$\Psi_\eta(x_2) = \inf_{|x_1|<\eta} \Phi(x) \le f(x_2) + \mathcal{O}(|x_2|^4) + \mathcal{O}(\mu)\,|x_2|^3\,, \quad |x_2| < \eta,$$

where

$$f(\zeta) = \frac{c}{3}\,(\mathrm{Im}\,\zeta)^3$$

is superharmonic for $\mathrm{Im}\,\zeta < 0$, (for $c > 0$). The largest subharmonic minorant U of f in the disc $|\zeta| < 1$ satisfies therefore

$$U(0) \le \frac{1}{\pi}\iint_{D(0,1)} U(\zeta)\,L(d\zeta) < \frac{1}{\pi}\iint_{D(0,1)} f(\zeta)\,L(d\zeta) = 0.$$

It follows then that

$$\widetilde{\Psi}_\eta(0) < \Phi(0) = 0,$$

for all $\eta > 0$ and $\mu > 0$ small enough.

Choosing the weight IV

Associated to the phase function φ is the complex canonical transformation

$$\kappa_\varphi : T^*\mathbb{C}^2 \ni (y, -\varphi'_y(x,y)) \mapsto (x, \varphi'_x(x,y)) \in T^*\mathbb{C}^2,$$

which satisfies

$$\kappa_\varphi(\mathrm{neigh}((0,0), T^*\mathbb{R}^2)) = \Lambda_\Phi \subset T^*\mathbb{C}^2,$$

where the manifold

$$\Lambda_\Phi = \left\{ \left(x, \tfrac{2}{i}\partial_x \Phi(x)\right); x \in \mathrm{neigh}(0, \mathbb{C}^2) \right\}.$$

is I-Lagrangian and R-symplectic.

We should incorporate the real canonical transformation giving the approximate model symbol into an FBI transform.

Choosing the weight V

Proposition

Let $\kappa : \mathrm{neigh}((y_0, \eta_0), T^*U) \to \mathrm{neigh}((0,0), T^*\mathbb{R}^2)$, $\kappa(y_0, \eta_0) = (0,0)$, *be a real analytic canonical transformation. Then the composition* $\kappa_\varphi \circ \kappa$ *is of the form*

$$\kappa_\varphi \circ \kappa = \kappa_\psi : T^*\mathbb{C}^2 \ni (y, -\psi'_y(x,y)) \mapsto (x, \psi'_x(x,y)) \in T^*\mathbb{C}^2,$$

where $\psi = \psi(x,y) \in \mathrm{Hol}\left(\mathrm{neigh}\left((0, y_0), \mathbb{C}^4\right)\right)$ *satisfies*

$$-\psi'_y(0, y_0) = \eta_0, \quad \mathrm{Im}\, \psi''_{yy}(0, y_0) > 0, \quad \det \psi''_{xy}(0, y_0) \neq 0.$$

J. Sjöstrand (1983). (Positivity of complex Lagrangian planes.)

Remark. We have

$$\kappa_\psi(\mathrm{neigh}((y_0, \eta_0), T^*U)) = \kappa_\varphi(\mathrm{neigh}((0,0), T^*\mathbb{R}^2)) = \Lambda_\Phi,$$

so the weight is unchanged.

One word about Step II

We have

$$\left(q(x,hD_x)\otimes 1_{\mathbb{C}^2}+hR(x,hD_x)\right)u=0 \quad \text{in} \;\; U,$$

where

$$||\,u\,||_{L^2(U)}\leq \mathcal{O}(1).$$

If $x_0\in U$ satisfies the assumptions of the Theorem, then Step I gives :

$$\mathrm{WF}_{a,h}(u)\cap q^{-1}(0)\cap \pi^{-1}(x_0)=\emptyset.$$

Here we recall that

$$q(x,\xi)=\sum_{|\alpha|\leq 2}a_\alpha(x)\xi^\alpha$$

is classically elliptic,

$$\left|\sum_{|\alpha|=2}a_\alpha(x)\xi^\alpha\right|\geq \frac{1}{C}\left|\xi\right|^2, \quad (x,\xi)\in T^*U.$$

Proposition

Let $(q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x, hD_x))\, u = 0$ *in* U, $x_0 \in U$, *and assume that*

$$\mathrm{WF}_{a,h}(u) \cap q^{-1}(0) \cap \pi^{-1}(x_0) = \emptyset,$$

where q *is classically elliptic. Then then there exists a neighborhood* Ω *of* x_0 *and* $C_0, c_0 > 0$ *such that*

$$\left|\partial^\beta u(x; h)\right| \le C_0(|\beta| C_0)^{|\beta|} e^{-c_0/h}, \quad x \in \Omega, \quad \beta \in \mathbb{N}^n.$$

This result is closely related to A. Martinez (2002) and can also be obtained as a consequence of the theory of global FBI transforms and global exponentially weighted spaces developed by J. Galkowski – M. Zworski (2021, 2022), allowing exponential weights which are not compactly supported in ξ.

Based on B. Helffer – J. Sjöstrand (1986), J. Sjöstrand (1996).

Back to the chiral model of TBG

We have

$$(q(x, hD_x) \otimes 1_{\mathbb{C}^2} + hR(x))\, u = 0,$$

where

$$q(x,\xi) = (2\bar{\zeta})^2 - U(z)U(-z), \quad z = x_1 + ix_2, \quad \zeta = \frac{1}{2}(\xi_1 - i\xi_2)\,.$$

Symplectic structure on $T^*\mathbb{R}^2$:

$$\sigma = d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2 = 2\mathrm{Re}\,(d\zeta \wedge dz) = d\zeta \wedge dz + d\bar{\zeta} \wedge d\bar{z}.$$

Poisson bracket :

$$\{a, b\} = a'_\zeta b'_z - b'_\zeta a'_z + a'_{\bar{\zeta}} b'_{\bar{z}} - b'_{\bar{\zeta}} a'_{\bar{z}}.$$

Exponential decay of solutions near x_0 is guaranteed by $q(x_0,\xi) = 0 \Longrightarrow$

$$\{q, \bar{q}\}(x_0,\xi) = 0, \quad \{q, \{q, \bar{q}\}\}(x_0,\xi) \neq 0, \quad H_{\mathrm{Re}\,q}(x_0,\xi) \not\parallel H_{\mathrm{Im}\,q}(x_0,\xi).$$

We have

$$q = 0 \Longleftrightarrow 2\bar{\zeta} = \pm\sqrt{U(z)U(-z)},$$

and

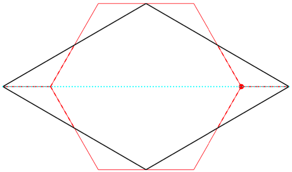
$$\{q, \bar{q}\}|_{q^{-1}(0)} = \pm 8i\mathrm{Im}\,\left((\overline{U(z)U(-z)})^{\frac{1}{2}} \partial_z (U(z)U(-z)) \right).$$

Let

$$H := \bigcup_{\pm} \bigcup_{k=0}^{2} \pm(1 + \omega^k [0, \tfrac{1}{2}]) z_S + \Lambda$$

be the hexagon spanned by the stacking points $\pm z_S + \Lambda$, $z_S = i/\sqrt{3}$, $\omega z_S \equiv z_S \mod \Lambda$. An elementary computation shows that

$$dq(\rho) \neq 0, \quad \{q, \bar{q}\}(\rho) = 0, \quad \rho \in \pi^{-1}(H) \cap q^{-1}(0).$$



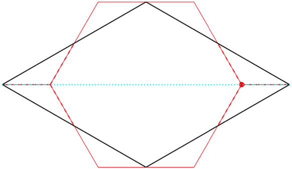
The second bracket

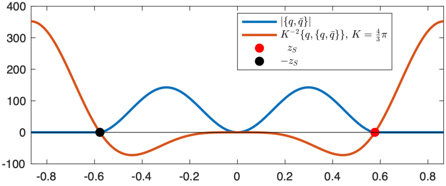
We have for $\rho \in q^{-1}(0) \cap \pi^{-1}(it)$, $it \in \pm z_S(1, 3/2]$,

$$\{q, \{q, \bar{q}\}\}(\rho) = -16V(\partial_z \partial_{\bar{z}} V + \overline{\partial_z^2 V}) + 8((\partial_z V)^2 - \partial_{\bar{z}} V \overline{\partial_z V}).$$

Here $V(z) = U(z)U(-z)$. It turns out that this expression can also be understood and we get

$$\{q, \{q, \bar{q}\}\}(\rho) = \tfrac{128}{9}\pi^2 (c-1)^2(2c+1)(2c-9) \neq 0, \quad c := \cos(2\pi\sqrt{3}t/3).$$





Conclusion : $\{q, \{q, \bar{q}\}\}(\rho) \neq 0$ for $\rho \in q^{-1}(0) \cap \pi^{-1}(z)$, for z along the open edges of the hexagon $\Longrightarrow$ the theorem applies there.

What about the corners?

We have

$$q^{-1}(0) \cap \pi^{-1}(\pm z_S) = \{(\pm z_S, 0)\}, \quad dq(\pm z_S, 0) \neq 0,$$

and

$$\{q, \{q, \bar{q}\}\}(\pm z_S, 0) = 0,$$

with the first non-vanishing bracket given by

$$\{q, \{q, \{q, \{q, \bar{q}\}\}\}\}(\pm z_S, 0) = H_q^4 \bar{q}(\pm z_S, 0) \neq 0.$$

We have

$$q(z_S + z, \zeta) = 4\bar{\zeta}^2 + ia\bar{z} - bz^2 + \mathcal{O}(|z|^3), \quad a, b > 0.$$

Z. Tao – M. Zworski were able to treat the case of corners, by means of a direct ad hoc analysis of the complex eikonal equation

$$\partial_{z_1}\varphi(z, w, v) = 4\left(\partial_v \varphi(z, w, v)\right)^2 + iav - bw^2 + \mathcal{O}((v, w)^3), \; z \in \mathbb{C}^2, \; w, v \in \mathbb{C}.$$

Explicit detailed analysis of the eikonal equation shows that

$$\Psi(z_2) = \inf_{z_1} \Phi(z)$$

is of the form

$$\Psi(z_2) = -\tfrac{1}{3}\mathrm{Im}\,(z_2^3) + |z_2|^2\,\mathrm{Im}\,(z_2^3) + \mathcal{O}(|z_2|^6).$$

Here the largest subharmonic minorant U of $|\zeta|^2\,\mathrm{Im}\,(\zeta^3)$ in the unit disk satisfies

$$U(0) < 0,$$

and hence we can proceed as before. Thus, the exponential decay holds also near the corners.

Exponential decay near the whole hexagon

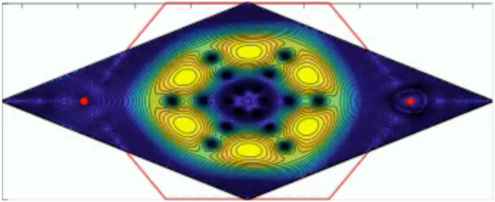
Theorem (Z. Tao – M. Zworski – M. H. (2023))

Assume that

$$(D(\alpha) + k)u = 0, \quad u \in H^1(\mathbb{C}/\Gamma; \mathbb{C}^2), \quad \|u\|_{L^2(\mathbb{C}/\Gamma;\mathbb{C}^2)} = 1.$$

Then there exists an α-independent open neighborhood Ω of the hexagon spanned by the stacking points and $C_0, c_0 > 0$ such that

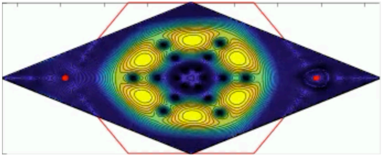
$$|u(z;\alpha)| \leq C_0 e^{-c_0\alpha}, \quad z \in \Omega, \ \alpha \geq 1.$$



What about the center of the hexagon ?

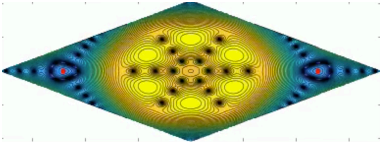
The origin $(x, \xi) = (0, 0)$ (the center of the hexagon) is a doubly characteristic point for q,

$$q(0,0) = 0, \quad dq(0,0) = 0.$$



chiral model

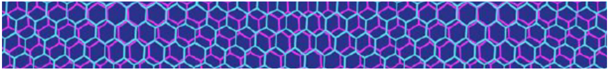
$$(q(x, hD_x) + hR(x))u = 0$$



scalar model

$$q(x, hD_x)u = 0$$

Lower order terms do seem to matter in this case !



Thank you very much for your attention !

CONGRATULATIONS MACIEJ !