From geodesic flow to wave dynamics on an Anosov manifold

Based on arxiv:2102.11196 about contact Anosov flows,
(and "work in progress" for some consequences for Anosov geodesic flows).

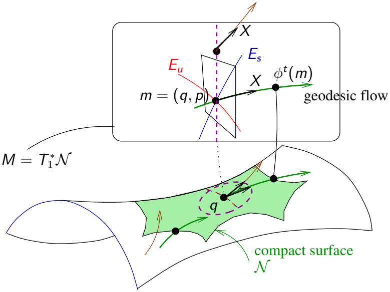
F.Faure (Grenoble) with M. Tsujii (Kyushu),

2024 May 29th, Orsay, conference in honor of **Maciej Zworski**

Definition

On $(\mathcal{N}, g)$ closed Riemannian manifold, the **geodesic flow** $\phi^t : T^*\mathcal{N} \setminus \{0\} \circlearrowleft$ is generated by the **vector field** X, defined by $\Omega(X, .) = dH$ with Hamiltonian function $H(q, p) = \|p\|_{g_q}$ with $p \in T^*_q\mathcal{N} \setminus \{0\}$.

- In local coord. $(q, p) \in T^*\mathbb{R}^{d+1} = \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ we have $X = \left(\frac{\partial H}{\partial p_j}, -\frac{\partial H}{\partial q_j}\right)_{j=0\ldots d}$.



- Energy shell $M := T^*_1\mathcal{N} = \left\{(q, p), \|p\|_{g_q} = 1\right\}$ is invariant.
- "geodesic flow = motion of a free particle or adhesive tape without crease"

Anosov theorem: if curvature $\kappa < 0$ then $TM = \mathbb{R}X \oplus E_u \oplus E_s$. called "sensitivity to initial conditions" in physics.

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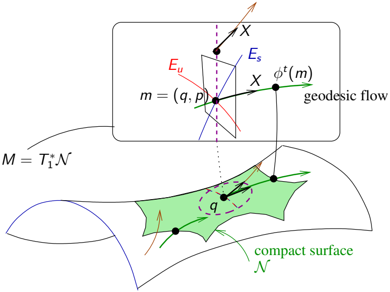

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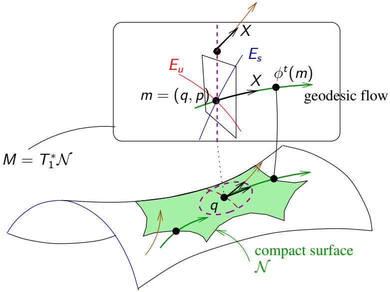

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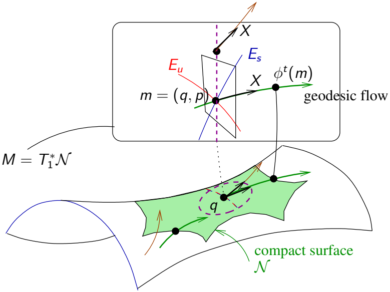

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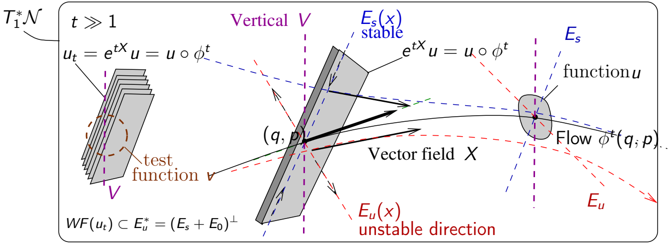
Observation of the Anosov geodesic flow

The geodesic **vector field** $X = \sum_j X_j(x) \frac{\partial}{\partial x_j}$ on $M = T_1^*\mathcal{N}$ is a derivation operator, generator of the **pull back** of functions $u \in C^\infty(M)$ by the flow $\phi^t : M \to M, t \in \mathbb{R}$:

$$u_t = u \circ \phi^t = e^{tX} u \qquad \Leftrightarrow \qquad \frac{du_t}{dt} = X u_t.$$



Rem: its dual $\left(e^{tX}\right)^*$ called "**Ruelle transfer operator**", transports probabilities, e.g.

$$\left(e^{tX}\right)^* \delta_m = \delta_{\phi^t(m)} \qquad : \text{particle dynamics.}$$

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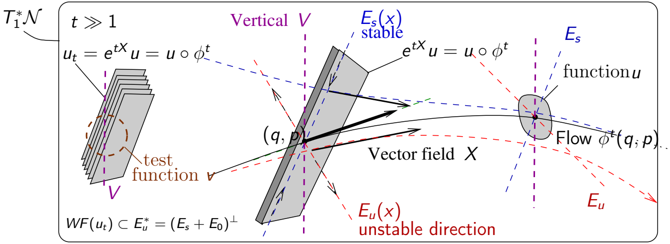

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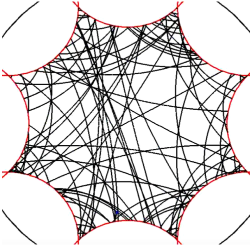

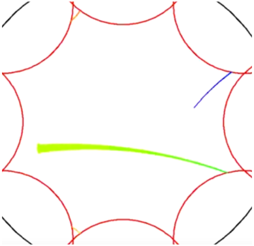

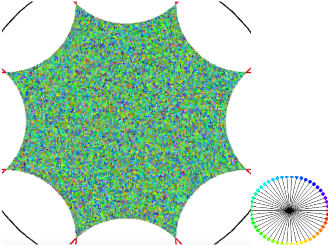

- **Mixing property:** (Anosov 60', Liverani 04, Tsujii 08, Nonnenmacher-Zworski 13) on $M = T_1^*\mathcal{N}$, $\forall u \in C^\infty(M), v \in C^\infty(M; \det(TM))$,

$$\langle v | u \circ \phi^t \rangle \underset{t \to +\infty}{\to} \langle v | 1 \rangle \langle \frac{1}{\mathrm{Vol}(M)} | u \rangle + O_{u,v}\left(e^{-t/2}\right) \quad \text{(for Bolza)}$$

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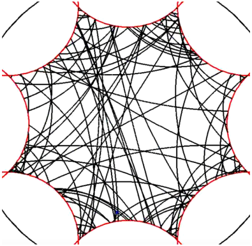
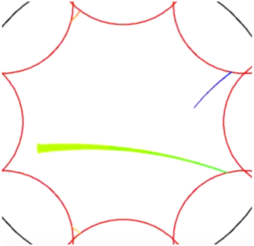
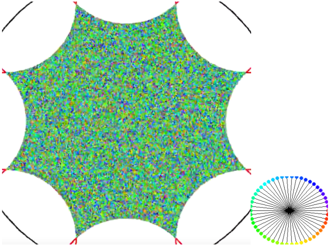




- **Question:** What is in the remainder $O_{u,v}\left(e^{-t/2}\right)$?
- Can we **describe the "fluctuations"** around equilibrium?
  (idem waves and storms on a deep ocean)

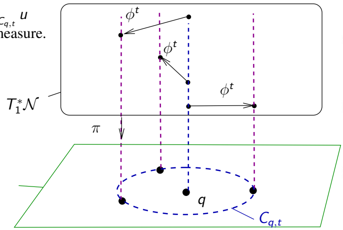
- On $(\mathcal{N}, g)$ closed, let $\pi : M = T_1^*\mathcal{N} \to \mathcal{N}$,
- **Pull back by** π: for $u \in C^\infty(\mathcal{N})$, let $v = (\pi^\circ u) = u \circ \pi \in C^\infty(M)$
- **Pull-back by the flow**: for $v \in C^\infty(M)$, $w = e^{tX} v = v \circ \phi^t \in C^\infty(M)$
- **Average on fibers**: for $w \in C^\infty(M)$, $\left((\pi^\circ)^\dagger w\right)(q) = \int_{\pi^{-1}(q)} w \in C^\infty(\mathcal{N})$

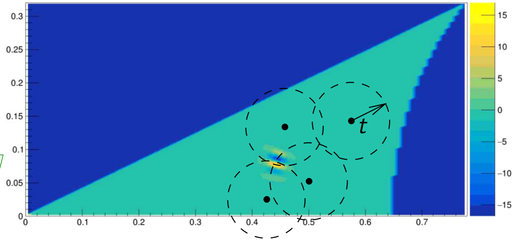
Definition

"Spherical mean operator". For $t \in \mathbb{R}$, let

$$\mathcal{L}_t := (\pi^\circ)^\dagger e^{tX} \pi^\circ \quad : L^2(\mathcal{N}) \to L^2(\mathcal{N}).$$

$(\mathcal{L}_t u)(q) = \int_{C_{q,t}} u$
with induced measure.





- Mixing for Bolza gives (Ratner 87): $\mathcal{L}_t = 1\langle \frac{1}{\mathrm{Vol}(\mathcal{N})}|.\rangle + O_{L^2\to L^2}\left(e^{-t/2}\right)$.

  but what is in this remainder $O_{L^2\to L^2}\left(e^{-t/2}\right)$?

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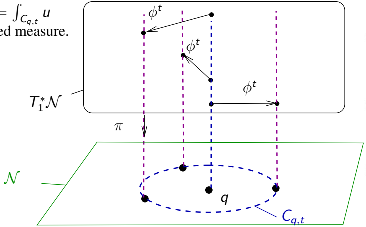

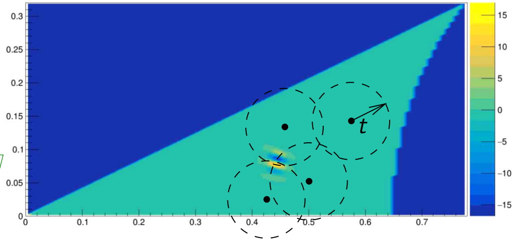

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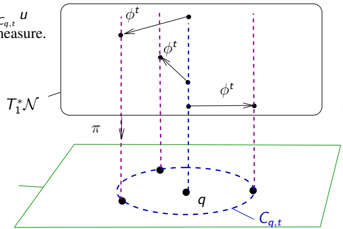

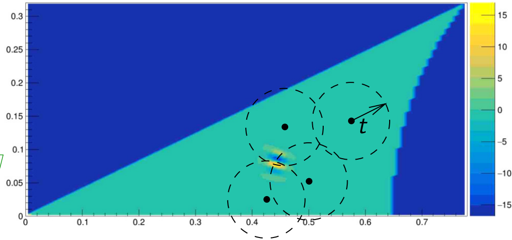

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On hyperbolic surfaces (special case)

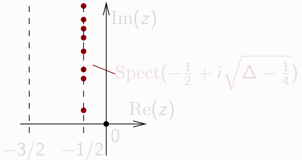
On an hyperbolic surface $\mathcal{N} = \Gamma\backslash \mathrm{SL}_2\mathbb{R}/SO_2$, with co-compact Γ,

Theorem (Spherical mean on hyperbolic surface)

For $t \gg 1$, on $L^2(\mathcal{N})$,

$$\mathcal{L}_t = \underbrace{R_t}_{\text{finite rank}} + e^{-\frac{1}{2}t}\left(W \underbrace{e^{it\sqrt{\Delta-\frac{1}{4}}}}_{\text{wave propagator}} + e^{-it\sqrt{\Delta-\frac{1}{4}}} W^\dagger + O_{L^2\to L^2}\left(e^{-t}\right)\right)$$

- $W : H^s(\mathcal{N}) \to H^{s+1/2}(\mathcal{N}), \forall s \in \mathbb{R}$, *invertible.*



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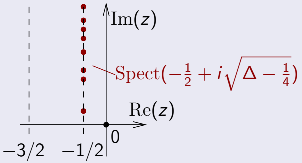

- Proof: use representation theory, principal series of $sl_2\mathbb{R}$.

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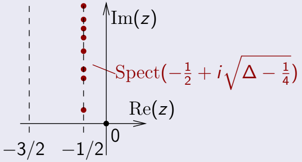


- Rem: $R_t = 1\langle \frac{1}{\mathrm{Vol}(\mathcal{N})} |.\rangle +$ other terms (compl. series),
- Rem: $u_t = e^{\pm it\sqrt{\Delta - \frac{1}{4}}} u_0$ implies $\frac{\partial^2 u_t}{\partial t^2} = -\left(\Delta - \frac{1}{4}\right) u_t$: **"wave equation"**

On Anosov manifold (more general case)

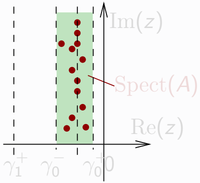
- Let $(\mathcal{N}, g)$ be a closed Riemanian manifold with an **Anosov geodesic flow** e^{tX} on $M = T_1^*\mathcal{N}$ $(TM = E_u \oplus E_s \oplus \mathbb{R}X)$
- Recall the **spherical mean** $\mathcal{L}_t = (\pi^\circ)^\dagger e^{tX}\pi^\circ$ bounded on $L^2(\mathcal{N})$, $\forall t \in \mathbb{R}$,
- For $k \in \mathbb{N}$, let $\mathcal{F}_k = |\det E_s|^{-1/2} \otimes \mathrm{Pol}_k(E_s) \to M$ and define $\gamma_k^\pm := \lim_{t\to\pm\infty} \log \left\| e^{tX_{\mathcal{F}_k}} \right\|^{1/t}_{L^\infty(M;\mathcal{F}_k)} < 0$. Rem: $\gamma_k^+ \underset{k\to\infty}{\to} -\infty$.
(for hyp. surf. $\gamma_1^\pm = -\frac{3}{2}, \gamma_0^\pm = -\frac{1}{2}, \gamma_k^\pm = -\frac{1}{2} - k$.)

Theorem (Spherical mean on Anosov manifold (F.T. 21 and in progress))

With pinching conditions $\gamma_1^+ < \gamma_0^- \le \gamma_0^+$ (discussed later), for $t \gg 1$,

$$\mathcal{L}_t = \underbrace{R_t}_{\text{finite rank}} + We^{tA} + e^{tA^\dagger}W^\dagger + O_{L^2\to L^2}\left(e^{\left(\gamma_1^+ + {}^\forall\epsilon\right)t}\right)$$

- $W : H^s(\mathcal{N}) \to H^{s+1/2}(\mathcal{N}), \forall s \in \mathbb{R}$, *invertible.*
- $A = i\sqrt{\Delta} + O\left(H^s \to H^{s-\frac{1}{2}}\right)$
- $\forall \epsilon > 0, \exists C > 0, \forall t \ge 0,$
$\left\|e^{tA}\right\|_{L^2} \le Ce^{t\left(\gamma_0^+ + \epsilon\right)}$, $\left\|e^{-tA}\right\|^{-1}_{L^2} \ge \frac{1}{C}e^{t\left(\gamma_0^- - \epsilon\right)}$, $\left\|e^{itA}\right\|_{L^2} \le C$
- *Operators W, A are **unique** (up to finite rank), given later.*



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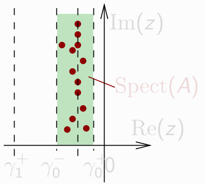

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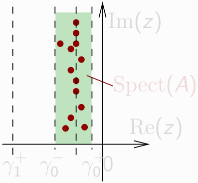
- $\forall \epsilon > 0, \exists C > 0, \forall t \geq 0,$
  $\left\|e^{tA}\right\|_{L^2} \leq Ce^{t\left(\gamma_0^+ + \epsilon\right)}, \left\|e^{-tA}\right\|_{L^2}^{-1} \geq \frac{1}{C} e^{t\left(\gamma_0^- - \epsilon\right)}, \left\|e^{itA}\right\|_{L^2} \leq C$
- *Operators W, A are **unique** (up to finite rank), given later.*



On Anosov manifold (more general case)

- Let $(\mathcal{N}, g)$ be a closed Riemanian manifold with an **Anosov geodesic flow** e^{tX} on $M = T_1^*\mathcal{N}$ $(TM = E_u \oplus E_s \oplus \mathbb{R}X)$
- Recall the **spherical mean** $\mathcal{L}_t = (\pi^\circ)^\dagger\, e^{tX}\pi^\circ$ bounded on $L^2(\mathcal{N})$, $\forall t \in \mathbb{R}$,
- For $k \in \mathbb{N}$, let $\mathcal{F}_k = |\det E_s|^{-1/2} \otimes \mathrm{Pol}_k(E_s) \to M$ and define $\gamma_k^\pm := \lim_{t\to\pm\infty} \log \left\| e^{tX_{\mathcal{F}_k}} \right\|_{L^\infty(M;\mathcal{F}_k)}^{1/t} < 0$. Rem: $\gamma_k^+ \underset{k\to\infty}{\to} -\infty$.
  (for hyp. surf. $\gamma_1^\pm = -\frac{3}{2}, \gamma_0^\pm = -\frac{1}{2}, \gamma_k^\pm = -\frac{1}{2} - k$.)

Theorem (Spherical mean on Anosov manifold (F.T. 21 and in progress))

With pinching conditions $\gamma_1^+ < \gamma_0^- \le \gamma_0^+$ (discussed later), for $t \gg 1$,

$$\mathcal{L}_t = \underbrace{R_t}_{\text{finite rank}} + We^{tA} + e^{tA^\dagger}W^\dagger + O_{L^2\to L^2}\left(e^{\left(\gamma_1^+ + {}^\forall\epsilon\right)t}\right)$$

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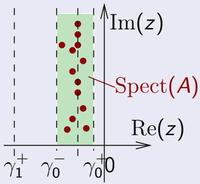

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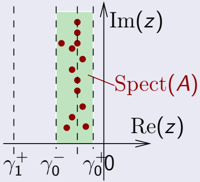
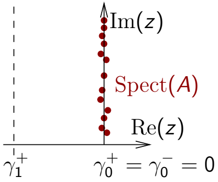
- *Operators* W, A *are* ***unique*** *(up to finite rank).*



- "by twisting" with the bundle $F = |\det E_s|^{1/2}$, we get $\gamma_1^+ < \gamma_0^\pm = 0$ (F.-Tsujii 2013)
- **More internal bands:** assuming $\gamma_{K+1}^+ < \gamma_K^-$, we can get remainder $O_{L^2 \to L^2}\left(e^{\left(\gamma_{K+1}^+ + {}^\forall\epsilon\right)t}\right)$, $\forall K \in \mathbb{N}$.



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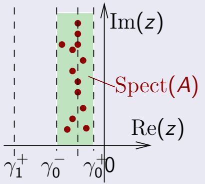

- **Eigenfunctions of A are in $C^\infty(\mathcal{N})$.**
We will see that $\mathrm{Spect(A)}$ = first band of **Pollicott-Ruelle spectrum** of X (discrete poles of $(z - X)^{-1} : C^\infty(M) \to \mathcal{D}'(M)$). (Ruelle, Pollicott, Baladi, Tsujii, Gouezel, Liverani, F. - Sjostrand, Datchev-Zworski Dyatlov, ...)
- So discrete **"Pollicott-Ruelle spectrum" has an intrinsic existence and manifestation in $L^2(\mathcal{N})$** (no anisotropic Sobolev space here!).

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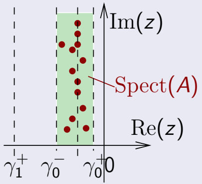

- From Atiyah-Bott trace formula, $\mathrm{Spect}(A)$ are **zeroes of a semi-classical zeta function** determined from the periodic orbits (Giulietti-Liverani-Pollicott 2012, Dyatlov-Zworski 2013, F.-Tsujii 2013).

Some related works

- In different situations, **band structure and Weyl law** for the spectrum of resonances:
  - Stefanov 1995, Sjöstrand-Zworski 1999 for **convex obstacles**
  - S. Dyatlov 2013 for **regular normally hyperbolic trapped sets**.

- **Emergence of quantum dynamics, band structure of Pollicott-Ruelle spectrum, using "Contact and Anosov properties".**
  - for **contact extension of linear cat map** on $\mathbb{T}^2$ (F. 2006)
    (this is a "normal form", and shows the main mechanism with symplectic spinors)
  - for **contact extension of Anosov diffeom.** (F.-Tsujii 2012)
  - for **geodesic flow on hyperbolic manifolds** (Dyatlov-F-Guillarmou 2014, Guillarmou-Hilgert-Weich 2016)
  - for **contact Anosov flows** (F-Tsujii 2016, 2021, Guillarmou-Cekic 2020)

- **Spherical mean**
  - on **Euclidean space with obstacles** (Y. Bonthonneau, Y. Chaubet, V.Dang, M.Léautaud, G.Riviere 2022-24)
  - ...

General remarks on "quantization" in mathematics

- **Quantization** $\mathrm{Op}(.)$, (e.g. $\mathrm{Op}(p_j) = -i\frac{\partial}{\partial q_j}$) applied to the geodesic flow gives the **"wave operator"** $\sqrt{\Delta} \approx \mathrm{Op}\left(\|p\|_g\right)$ (with the Hodge Laplacian $\Delta = d^\dagger d$), that generates the **wave equation**, for $u_t \in C^\infty(\mathcal{N}), t \in \mathbb{R}$:

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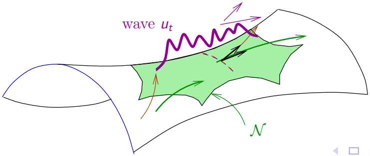
- **Semi-classical analysis** (WKB theory, Egorov’s Theorem etc) shows that for small wave-length $\lambda \ll 1$, function u_t is approximately transported by the geodesics:

$$\text{wave equation} \underset{t\ \text{fixed}, \lambda \to 0}{\Longrightarrow} \text{geodesic flow}$$

- **Ex:** geometrical optics is a limit of wave optics with $\lambda \approx 0.5\mu m$.
  Classical Newtonian mechanics is a limit of quantum Schrödinger mechanics. **movie of wave packet**



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- Curiously, Thm 4 concerns the **opposite direction**:

$$\text{geodesic flow} \underset{t \gg 1}{\Longrightarrow} \text{wave equation}$$

What does it mean?

General remarks on quantization(s) in mathematics

- Quantization is not unique: **many quantum operators (PDO) have the same classical limit (principal symbol)** but have **different spectra**.
  - For example with Weyl quantization, the operator depends on the choice of coordinates. In geometric quantization, the operator depends on the choice of polarization.
  - Hence the **classical dynamics does not determine the quantum spectrum** in general.
- The operator A in Thm 4 is **one quantization among others** but **uniquely defined from the Anosov geodesic flow** and has therefore **special properties w.r.t. the dynamics**, e.g. almost exact Trace formula, Van-Vleck formula (remainders are $O_{\hbar}\left(e^{-Ct}\right) \underset{t\to\infty}{\longrightarrow} 0$), Exact Egorov ...
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Physical meaning? (informal discussion)

Let us observe the following similarities:

1. Thm 4 shows that the **propagation of probability measures** under a **deterministic but chaotic dynamics** (Anosov geodesic flow) is an equilibrium measure + small fluctuations governed by the Schrödinger wave equation, i.e. **"quantum dynamics emerges"**.
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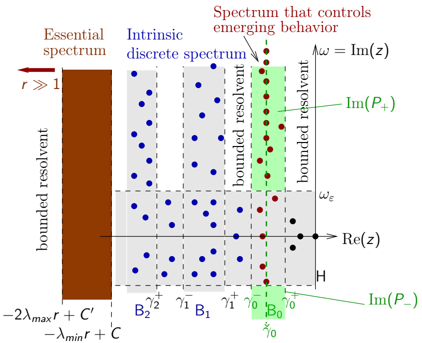
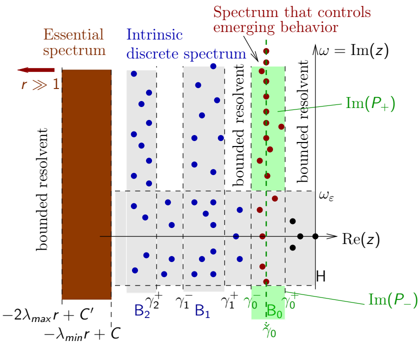
Ingredients of proof of thm 4



Based on:

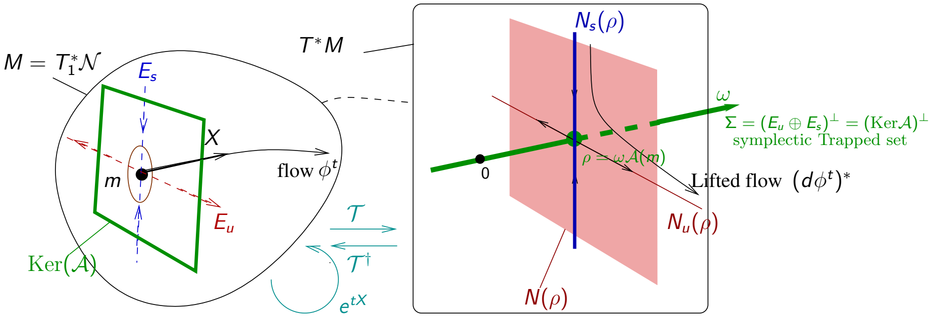
1. arxiv:2102.11196, with M. Tsujii that concerns **contact Anosov flows**
2. (Work in progress) "spherical mean" for **geodesic Anosov flows**

Ingredients of proof of thm 4



- **Step 1: *"discrete Pollicott-Ruelle spectrum with bands".***
  - **Anisotropic Sobolev spaces** $\mathcal{H}_W(M)$ (from a weight function W on $T^* T_1^* \mathcal{N}$ adapted to the dynamics)
  - We deduce **discrete Pollicott-Ruelle spectrum of** X in $\mathcal{H}_W(M)$, with gaps if $\gamma_1^+ < \gamma_0^-$.

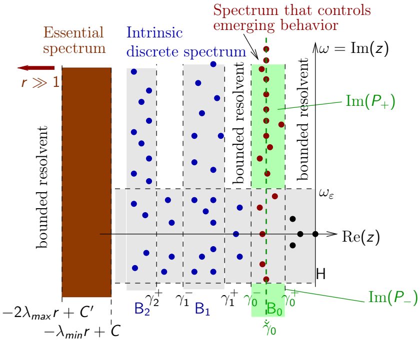
Ingredients of proof of thm 4



- We use **microlocal analysis** of X using **symplectic geometry**,
- At the heart of the proof (more details later): **symplectic spinors** and **emergence of quantum dynamics** for the bundle of linear symplectic maps

$$d\left(d\phi^t\right)^* : TT^*M \circlearrowleft$$

Steps of the proof

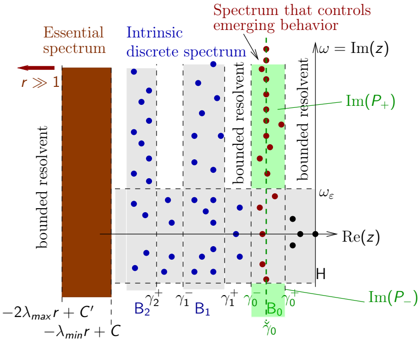


- **Step 2: *"spherical mean"*.** Deduce asymptotics $t \gg 1$ of

$$\mathcal{L}_t = (\pi^\circ)^\dagger \, e^{tX} \pi^\circ$$

  - Use that the vertical direction $V = \mathrm{Ker}\,(d\pi)$ is transverse to E_u, E_s (Klingenberg 74). Hence **averaging erases the wave-front set of Pollicott-Ruelle distributions.**

Steps of the proof



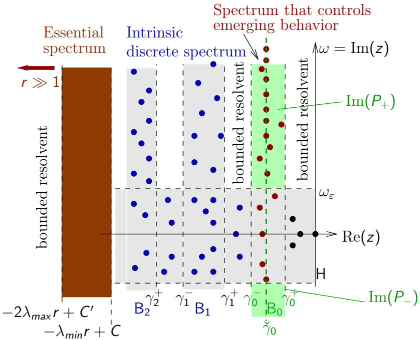
- Define a spectral (bounded) projector for the first band

$$P_\pm : \mathcal{H}_W(M) \to \mathrm{Im}(P_\pm) \subset \mathcal{H}_W(M).$$

- Using transversality $V \perp (E_u, E_s)$, the **pull back is Fredholm**:

$$B_\pm := P_\pm \pi^\circ : L^2(\mathcal{N}) \to \mathrm{Im}(P_\pm)$$

Steps of the proof



Then (roughly),

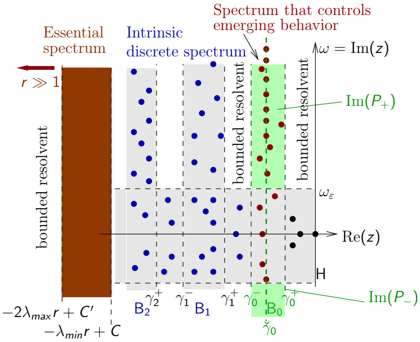
$$\mathcal{L}_t = (\pi^\circ)^\dagger \, e^{tX} \pi^\circ = \mathcal{L}_t^+ + \mathcal{L}_t^- + R_t + O_{L^2}\left(e^{\gamma_1^+ t}\right)$$

with

$$\mathcal{L}_t^\pm = (\pi^\circ)^\dagger \, e^{tX} P_\pm \pi^\circ = (\pi^\circ)^\dagger \, B_\pm B_\pm^{-1} e^{tX} B_\pm = W_\pm e^{tA_\pm}$$

$$B_\pm := P_\pm \pi^\circ, \qquad A_\pm := B_\pm^{-1} X B_\pm, \qquad W_\pm = (\pi^\circ)^\dagger \, B_\pm.$$

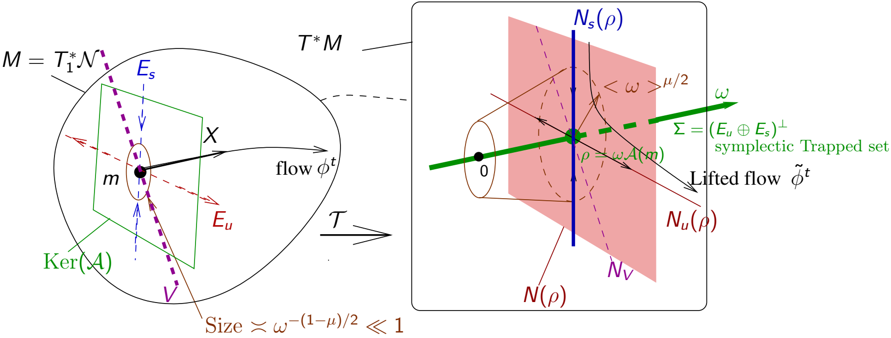
Steps of the proof



- **"Exact Egorov property"**: for $a \in C^\infty(M)$, we have $e^{tX}\mathcal{M}_a e^{-tX} = \mathcal{M}_{a\circ\phi^t}$. Define $\mathrm{Op}(a) := B^{-1}\mathcal{M}_a B : L^2(\mathcal{N}) \to L^2(\mathcal{N})$. Then

$$e^{tA}\mathrm{Op}(a)\,e^{-tA} = \left(B^{-1}e^{tX}B\right)\left(B^{-1}\mathcal{M}_a B\right)\left(B^{-1}e^{-tX}B\right)$$
$$= \mathrm{Op}\left(a \circ \phi^t\right)$$

Step 1 of the proof (more details)



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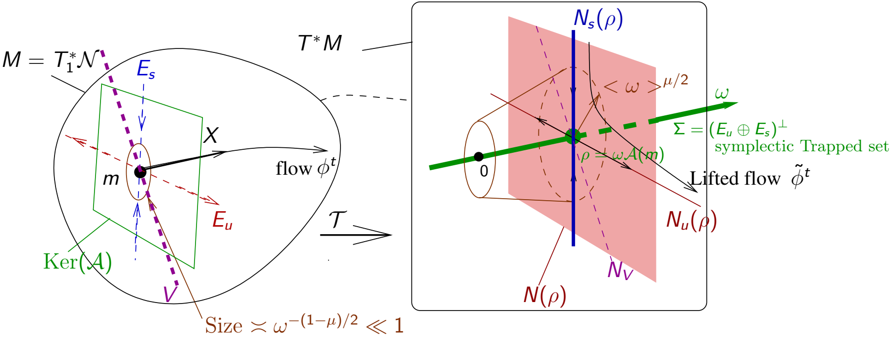
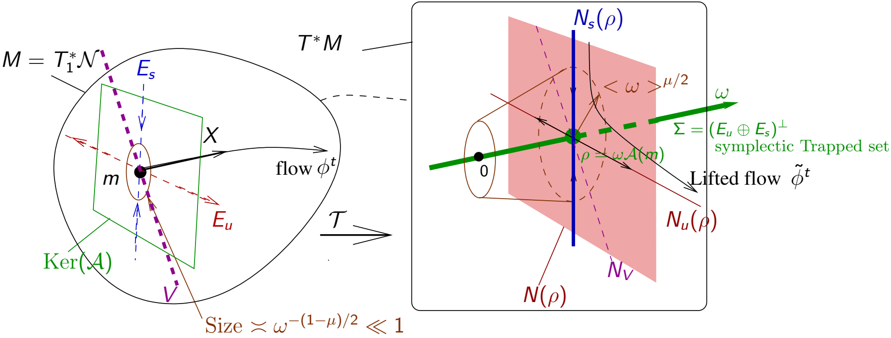


- e^{tX} is a **Fourier integral operator:** in the limit of high frequencies, its action is well described on the cotangent bundle T^*M with the induced flow $\tilde{\phi}^t := (d\phi^t)^*$, $t \in \mathbb{R}$.

Step 1 of the proof (more details)



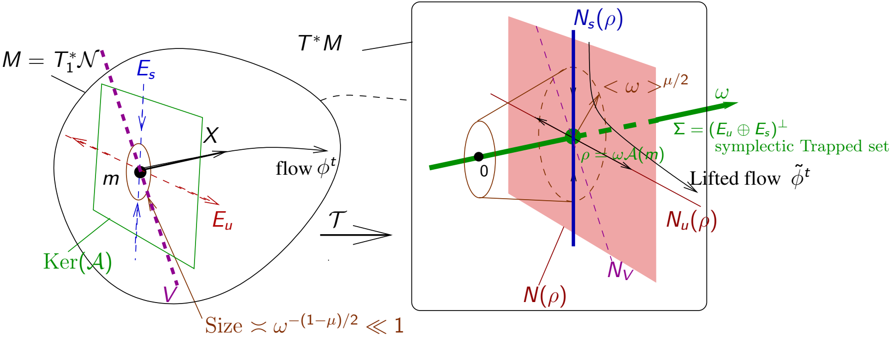
- Introduce a **Hörmander metric** g on T^*M, $\Omega-$compatible.
- define an L^2-isometric "**wave-packet transform**"

$$\mathcal{T} : C^\infty(M;F) \to \mathcal{S}(T^*M;F)$$

to use **micro-local analysis** on T^*M for the pull back operator e^{tX}.

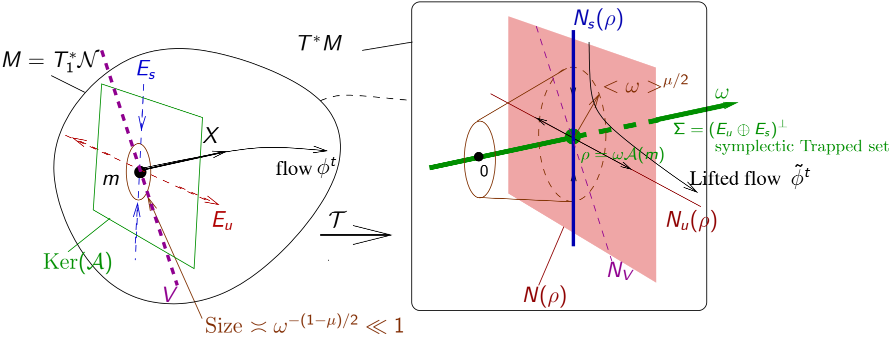
- The unit boxes for the metric g correspond to the effective size of wave-packets and reflect the **uncertainty principle**.

Step 1 of the proof (more details)



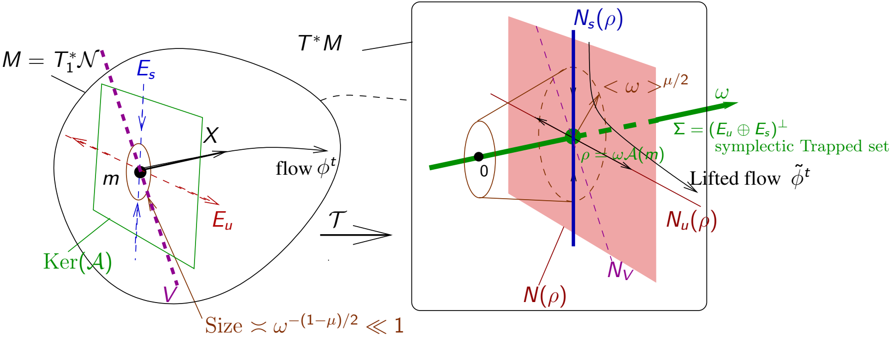
- The dynamics $\tilde{\phi}^t$ is a **"scattering dynamics"** on the **trapped set** $\Sigma = \mathbb{R}^* \mathscr{A} \subset T^*M$ (Liouville 1-form)
- Σ is **symplectic** and **normally hyperbolic**.
- **In the outer part of Σ,** we put a weight W such that $W\left(\tilde{\phi}^t(\rho)\right)$ decays with $t \to +\infty$. Hence the operator e^{tX} has a negligible contribution in some anisotropic Sobolev space $\mathcal{H}_W$.
- So only the **dynamics in a vicinity of Σ** plays a role for our purpose.

Step 1 of the proof (more details)



- We consider **a vicinity of** Σ of a given $g-$size $\omega^{\mu/2}$, at $\rho = \omega\mathcal{A}(m) \in \Sigma$, with some $0 < \mu < 1$.
- The projection on M has size $\asymp \omega^{-(1-\mu)/2} \ll 1$ if $\omega \gg 1$.
- This will allow us to use the **linearization of the dynamics** $\tilde{\phi}^t$ as a local approximation.

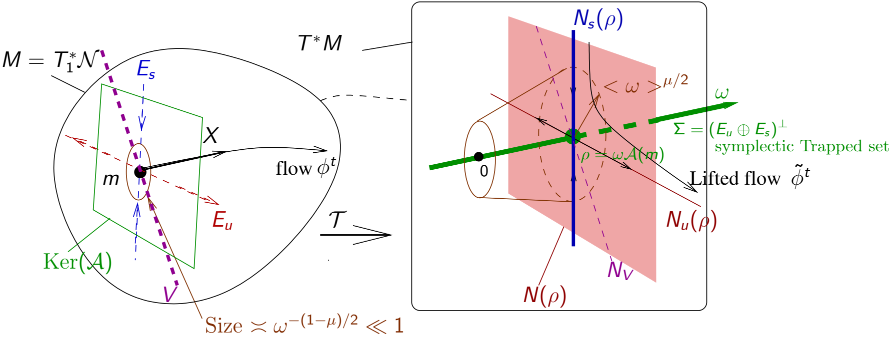
Step 1 of the proof (more details)



- At $\rho = \omega\mathcal{A}(m) \in \Sigma$, there is a **micro-local decoupling** (idem symplectic spinors)

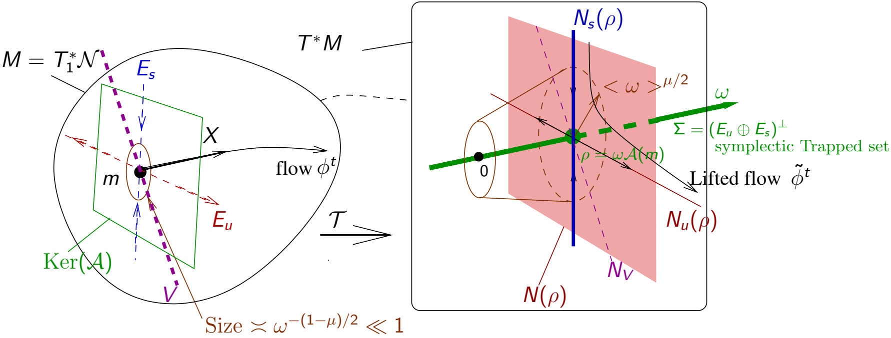
$$T_\rho T^*M = \underbrace{T_\rho \Sigma}_{\text{Tangent}} \overset{\perp}{\oplus} \underbrace{(N_u(\rho) \oplus N_s(\rho))}_{\text{normal}} \quad : \text{invariant decomp.}$$

Step 1 of the proof (more details)



- The dynamics on the normal direction N is hyperbolic and responsible for the emergence of polynomial functions along the stable direction $N_s \equiv E_s$ idem $V = -x\frac{d.}{dx}$, $Vx^k = (-k)\,x^k$ on $\mathbb{R}$.
- We introduce **approximate projectors** $\mathrm{Op}_\Sigma\,(T_k)$ that restricts functions to the symplectic trapped set Σ and that are polynomial valued along N_s with degree k.
- Rem: this projector plays a similar role as the Bergman projector (or Szegö projector) in **geometric quantization**.

Step 1 of the proof (more details)



- What remains for large time, is an **effective Hilbert space** of functions (or quantum waves) that live on the trapped set Σ, valued in the vector bundle $\mathcal{F}_k = |\det E_s|^{-1/2} \otimes \mathrm{Pol}_k(E_s)$.
- We deduce band structure of X and other properties.
- Rem: the **main geometrical object** considered in this paper is this fibration $N_s \to \Sigma \to M = T_1^*\mathcal{N} \to \mathcal{N}$.

What is the meaning of going beyond the equilibrium description for the Pollicott-Ruelle spectrum?

Illustration: **the ocean** is quite, deep, flat, gentle $\equiv$ **Equilibrium state,** but with a better look, the behavior at the surface may be furious, wavy, and never stop to move. Are they neglictible?

Some very speculative question: **can quantum dynamics in the physics world emerges from an underlying chaotic deterministic yet unknown system**?

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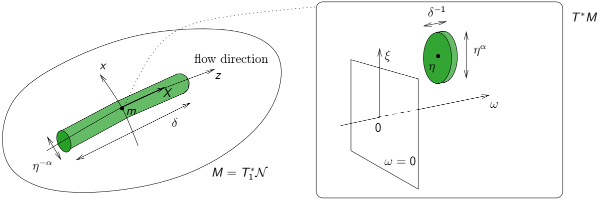
Thank you for your attention!

(*) Wave packets

- Local flow box coordinates on M: $y = (x, z) \in \mathbb{R}^n \times \mathbb{R}$ s.t. $X = \frac{\partial}{\partial z}$ and dual coordinates $\eta = (\xi, \omega) \in \mathbb{R}^n \times \mathbb{R}$ on $T_y^* M$.



- Let $\frac{1}{2} \leq \alpha < 1$ and $0 < \delta \ll 1$. **Wave packet function** is:

$$\varphi_{(y,\eta)}(y') \underset{|\eta|\gg 1}{\approx} a \exp\left(i\eta.y' - \left| \frac{x'-x}{\langle \eta \rangle^{-\alpha}} \right|^2 - \left| \frac{z'-z}{\delta} \right|^2 \right), \qquad \left\| \varphi_{(y,\eta)} \right\|_{L^2(M)} \underset{|\eta|\gg 1}{\approx} 1$$

- **Metric g on** T^*M, compatible with $\Omega = dy \wedge d\eta$:

$$g_{y,\eta} = \left(\frac{dx}{\langle \eta \rangle^{-\alpha}} \right)^2 + \left(\frac{d\xi}{\langle \eta \rangle^{\alpha}} \right)^2 + \left(\frac{dz}{\delta} \right)^2 + \left(\frac{d\omega}{\delta^{-1}} \right)^2$$

- Rem: $\alpha \geq \frac{1}{2} \Leftrightarrow g$ remains equivalent uniformly/η after change of flow box coordinates.

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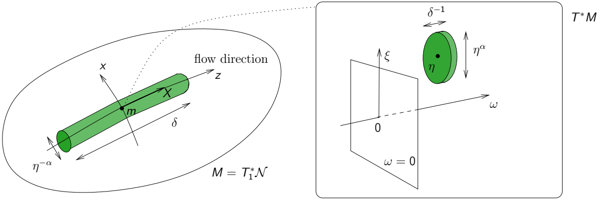

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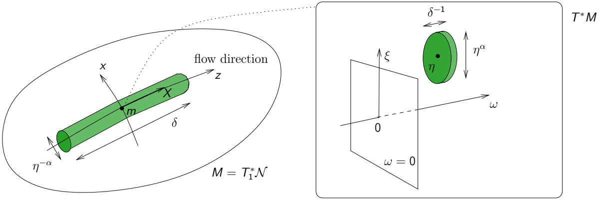

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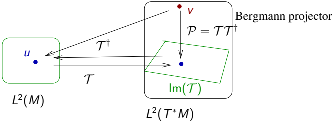
(*) Wave packet transform (or FBI, wavelet, Bargmann, Anti-Wick... transform)

(Abuse of notations that forget charts and partitions of unity.)

$$\mathcal{T} : \begin{cases} C^\infty(M) & \rightarrow \mathcal{S}(T^*M) \\ u(y') & \rightarrow (\mathcal{T}u)(y,\eta) := \langle \varphi_{y,\eta}, u \rangle_{L^2(M)} \end{cases}$$

Lemma (fundamental 1. "Resolution of identity")

$$\mathcal{T}^* \circ \mathcal{T} = \mathrm{Id}$$



Remarks: $\forall u \in C^\infty(M), \quad u(y') = \int_{T^*M} \varphi_{y,\eta}(y') \langle \varphi_{y,\eta}, u \rangle \frac{dy d\eta}{(2\pi)^{n+1}}$.
$\mathcal{T} : L^2(M) \rightarrow \mathrm{Im}(\mathcal{T}) \subset L^2(T^*M)$ is an isomorphism. Hence **we "lift the analysis to T^*M"**.
$\Pi = \mathcal{T} \circ \mathcal{T}^* : L^2(T^*M) \rightarrow \mathrm{Im}(\mathcal{T})$ is an orthogonal projector.