Bulk edge correspondence for curved interfaces

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Orsay conference on Analysis and PDE
In honor of Maciej Zworski

Electronic evolution

Equation for electrons moving through a 2D crystal:

$$i\frac{\partial \psi}{\partial t} = H\psi, \qquad \psi \in \ell^2(\mathbb{Z}^2, \mathbb{C}^d), \qquad \text{where:}$$

- ψ is the wavefunction ($|\psi(t,n)|^2$ is probability that electron at t is at n)
- H is the Hamiltonian of the crystal (typically graph Laplacian weighted accorded to tunnelling probabilities)

Assumption: H is selfadjoint and short-range $\left(\left|H(n,m)\right| \leq e^{-\nu|n-m|}\right)$

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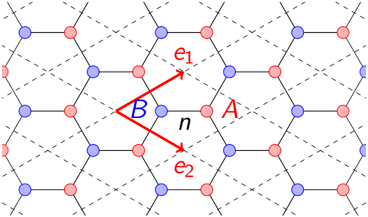
Example: Wallace's model for graphene



$$\psi = \begin{bmatrix}\psi^A \\ \psi^B\end{bmatrix}_n \in \ell^2(\mathbb{Z}^2, \mathbb{C}^2),$$

$$\left(H_0 \begin{bmatrix}\psi^A \\ \psi^B\end{bmatrix}\right)_n = \begin{bmatrix}\psi^B_{n+v_1} + \psi^B_{n+v_2} + \psi^B_n \\ \psi^A_{n-v_1} + \psi^A_{n-v_2} + \psi^A_n\end{bmatrix}.$$

Conductors versus insulators

We say that a system with Hamiltonian H is:

$$\text{conducting at energy } E \;\Leftrightarrow\; E \in \Sigma(H) \qquad (\Sigma(H)\text{: spectrum of } H)$$

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Example: the Wallace model is conducting at energy 0 (easy to see using Fourier transform)

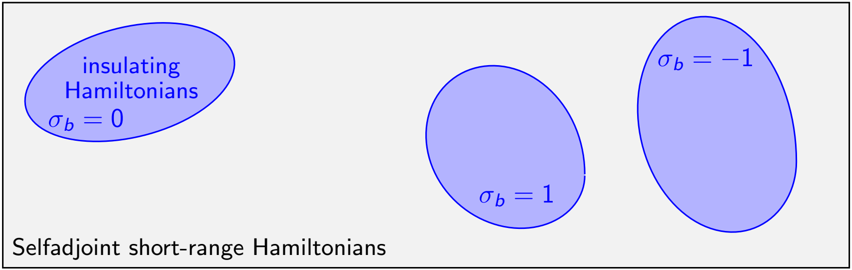
Example: Haldane's model (modified for simplicity)

$$H_s = H_0 + s \cdot D, \qquad D\psi_n = i \begin{bmatrix} \psi^A_{n+e_1} - \psi^A_{n-e_1} \\ \psi^B_{n-e_1} - \psi^B_{n+e_1} \end{bmatrix}, \qquad s \in \mathbb{R}$$

D: second-nearest neighbor coupling that breaks time-reversal invariance.
There is a spectral gap at energy 0: H_s is insulating at energy 0.

Topology in insulators

Fact: The space of insulating Hamiltonians at a fixed energy E is disconnected.



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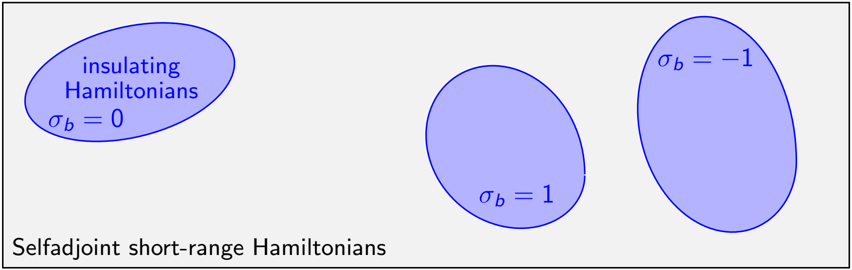

Fact: Its connected components are indexed by the Hall conductance:

$$\sigma_b(H) \overset{\text{def}}{=} \operatorname{Tr} iP\big[[P, \mathbf{1}_{n_1>0}], [P, \mathbf{1}_{n_2>0}]\big],$$

where $P = \mathbf{1}_{(-\infty,E]}(H)$ is the spectral projection below energy E. This trace is well-defined, the result is an integer [Thouless–Kohmoto–Nightingale–den Nijs '82, ... Elgart–Graf–Schecker '05].

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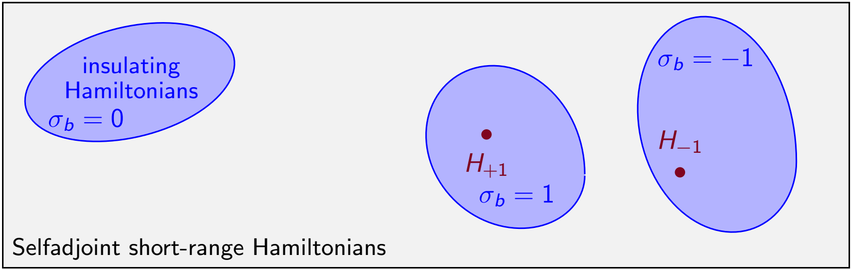

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This gives rise to **topological phases of matter (insulators).**

Example: for Haldane's model $\sigma_B(H_s) = \mathrm{sgn}(s)$.

Connecting topological insulators

We consider Hamiltonians H of the form:

$$H = \begin{cases} H_+ & \text{inside } \Omega \\ H_- & \text{outside } \Omega \end{cases}, \qquad \Omega \subset \mathbb{Z}^2$$

where H_-, H_+ **are insulators at** E **with distinct Hall conductances**.

Precise assumption: $H = \mathbf{1}_\Omega H_+ \mathbf{1}_\Omega + \mathbf{1}_{\Omega^c} H_- \mathbf{1}_{\Omega^c} + E$,

$$E \text{ selfadjoint, short range and } \quad \left| E(n,m) \right| \leq e^{-\nu d(n, \partial\Omega)}.$$

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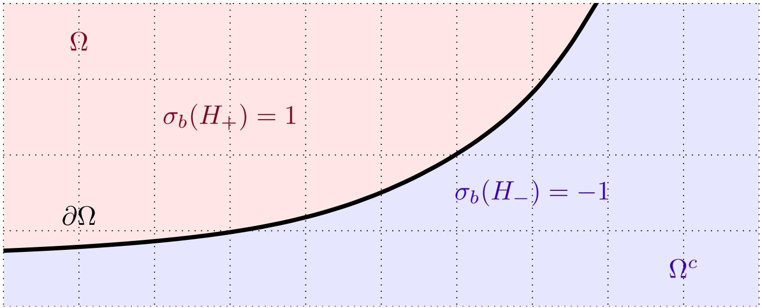

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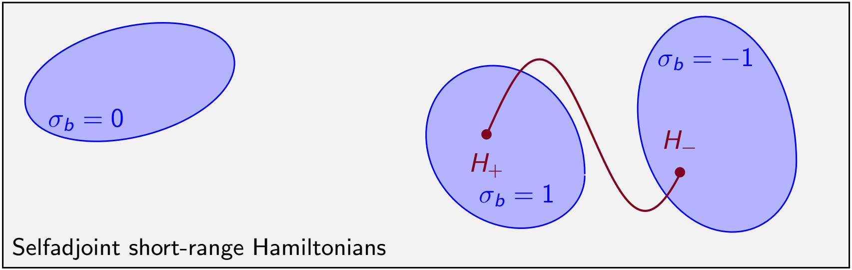

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Numerical evidence for $H_\pm = H_{\pm s}$ and Ω the top-right corner:

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Theorem [DZ '23]

If Ω and Ω^c contain arbitrarily large balls and $\sigma_b(H_+) \neq \sigma_b(H_-)$ then $E \in \Sigma(H)$: H is a conductor at energy E.

Interpretation: The interface between two "large" topological phases supports currents: **the famous edge states.**

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Related results: [Fröhlich–Graf–Walcher '00, Thiang '20, Ojito '22] for quantum Hall Hamiltonian in asymptotically sector-like regions; [Thiang–Ledewig '22] for periodic operators.

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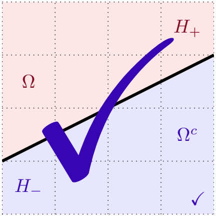
A condition on Ω is necessary: for instance **if Ω fits in a strip we constructed examples with** $E \in \Sigma(H)$**!**



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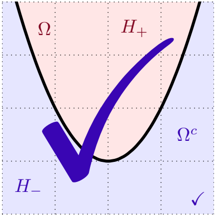

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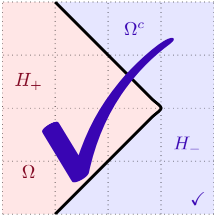

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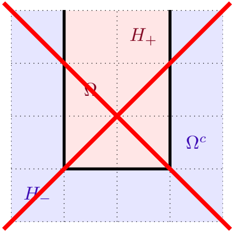

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Let H_1, H_2 be two operators with $E \notin \Sigma(H_1) \cup \Sigma(H_2)$. There exists $R > 0$, $\varepsilon > 0$ such that if for some x,

$$\left\| \mathbf{1}_{B(x,R)} (H_1 - H_2) \mathbf{1}_{B(x,R)} \right\| \leq \varepsilon$$

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"The Hall conductance is locally determined."

How could it be unambiguously defined? Thanks to the **global assumption** $E \notin \Sigma(H)$**!**

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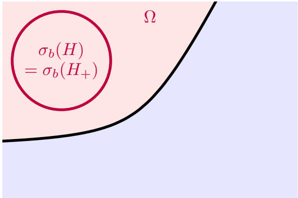

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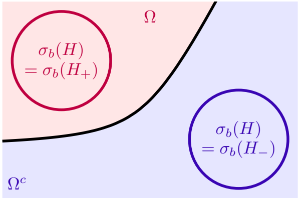

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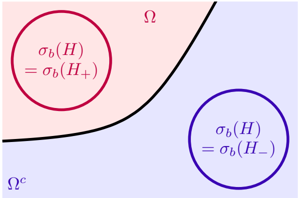
- So $\sigma_b(H_+) = \sigma_b(H_-)$, **contradiction!**



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then $\sigma_b(H_1) = \sigma_b(H_2)$.

Proof: 1. Observe: for all $x = (x_1, x_2)$, for R large:

$$\begin{aligned}
\sigma_b(H) &= \mathrm{Tr}\, iP\big[[P, \mathbf{1}_{n_1>0}], [P, \mathbf{1}_{n_2>0}]\big] = \mathrm{Tr}\left[P\mathbf{1}_{n_1>0}P, P\mathbf{1}_{n_2>0}P\right] \\
&= \mathrm{Tr}\left[P\mathbf{1}_{n_1>x_1}P, P\mathbf{1}_{n_2>0}P\right] + \mathrm{Tr}\left[P\mathbf{1}_{x_1\geq n_1>0}P, P\mathbf{1}_{n_2>0}P\right] \\
&= \mathrm{Tr}\left[P\mathbf{1}_{n_1>x_1}P, P\mathbf{1}_{n_2>x_2}P\right] = \mathrm{Tr}\, iP\big[[P, \mathbf{1}_{n_1>x_1}], [P, \mathbf{1}_{n_2>x_2}]\big] \\
&\simeq \mathrm{Tr}\, iP^{x,R}\big[[P^{x,R}, \mathbf{1}_{n_1>x_1}], [P^{x,R}, \mathbf{1}_{n_2>x_2}]\big], \quad P^{x,R} \overset{\text{def}}{=} \mathbf{1}_{B(x,R)}P\mathbf{1}_{B(x,R)}.
\end{aligned}$$

2. Note (1) $\Rightarrow P_1^{B_r(x)} \simeq P_2^{x,R}$.

3. So $\sigma_b(H_1) \simeq \sigma_b(H_2)$. But these are both integers, so equality holds for ε small, R large. □

What is next?

Theorem [DZ '23]

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How much is the conductance?

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How much is the conductance?

Bulk-edge correspondence for half-spaces

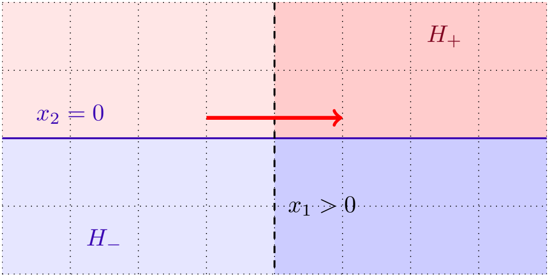
Say $\Omega = \{x_2 > 0\}$ so that: $H = \begin{cases} H_+ & \text{for } n_2 \gg +1 \\ H_- & \text{for } n_2 \ll -1 \end{cases}$.

The **conductance** along $\partial\Omega = \{n_2 = 0\}$ is:

$$\sigma_e(H) = \mathrm{Tr}\, i[H, \mathbf{1}_{n_1>0}]\, g'(H)$$

where $g(\lambda)$ switches from 0 to 1 as λ crosses E. Hence:

- $i[H, \mathbf{1}_{n_1>0}]$: charge moving left to right
- $\mathrm{Tr}(\,\bullet\, g'(H))$: density of states near energy E



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Theorem

If $\Omega = \{x_2 > 0\}$ *then* $\sigma_e(H) = \sigma_b(H_+) - \sigma_b(H_-)$.

Long history: [Hatsugai '93, Kellendonk–Richter–Schulz-Baldes '02, Elbau–Graf '02], many extensions: disorder [Elgart–Graf–Schencker '05], Floquet systems [Graf–Tauber '18], continuous models [Drouot '20, Faure '23], etc.

What if the edge is curved?

Return to: $H = \begin{cases} H_+ & \text{inside } \Omega \\ H_- & \text{outside } \Omega \end{cases}, \qquad \Omega \subset \mathbb{Z}^2, \qquad E \notin \Sigma(H_\pm).$

Conductance along $\partial\Omega$, across ∂W:

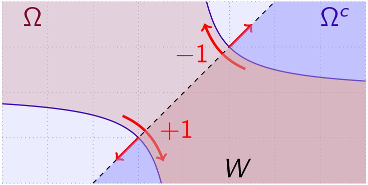
$$\sigma_e^{\Omega W}(H) = \mathrm{Tr}\, i[H, \mathbf{1}_W]\, g'(H) \qquad \text{[Ludewig–Thiang '22]}$$

Necessary condition for $\sigma_e^{\Omega W}(H)$ to be well-defined:

$$\lim_{|x|\to\infty} \frac{\Psi_{\Omega W}(x)}{\ln|x|} = +\infty, \qquad \Psi_{\Omega W}(x) \overset{\text{def}}{=} d(x, \partial\Omega) + d(x, \partial W).$$

"Non-tunneling condition between $\partial\Omega$ and ∂W near spatial infinity"

Define an **intersection number** $\chi_{\Omega W}$ between $\partial\Omega$ and ∂W:



- Orient Ω according to outward-pointing normal
- $\chi_{\Omega W}$ counts (in a signed way) how many times a particle traveling along $\partial\Omega$, with direction of the orientation, enters W. **Here $\chi_{\Omega W} = 1 - 1 = 0$.**

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Theorem [DZ '24]

Under the non-tunneling condition: $\sigma_e^{\Omega W}(H) = \chi_{\Omega W} \cdot \big(\sigma_b(H_+) - \sigma_b(H_-)\big)$.

"The edge conductance is the difference of Hall conductivity times the intersection number between $\partial\Omega$ and ∂W."

This is an index theorem for condensed matter physics!

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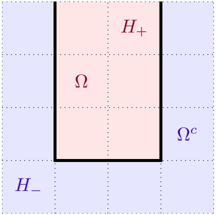

Some condition is necessary: the "tubed" Haldane model remains insulating, so $\sigma_e(H) = 0$ **but** $\Delta\sigma_b \neq 0$!

What if the edge is curved?

Return to: $H = \begin{cases} H_+ & \text{inside } \Omega \\ H_- & \text{outside } \Omega \end{cases}, \qquad \Omega \subset \mathbb{Z}^2, \qquad E \notin \Sigma(H_\pm).$

Conductance along $\partial\Omega$, across ∂W:

$$\sigma_e^{\Omega W}(H) = \mathrm{Tr}\, i[H, \mathbf{1}_W]\, g'(H) \qquad \text{[Ludewig–Thiang '22]}$$

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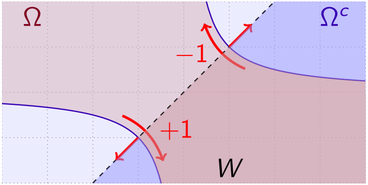

In physical situations the support of the material Ω is fixed. But there is flexibility on W **and this informs us on the nature of edge currents!**

Here $\chi_{\Omega W} = 0$: there is as much current entering W as there is leaving W.

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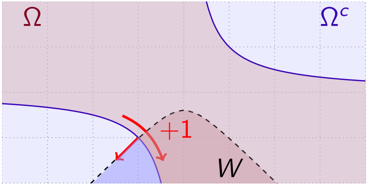


Here $\chi_{\Omega W} = 1$: $\Delta\sigma_b$ edge states enter W along the lower branch of $\partial\Omega$ – so $\Delta\sigma_b$ currents exit W along the upper branch!

Each connected component of $\partial\Omega$ has conductance $\pm\Delta\sigma_b$, with $\pm$ depending on the side of $\partial\Omega$ that Ω lies on. **Quantumly a bit counter-intuitive...**

Proof of $\sigma_e^{\Omega W}(H) = \chi_{\Omega W}\big(\sigma_b(H_+) - \sigma_b(H_-)\big)$

1. After some manipulations: $\partial\Omega, \partial W$ are connected. So $\chi_{\Omega W} \in \{-1, 0, +1\}$. Say $\chi_{\Omega W} = +1$. So we need $\sigma_e^{\Omega W}(H) = \sigma_b(H_+) - \sigma_b(H_-)$.

2. By adapting manipulations due to [Elgart–Graf–Schencker '05]:

$$\sigma_e^{\Omega W}(H) = \sigma_b^{\Omega W}(H_+) - \sigma_b^{\Omega W}(H_-),$$

$$\sigma_b^{\Omega W}(H_\pm) = \operatorname{Tr} K_{\Omega W}^{\pm}, \qquad K_{\Omega W}^{\pm} \overset{\text{def}}{=} iP_\pm\big[[P_\pm, \mathbf{1}_\Omega], [P_\pm, \mathbf{1}_W]\big].$$

Lemma [DZ '24]

We have the bound: *(where ν is the short range rate)*

$$\left|K_{\Omega W}^{\pm}(x,y)\right| \leq e^{-\nu\Psi_{\Omega W}(x) - \nu\Psi_{\Omega W}(y)}.$$

In particular $\sigma_b^{\Omega W}(H_\pm)$ is well defined because

$$\lim_{|x|\to\infty} \frac{\Psi_{\Omega W}(x)}{\ln|x|} = \infty.$$

So we need $\sigma_b^{\Omega W}(H_+) = \sigma_b(H_+)$.

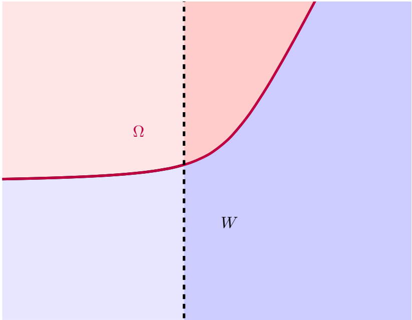
Proof of $\sigma_b^{\Omega W}(H_+) = \sigma_b(H_+)$

3. Given n, we deform Ω, W to Ω_n, W_n in a compact set such that:

a. $\Omega_n = \{x_1 > 0\}$ and $W_n = \{x_2 > 0\}$ in $B_n(0)$.

b. $\Psi_{\Omega_n W_n}$ satisfies

$$\lim_{|x|\to\infty} \frac{\Psi_{\Omega_n W_n}(x)}{\ln|x|} = +\infty \qquad \textbf{uniformly in } n.$$



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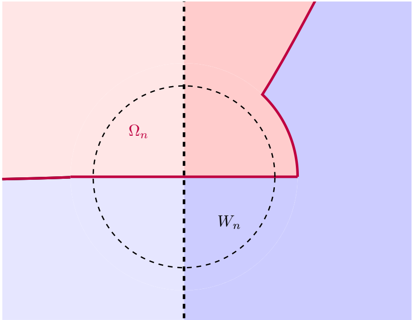

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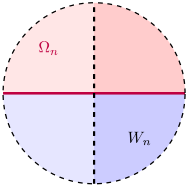

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This construction is the hard part!

Now:

$$\begin{aligned}
\sigma_b^{\Omega W}(H_+) &= \sigma_b^{\Omega_n W_n}(H_+) = \lim_{n\to\infty} \sigma_b^{\Omega_n W_n}(H_+) \\
&= \lim_{n\to\infty} \sum_{x\in\mathbb{Z}^2} K_{\Omega_n W_n}(x,x) \\
&= \sum_{x\in\mathbb{Z}^2} \lim_{n\to\infty} K_{\Omega_n W_n}(x,x) \\
&= \sum_{x\in\mathbb{Z}^2} K_{\{x_1>0\}\{x_2>0\}}(x,x) = \sigma_b(H_+).
\end{aligned}$$

Proof summary

$$\underbrace{\sigma_e^{\Omega W}(H)}_{\text{conductivity of } \partial\Omega \text{ across } W} = \underbrace{\sum_j}_{\text{surgery}} \sigma_e^{\Omega_j W_j}(H)$$

$$= \sum_j \underbrace{\chi_{\Omega_j W_j}}_{\in\{0,\pm 1\}} \cdot \underbrace{\left(\sigma_b^{\Omega_j W_j}(H_+) - \sigma_b^{\Omega_j W_j}(H_-)\right)}_{\text{adapted from [Graf-Elgart-Schencker]}}$$

$$= \sum_j \chi_{\Omega_j W_j} \cdot \underbrace{\left(\sigma_b(H_+) - \sigma_b(H_-)\right)}_{\text{deformation}}$$

$$= \underbrace{\chi_{\Omega W}}_{\cap \# \, \partial\Omega, \partial W} \cdot \underbrace{\left(\sigma_b(H_+) - \sigma_b(H_-)\right)}_{\text{difference of bulk indices}} \; ! \qquad \square$$

Conclusion

- We investigated **conduction between distinct topological phases.**
- If the topological phases fill large enough regions, **edge spectrum fill the bulk gap.**
- **The conductance** (number of edge modes for translation-invariant settings) of each connected component of the interface is σ_b**.**
- These are **extensions of well-known results when the boundary is straight.**

Happy birthday, Maciej!