Control of eigenfunctions on negatively curved manifolds

Semyon Dyatlov (MIT)

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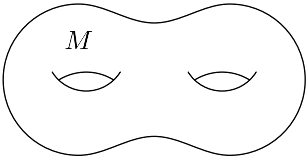
Control of eigenfunctions on surfaces

- (M, g) negatively curved surface
- Geodesic flow $\varphi^t : SM \to SM$ is a standard model of classical chaos
- Eigenfunctions of the Laplacian $-\Delta_g$ studied by quantum chaos



$$(-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2} = 1$$

Theorem 1

Let $\Omega \subset M$ be an arbitrary nonempty open set. Then

$$\|u\|_{L^2(\Omega)} \geq c > 0$$

where c depends on M, Ω but not on λ

Constant curvature: D–Jin '18, using D–Zahl '16 and Bourgain–D '18

Variable curvature: D–Jin–Nonnenmacher '22, using Bourgain–D '18

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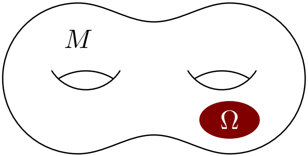

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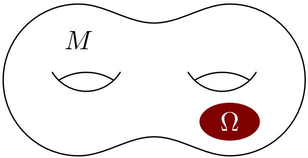

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Logunov–Malinnikova '18: $c = c(\lambda, \Omega) \sim (\mathrm{vol}(\Omega)/C)^{\lambda}$ for any (M, g)

Our result is interesting for fixed Ω in the high frequency limit $\lambda \to \infty$

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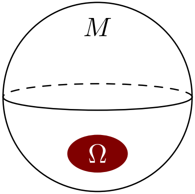

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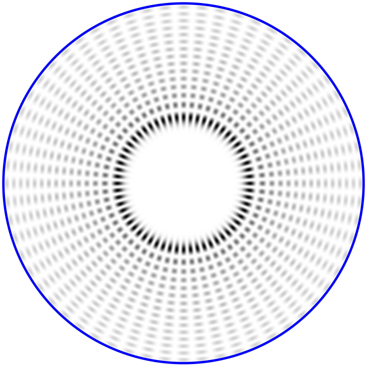
The chaotic nature of geodesic flow is important

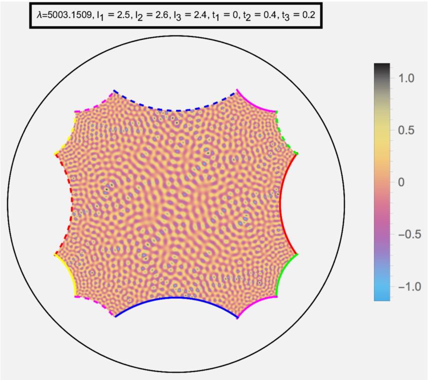
For example, Theorem 1 is false if M is the round sphere

An illustration

Picture on the right courtesy of Alex Strohmaier, using Strohmaier–Uski '12



Disk (Dirichlet b.c.)
Whitespace in the middle



Hyperbolic surface
No whitespace

Microlocal analysis

Localization in position and frequency using semiclassical quantization

$$a(x,\xi) \in C^\infty(T^*M) \;\mapsto\; \operatorname{Op}_h(a) = a\Big(x, \frac{h}{i}\partial_x\Big) : C^\infty(M) \to C^\infty(M)$$

Examples (on $\mathbb{R}^n$): $\operatorname{Op}_h(x_j)u = x_j u$, $\operatorname{Op}_h(\xi_j)u = \frac{h}{i}\partial_{x_j} u$

We put $h := \lambda^{-1} \ll 1$ where λ^2 is the Laplace eigenvalue

Properties of quantization in the semiclassical limit $h \to 0$

- Product Rule: $\operatorname{Op}_h(a)\operatorname{Op}_h(b) = \operatorname{Op}_h(ab) + \mathcal{O}(h)$
- Adjoint Rule: $\operatorname{Op}_h(a)^* = \operatorname{Op}_h(\bar{a}) + \mathcal{O}(h)$
- L^2 boundedness: $\sup|a| < \infty \quad \Longrightarrow \quad \|\operatorname{Op}_h(a)\|_{L^2\to L^2} = \mathcal{O}(1)$
- Egorov's Theorem: $U(-t)\operatorname{Op}_h(a)U(t) = \operatorname{Op}_h(a\circ\varphi^t) + \mathcal{O}(h)$

  where $U(t) = e^{-it\sqrt{-\Delta}}$ is the half-wave propagator

  and $\varphi^t : T^*M \to T^*M$ is the homogeneous geodesic flow

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Semiclassical measures

Take a high frequency sequence of Laplace eigenfunctions

$$(-\Delta_g - \lambda_j^2)u_j = 0, \quad \|u_j\|_{L^2(M)} = 1, \quad \lambda_j \to \infty$$

We say u_j converges weakly to a measure μ on T^*M if, with $h_j := \lambda_j^{-1}$,

$$\forall a \in C_c^\infty(T^*M) : \quad \langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle_{L^2} \to \int_{T^*M} a\, d\mu \quad \text{as } j \to \infty$$

Call such limits μ semiclassical measures

Basic properties

- μ is a probability measure, $\operatorname{supp}\mu \subset S^*M$
- μ is invariant under the geodesic flow $\varphi^t : S^*M \to S^*M$
- The pushforward of μ to M is the weak limit of $|u_j(x)|^2\, d\,\mathrm{vol}_g$
- Natural candidate: Liouville measure $\mu_L \sim d\,\mathrm{vol}$ (equidistribution)
- Natural enemy: delta measure δ_γ on a closed geodesic (scarring)

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QE, QUE, and entropy bounds

- Quantum Ergodicity: if φ^t is ergodic with respect to μ_L then there exists a density 1 sequence of eigenfunctions u_j converging to μ_L [Shnirelman '74, Zelditch '87, Colin de Verdière '85, Zelditch–Zworski '96]

  "Most eigenfunctions equidistribute"

- Quantum Unique Ergodicity conjecture: if φ^t is an Anosov flow (e.g. (M, g) is negatively curved) then the whole sequence of eigenfunctions converges to μ_L [Rudnick–Sarnak '94] Known for arithmetic hyperbolic surfaces [Lindenstrauss '06]

  "All eigenfunctions equidistribute (maybe)"

- Entropy bounds: if φ^t is an Anosov flow then $h_{\mathrm{KS}}(\mu) > 0$, in particular $h_{\mathrm{KS}}(\mu) \geq \frac{d-1}{2}$ for hyperbolic d-manifolds [Anantharaman '08, Anantharaman–Nonnenmacher '07]

  "Eigenfunctions cannot be too concentrated"

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Support of semiclassical measures

Theorem 2 [D–Jin '18, D–Jin–Nonnenmacher '22]

Assume that (M, g) is a negatively curved surface. Then every semiclassical measure μ satisfies $\operatorname{supp}\mu = S^*M$.

"Eigenfunctions cannot be too concentrated (in a different way than entropy bounds)"

- Implies Theorem 1: the weak limit of any $|u_j|^2\,d\operatorname{vol}_g$ is supported on the entire M and thus charges any nonempty open set
- Key tool: Fractal Uncertainty Principle [Bourgain–D '18]

If $X, Y \subset \mathbb{R}$ are porous up to scale h, then $\exists C, \beta > 0$:

$$f \in L^2(\mathbb{R}), \quad \operatorname{supp}\widehat{f} \subset h^{-1}Y \quad \Longrightarrow \quad \|\mathbb{1}_X f\|_{L^2(\mathbb{R})} \leq Ch^\beta \|f\|_{L^2(\mathbb{R})}$$

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Sketch of the proof

- Assume (M, g) is a hyperbolic surface and $u = u_j$ is a sequence of eigenfunctions converging to a measure μ with $\operatorname{supp}\mu \neq S^*M$
- Take nonempty open $\mathcal{U} \subset S^*M$ such that $\mu(\mathcal{U}) = 0$
- u is small on $\mathcal{U}$: $\operatorname{supp} b \subset \mathcal{U} \implies \|\operatorname{Op}_h(b)u\| = o(1)$
- By Egorov's Theorem, u is also small on $\varphi^t(\mathcal{U})$ for all $|t| \leq \log(1/h)$. So $u = \operatorname{Op}_h(a_+)u + o(1) = \operatorname{Op}_h(a_-)u + o(1)$ where $(T = \log(1/h))$

$$\operatorname{supp} a_\pm \ \subset \ \Gamma_\pm(T) := \{\rho \in S^*M \mid \forall t \in [0, T],\ \varphi^{\mp t}(\rho) \notin \mathcal{U}\}$$

  $\Gamma_\pm(T)$ consist of trajectories not intersecting the 'hole' $\mathcal{U}$ for time T
- The sets $\Gamma_\pm(T)$ have fractal structure. Then Fractal Uncertainty Principle implies that $\|\operatorname{Op}_h(a_+)\operatorname{Op}_h(a_-)\| = o(1)$, so $u = o(1)$, a contradiction!

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Sketch of the proof: applying FUP

$$\operatorname{supp} a_\pm \subset \Gamma_\pm(T) := \{\rho \in S^*M \mid \forall t \in [0, T],\ \varphi^{\mp t}(\rho) \notin \mathcal{U}\}$$

Here is a numerical illustration in the related case of cat maps:

$\Gamma_-(T)$, $T = 0$ $\mathcal{U}$ (in white) $\Gamma_+(T)$, $T = 0$

- Using the unstable/stable directions of the geodesic flow φ^t on S^*M
- $\Gamma_+(T)$ smooth in unstable direction, porous in the stable direction
- $\Gamma_-(T)$ smooth in stable direction, porous in the unstable direction
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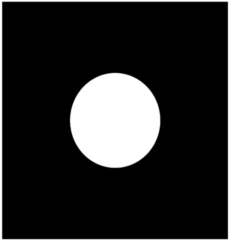

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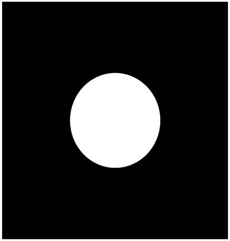
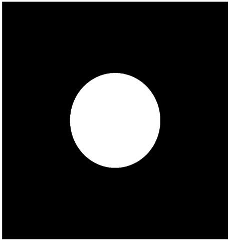


$\mathcal{U}$ (in white)



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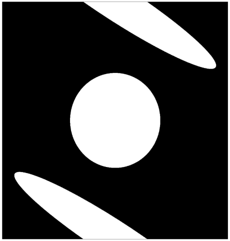
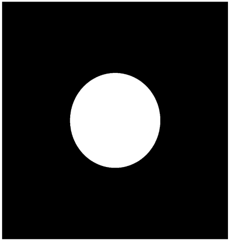


$\Gamma_-(T)$, $T = 1$



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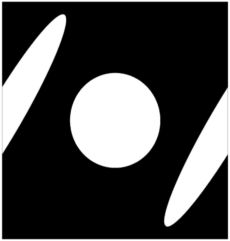

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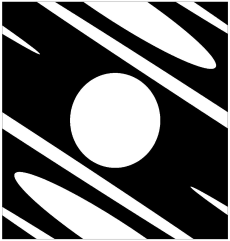
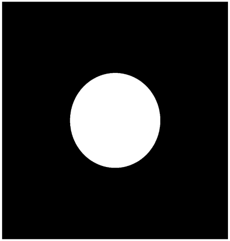
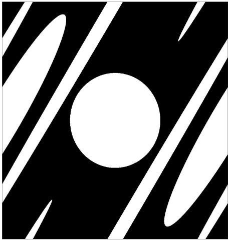


$\Gamma_-(T)$, $T = 2$



$\mathcal{U}$ (in white)



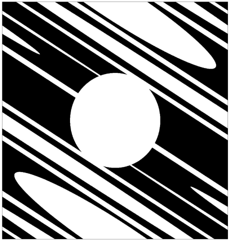
$\Gamma_+(T)$, $T = 2$

- Using the unstable/stable directions of the geodesic flow φ^t on S^*M
- $\Gamma_+(T)$ smooth in unstable direction, porous in the stable direction
- $\Gamma_-(T)$ smooth in stable direction, porous in the unstable direction
- FUP $\Longrightarrow$ $\|\mathrm{Op}_h(a_+)\,\mathrm{Op}_h(a_-)\| = o(1)$, finishing the proof

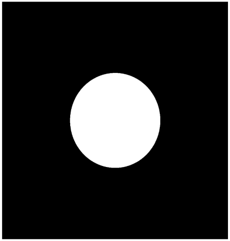
Sketch of the proof: applying FUP

$$\operatorname{supp} a_\pm \ \subset \ \Gamma_\pm(T) := \{\rho \in S^*M \mid \forall t \in [0, T],\ \varphi^{\mp t}(\rho) \notin \mathcal{U}\}$$

Here is a numerical illustration in the related case of cat maps:



$\Gamma_-(T)$, $T = 3$



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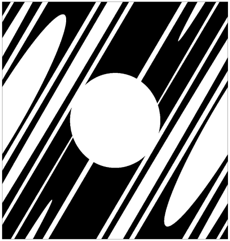

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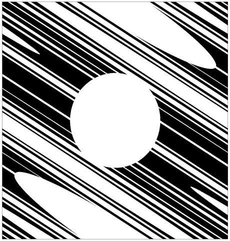
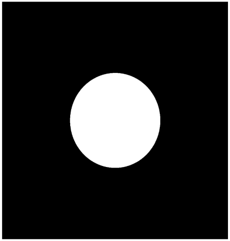


$\Gamma_-(T)$, $T = 4$



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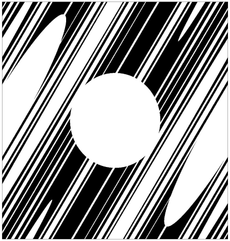

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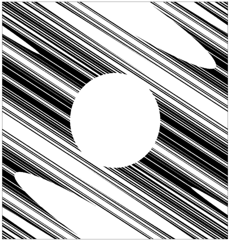
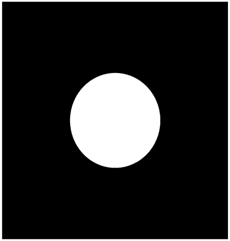


$\Gamma_-(T)$, $T = 5$



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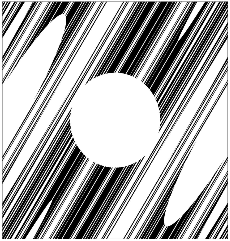

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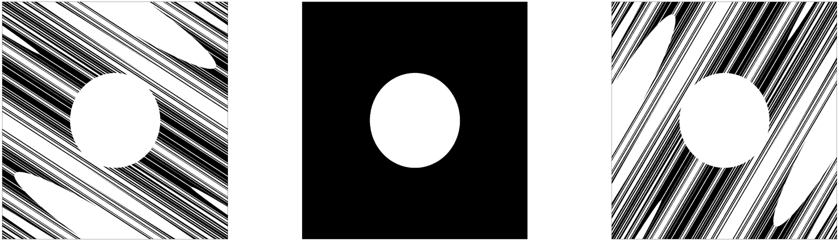

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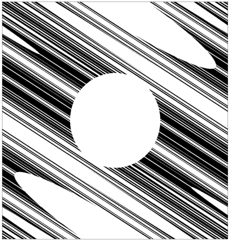

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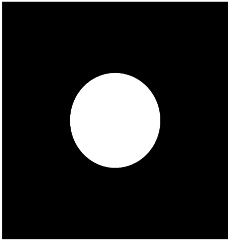

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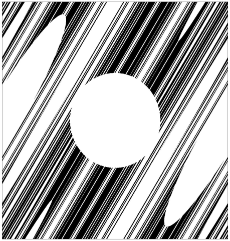

$\Gamma_+(T)$, $T = 5$

- The proof of porosity uses that each global stable/unstable flow line is dense and thus intersects the nonempty open set $\mathcal{U}$

Higher dimensions

Conjecture

Assume that (M, g) is a negatively curved manifold (of any dimension). Then every semiclassical measure μ satisfies $\operatorname{supp}\mu = S^*M$.

- Theorems 1–2 only applied to surfaces because the FUP of Bourgain–D '18 was only valid for subsets of $\mathbb{R}$
- Basic counterexample to FUP in $\mathbb{R}^2$: $X = \mathbb{R} \times \{0\}$, $Y = \{0\} \times \mathbb{R}$ are porous on balls, and $\widehat{\delta_X} = \delta_Y$
- FUP in higher dimensions [Cohen '25]: if X is porous on balls and Y is porous on lines then

$$f \in L^2(\mathbb{R}^d), \quad \operatorname{supp}\hat{f} \subset h^{-1}Y \quad \Longrightarrow \quad \|\mathbb{1}_X f\|_{L^2(\mathbb{R}^d)} \leq Ch^{\beta}\|f\|_{L^2(\mathbb{R}^d)}$$

- We are still very far from the conjecture but…

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Higher dimensions: locally symmetric spaces

Theorem [Kim–Miller '25]

Assume that (M, g) is a compact hyperbolic manifold and μ a semiclassical measure on S^*M. Then $\operatorname{supp}\mu \supset S^*\Sigma$ for some $\Sigma \subset M$ compact immersed totally geodesic submanifold of dimension ≥ 2.

- Uses Cohen's FUP. To get porosity on lines, need $\mathcal{U}$ to intersect each global (un)stable line.
- The closures of these lines are classified using Ratner's Theorem.

Theorem [Athreya–D–Miller '25]

Assume that (M, g) is a compact complex hyperbolic manifold and μ a semiclassical measure on S^*M. Then $\operatorname{supp}\mu \supset S^*\Sigma$ for some $\Sigma \subset M$ compact immersed totally geodesic complex submanifold.

- Uses the 1D FUP but only in the fast stable/unstable directions.

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- Those are families of matrices quantizing hyperbolic toral automorphisms $A : \mathbb{T}^{2d} \to \mathbb{T}^{2d}$, $A \in \mathrm{Sp}(2d, \mathbb{Z})$.
- Have corresponding semiclassical measures on $\mathbb{T}^{2d}$.
- For $d = 1$, these measures have $\operatorname{supp} \mu = \mathbb{T}^2$ [Schwartz '24] but QUE fails [Faure–Nonnenmacher–De Bièvre '03]

More recent results in $d \geq 2$ by Kim–Anderson–Lemke Oliver '26 (see also D–Jézéquel '24):

- For a generic A, we have $\operatorname{supp} \mu = \mathbb{T}^{2d}$ (more precisely, need $p_k(z) = \det(A^k - zI)$ irreducible over $\mathbb{Q}$ for all $k \geq 1$)
- Can have $\operatorname{supp} \mu \subset \Sigma$ where $\Sigma \subset \mathbb{T}^{2d}$ is a Lagrangian A-invariant subtorus [Kelmer '10]
- Cannot have $\operatorname{supp} \mu \subset \Sigma$ where Σ is a (proper) symplectic subtorus
- Can have $\operatorname{supp} \mu \subset \Sigma_1 \cup \Sigma_2$ where Σ_1, Σ_2 are symplectic subtori

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