Uncertainty Principles in Quantum Chaos

Semyon Dyatlov (MIT)

Jan 11, 2025

Laplace eigenfunctions

- The topic: high energy behavior of Laplace eigenfunctions
- 'Simplest' setting: bounded planar domain $\Omega \subset \mathbb{R}^2$
- Complete system of eigenfunctions of $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, Dirichlet b.c.:

$$-\Delta u_j = \lambda_j^2 u_j, \quad u_j|_{\partial\Omega} = 0, \quad \|u_j\|_{L^2(\Omega)} = 1, \quad \lambda_j \to \infty$$

- Quantum mechanical interpretation:
  $u_j =$ pure quantum state of a particle constrained to Ω
  $|u_j|^2\,dx =$ probability distribution of the location of the particle
- Study $|u_j|^2\,dx$ in the high energy limit $\lambda_j \to \infty$
  in the sense of weak convergence of measures on Ω
- Looking for equidistribution: weak limit = volume measure

$$\int_\Omega a|u_j|^2\,dx \to \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega a\,dx \quad \text{for all } a \in C^0(\Omega)$$

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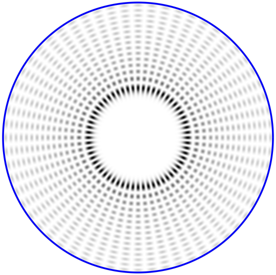
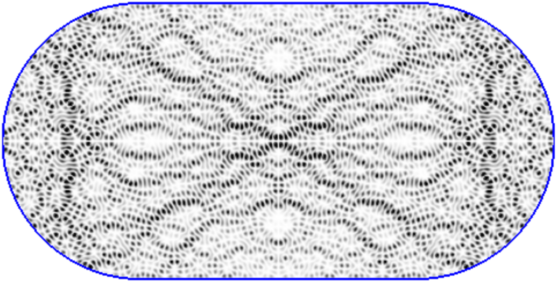
Two examples: quantum side

Eigenfunction concentration
(picture on the right by Alex Barnett)



No equidistribution



Equidistribution

What is the ‘classical’ difference between the domains?

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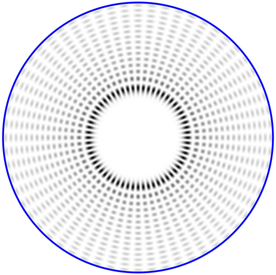

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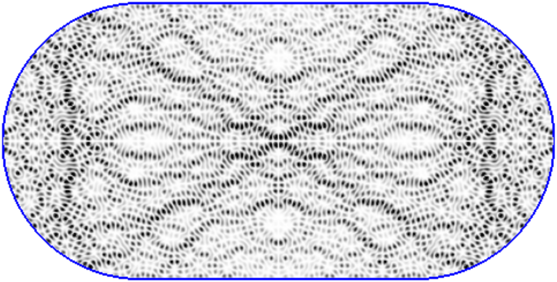


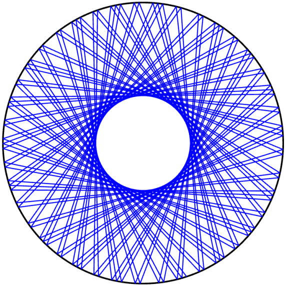
Equidistribution

What is the ‘classical’ difference between the domains?

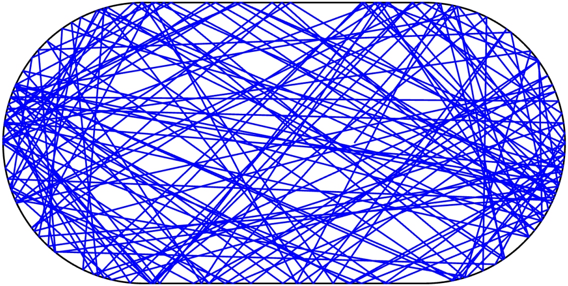
It is the long time behavior of billiard trajectories

Two examples: classical side

A long billiard trajectory



Completely integrable



Ergodic (by Bunimovich)

Ergodicity is a weak way to define chaotic behavior:
almost every trajectory equidistributes as time $\to \infty$

Quantum chaos: chaotic classical flow $\Rightarrow$ equidistribution of eigenfunctions

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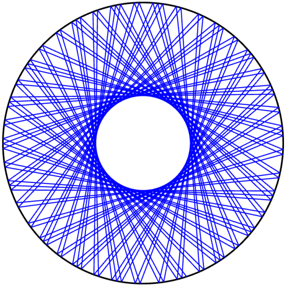

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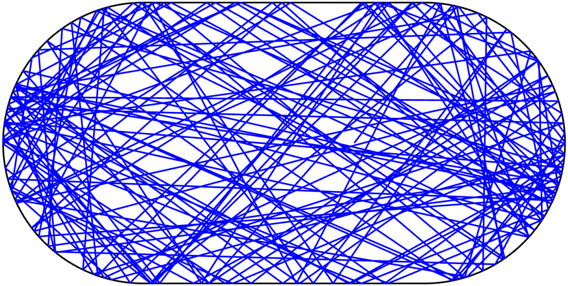

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Quantum Ergodicity

$\Omega \subset \mathbb{R}^2$ a planar domain, u_j a complete system of eigenfunctions:

$$-\Delta u_j = \lambda_j^2 u_j, \quad u_j|_{\partial\Omega} = 0, \quad \|u_j\|_{L^2(\Omega)} = 1, \quad \lambda_j \to \infty$$

Theorem 1

Assume that Ω has ergodic billiard flow. Then there exists a density 1 subsequence λ_{j_k} such that u_{j_k} equidistribute:

$$\int_\Omega a|u_{j_k}|^2\,dx \to \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega a\,dx \quad \text{for all } a \in C^0(\Omega).$$

- Shnirelman '74, Zelditch '87, Colin de Verdière '85, Gérard–Leichtnam '93, Zelditch–Zworski '96
- Applies to general Riemannian manifolds (use the geodesic flow)
- Do we have equidistribution for all eigenfunctions?

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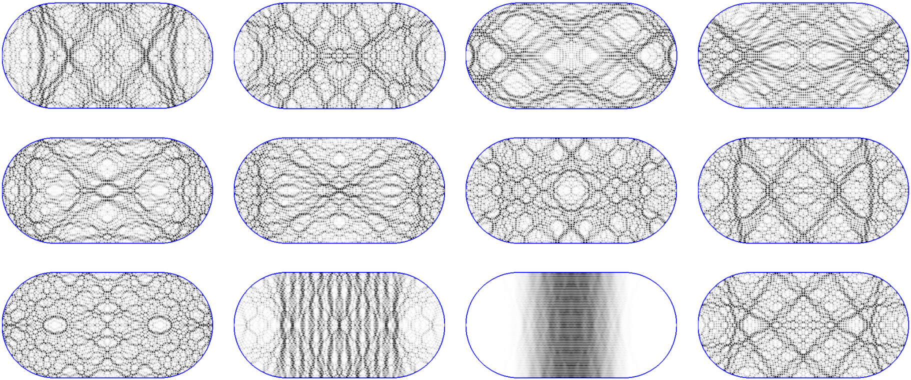
Eigenfunctions for the stadium

A selection of high energy eigenfunctions (by Alex Barnett):



- Most eigenfunctions equidistribute by Quantum Ergodicity
- Some eigenfunctions do not equidistribute: Hassell '10

Quantum Unique Ergodicity

Setting: boundaryless compact Riemannian manifold (M, g)

Eigenfunctions of Laplace–Beltrami operator Δ_g on M:

$$-\Delta_g u_j = \lambda_j^2 u_j, \quad \|u_j\|_{L^2(M, d\,\mathrm{vol}_g)} = 1, \quad \lambda_j \to \infty$$

QUE conjecture [Rudnick–Sarnak '94]

Assume that g has negative sectional curvature. Then the entire sequence of eigenfunctions equidistributes:

$$\int_M a|u_j|^2\, d\,\mathrm{vol}_g \to \frac{1}{\mathrm{vol}_g(M)} \int_M a\, d\,\mathrm{vol}_g \quad \text{for all } a \in C^0(M).$$

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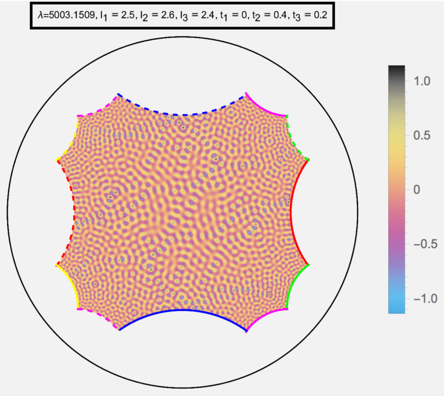
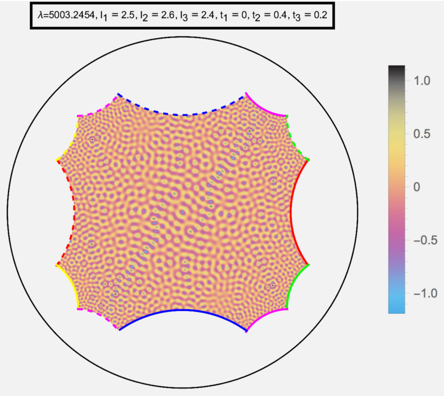
Eigenfunctions on hyperbolic surfaces

Hyperbolic surfaces: $\dim M = 2$ and g has curvature -1

Pictures courtesy of Alex Strohmaier, using Strohmaier–Uski '12





Hyperbolicity of the geodesic flow

Will specialize to hyperbolic surfaces

Geodesic flow on the unit tangent bundle:

$$\varphi^t : SM \to SM, \quad SM = \{(x,\xi) \colon x \in M,\ \xi \in T_xM,\ |\xi|_g = 1\}$$

The flow φ^t is hyperbolic: there is a frame of 3 vector fields on SM

- Flow field V_0, the generator of $\varphi^t = e^{tV_0}$
- Stable field V_s, with $d\varphi^t(\rho)V_s(\rho) = e^{-t}V_s(\varphi^t(\rho))$
- Unstable field V_u, with $d\varphi^t(\rho)V_u(\rho) = e^{t}V_u(\varphi^t(\rho))$

The strongly chaotic behavior of φ^t as $t \to \infty$ is caused by

- exponential contraction in the stable direction,
- exponential expansion in the unstable direction,
- and wrapping around the compact manifold M

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A simpler example of hyperbolicity: Arnold cat map

To illustrate the geodesic flow φ^t on SM, look instead at the following map on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$:

$$\Phi : x \mapsto Ax \bmod \mathbb{Z}^2,$$
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

A has eigenvalues $\lambda^{-1} < 1 < \lambda$

Exponential expansion/contraction along the eigenspaces and wrapping around the torus cause the trajectories $\Phi^n(x)$ to behave chaotically as $n \to \infty$:

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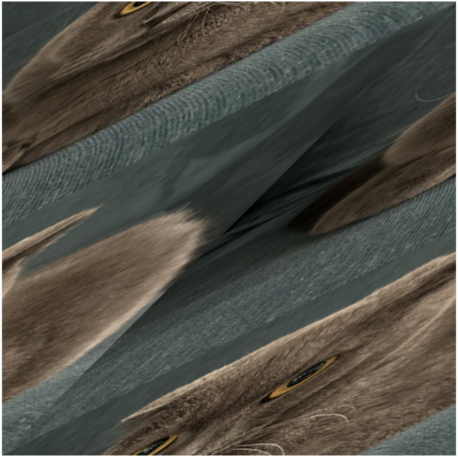

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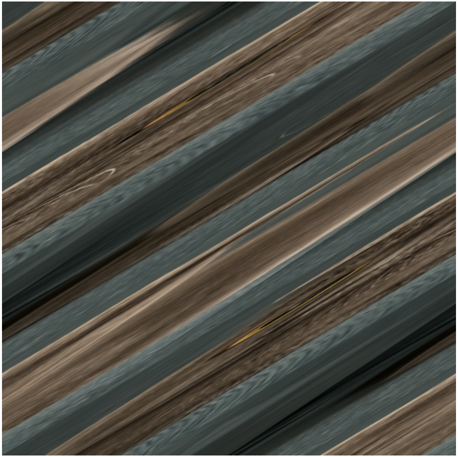

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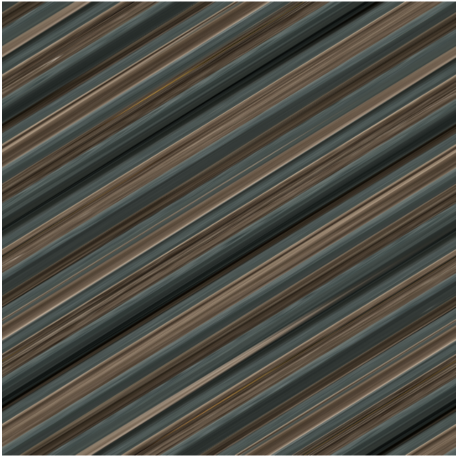

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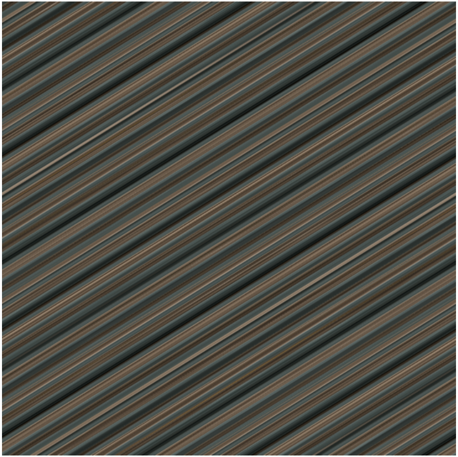

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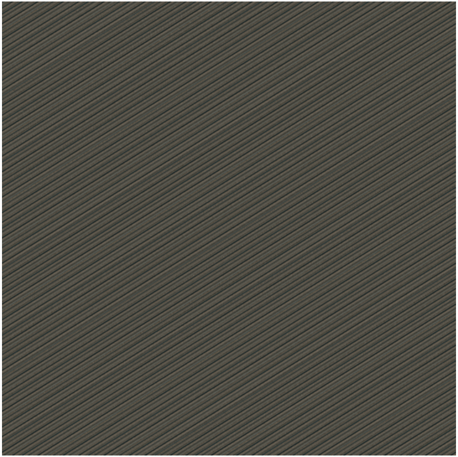

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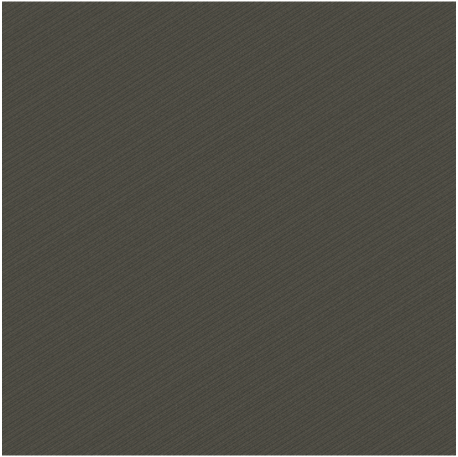

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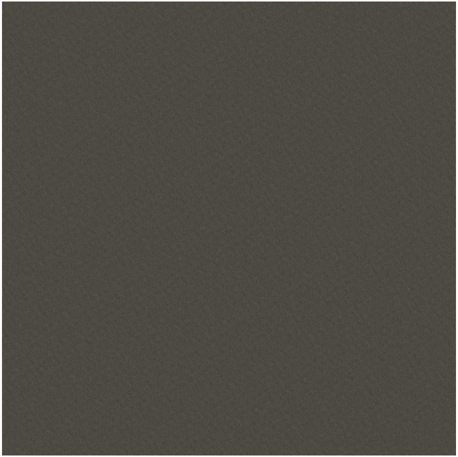

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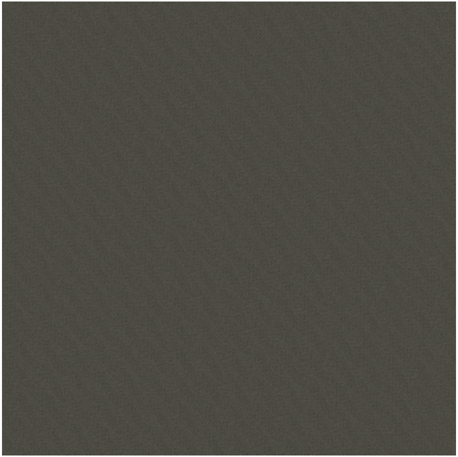

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Microlocalization

- We studied localization of eigenfunctions u_j via the integrals

$$\int_M a|u_j|^2\, d\,\mathrm{vol}_g = \langle a u_j, u_j\rangle_{L^2} \to \dots, \qquad a \in C^0(M)$$

- Now we study localization of u_j in position x and momentum ξ via semiclassical quantization $\mathrm{Op}_h(a) = a(x, -ih\partial_x) : L^2(M) \to L^2(M)$

$$\langle \mathrm{Op}_{h_j}(a) u_j, u_j\rangle_{L^2} \to \dots, \qquad h_j = \lambda_j^{-1} \to 0, \quad a(x,\xi) \in C_c^\infty(T^*M)$$

where $-\Delta_g u_j = \lambda_j^2 u_j$, $\|u_j\|_{L^2} = 1$, u_j oscillates at frequency $\sim h_j^{-1}$

- $a = a(x) \implies \mathrm{Op}_h(a)$ is the multiplication operator by a
- On $\mathbb{R}^n$, $a = a(\xi) \implies \mathrm{Op}_h(a)$ is a Fourier multiplier:

$$\widehat{\mathrm{Op}_h(a)u}(\eta) = a(h\eta)\hat{u}(\eta)$$

That is: frequency $\eta = \xi/h$, momentum $= \xi$

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Definition

The sequence u_j converges microlocally to a measure μ on T^*M if

$$\langle \mathrm{Op}_{h_j}(a) u_j, u_j \rangle_{L^2(M)} \to \int_{T^*M} a\, d\mu \quad \text{for all } a \in C_c^\infty(T^*M)$$

Semiclassical measures: weak limits of sequences of eigenfunctions

Note: $|u_j|^2\, d\,\mathrm{vol}_g \to \pi_* \mu$ weakly where $\pi : T^*M \to M$

Properties of semiclassical measures

- μ probability measure
- $\operatorname{supp} \mu$ contained in the unit cotangent bundle $S^*M \simeq SM$
- μ invariant under the geodesic flow $\varphi^t : S^*M \to S^*M$

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The sequence u_j converges microlocally to a measure μ on T^*M if

$$\langle \mathrm{Op}_{h_j}(a) u_j, u_j \rangle_{L^2(M)} \to \int_{T^*M} a\, d\mu \quad \text{for all } a \in C_c^\infty(T^*M)$$

Semiclassical measures: weak limits of sequences of eigenfunctions

Note: $|u_j|^2\, d\,\mathrm{vol}_g \to \pi_*\mu$ weakly where $\pi : T^*M \to M$

Properties of semiclassical measures

- μ probability measure
- $\operatorname{supp}\mu$ contained in the unit cotangent bundle $S^*M \simeq SM$
- μ invariant under the geodesic flow $\varphi^t : S^*M \to S^*M$

Main results

- A stronger equidistribution property: u_j converges microlocally to the Liouville measure $\mu_L = c\,d\,\mathrm{vol}_g(x)dS(\xi)$ on S^*M
- Implies equidistribution for $|u_j|^2\,d\,\mathrm{vol}_g$
- QE and QUE actually feature microlocal equidistribution

- Plenty of φ^t-invariant measures, e.g. δ-measure on a closed geodesic
- QUE conjecture: Liouville measure is the only semiclassical measure

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Main results: full support property

Theorem 2 [D–Jin '18, D–Jin–Nonnenmacher '21]

If μ is a semiclassical measure on a negatively curved surface, then

$$\operatorname{supp}\mu = S^*M.$$

That is, $\mu(\mathcal{U}) > 0$ for any open nonempty $\mathcal{U} \subset S^*M$.

Corollary: $\|u_j\|_{L^2(\Omega)} \geq c_\Omega > 0$ for any open nonempty $\Omega \subset M$ where c_Ω depends on M, Ω but not on λ_j

Theorem 2 is only known for surfaces because the main new tool, Fractal Uncertainty Principle, was only known for subsets of $\mathbb{R}$. Recent work: Han–Schlag '20, Jaye–Mitkovski '22, D–Jézéquel '24, Athreya–D–Miller '24, Cohen '23, Kim '24

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Main results: entropy bound

Theorem 3 [Anantharaman–Nonnenmacher '07]

On a hyperbolic surface, each semiclassical measure μ has entropy

$$h_{KS}(\mu) \geq \frac{1}{2}.$$

Holds for any negatively curved manifold with some constant > 0
Anantharaman '08, Rivière '10, Anantharaman–Silberman '13

- Liouville measure μ_L has entropy 1, delta-measure has entropy 0
- Counterexample of Faure–Nonnenmacher–de Bièvre '03: in the toy model of quantum cat maps, can have semiclassical measure $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\mu_L$ of entropy $= \frac{1}{2}$

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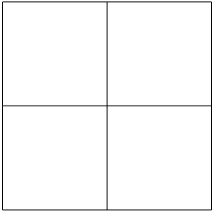
Definition of Kolmogorov–Sinai entropy

How to define the entropy $h_{KS}(\mu)$ of a φ^t-invariant measure μ on S^*M:

- Start with a fixed fine partition of S^*M
- Refine it using the flow φ^t for times $t = 0, 1, \dots, N-1$
- Take a μ-random point $\rho \in S^*M$ and let $\mathcal{A}$ be the element of the refined partition containing ρ. Then

$$\mathbb{E} \log \mu(\mathcal{A}) \approx -h_{KS}(\mu)N \quad \text{as } N \to \infty$$



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(using Arnold cat map model for the figures)

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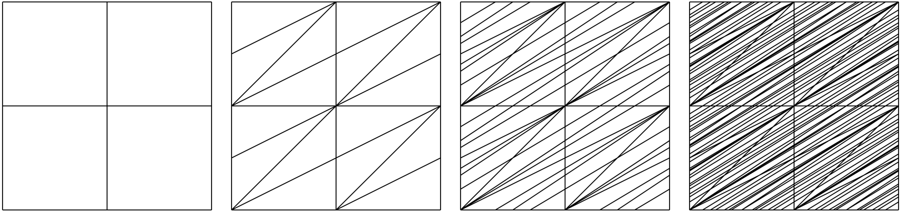

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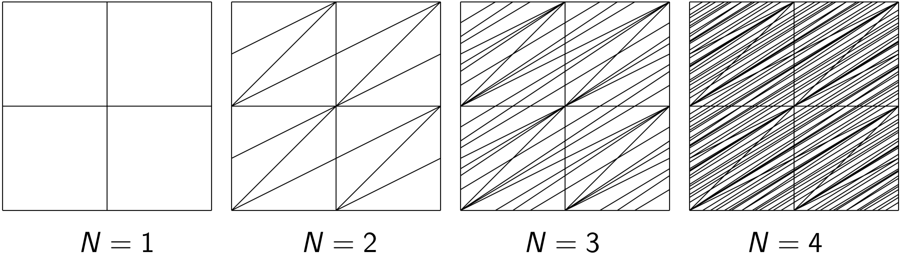


(using Arnold cat map model for the figures)

Proofs and uncertainty principles

We now briefly discuss the proofs of Theorems 2 and 3:

- Relate macroscopic information (semiclassical measure) to microscopic information (microlocalization in h-dependent sets)
- This uses that microlocalization of Laplace eigenfunctions u_j is invariant under the geodesic flow φ^t
- If the semiclassical measure is 'too concentrated' then u_j has microlocalization inconsistent with an uncertainty principle

Semiclassical measures: three examples

Example 1:

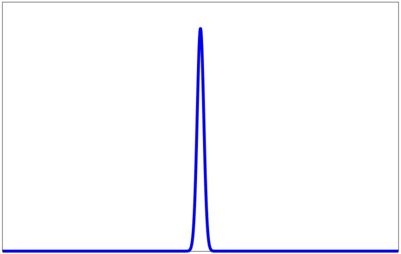
$$u_h(x) = \pi^{-\frac{1}{4}} h^{-\frac{1}{2}} \exp\Big(-\frac{x^2}{2h^2}\Big)$$

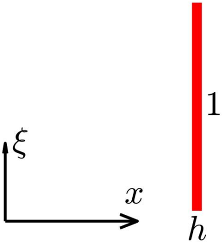
$$\widehat{u_h}(\eta) = \sqrt{2}\pi^{\frac{1}{4}} h^{\frac{1}{2}} \exp\Big(-\frac{h^2\eta^2}{2}\Big)$$

Microlocalized at position $\sim h$ and frequency $\sim h^{-1}$, i.e. momentum ~ 1

Converges microlocally to the measure

$$\mu = \pi^{-\frac{1}{2}} \exp(-\xi^2)\delta_0(x) \times d\xi$$





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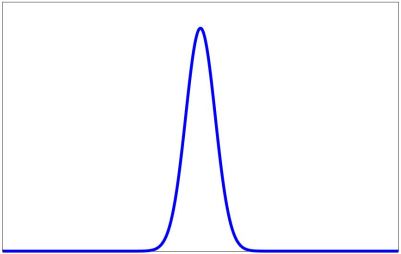

Microlocalized at position $\sim h^{\frac{1}{2}}$ and frequency $\sim h^{-\frac{1}{2}}$, i.e. momentum $\sim h^{\frac{1}{2}}$

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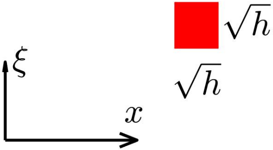
$$\mu = \delta_0(x) \times \delta_0(\xi)$$



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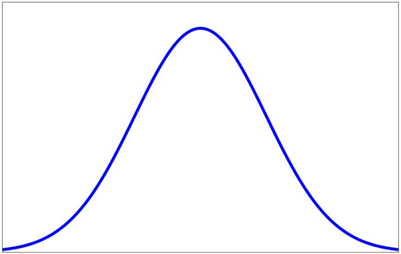
Microlocalized at position ~ 1 and frequency ~ 1, i.e. momentum $\sim h$



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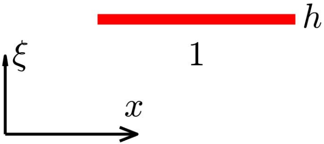
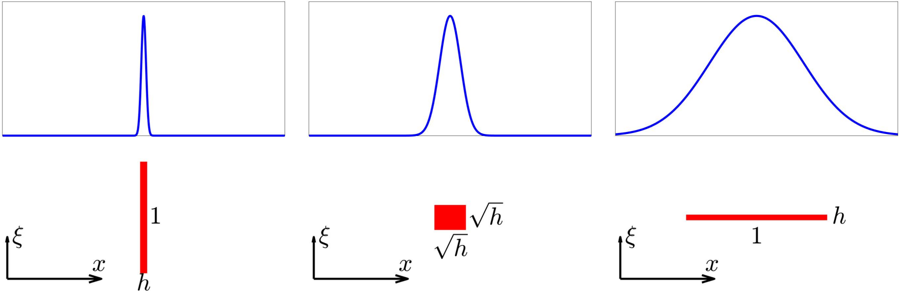


Basic Uncertainty Principle

Uncertainty Principle

If $u \in L^2(\mathbb{R})$ is microlocalized in an interval of size Δx in position and an interval of size $\Delta\xi$ in momentum (= frequency $\times h$) then

$$\Delta x \cdot \Delta\xi \gtrsim h.$$



Proving the full support property (Theorem 2)

Key ingredient: Fractal Uncertainty Principle [Bourgain–D '18]

No $u \in L^2(\mathbb{R})$ can be localized in position and frequency near a fractal set

- Argue by contradiction: assume $\mu(\mathcal{U}) = 0$ for some open nonempty $\mathcal{U} \subset S^*M$, then u_j is microlocalized away from $\mathcal{U}$
- Then u_j also microlocalized away from $\varphi^t(\mathcal{U})$ for all t. Use for $|t| \leq \log(1/h)$, get microlocalization incompatible with FUP

$\Gamma_-(N)$, $N = 0$ $\mathcal{U}$ (in white) $\Gamma_+(N)$, $N = 0$

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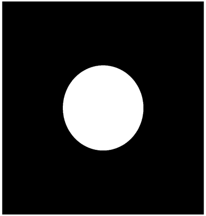

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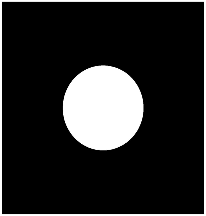

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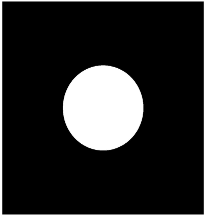

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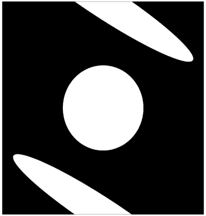

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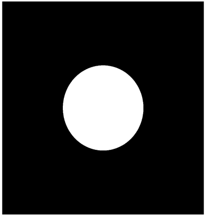

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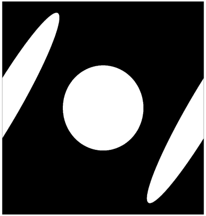

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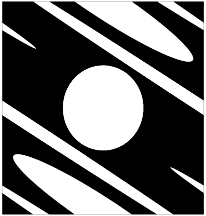
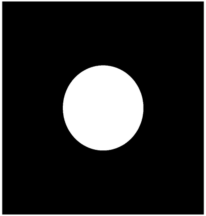


$\Gamma_-(N)$, $N = 2$



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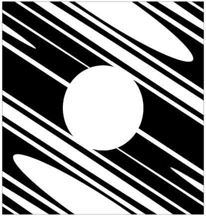

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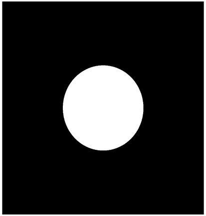
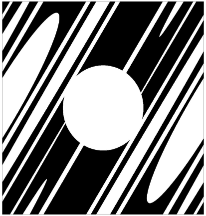


$\mathcal{U}$ (in white)



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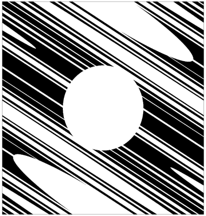
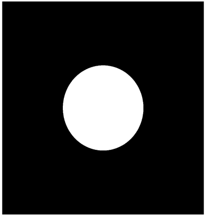


$\Gamma_-(N)$, $N = 4$



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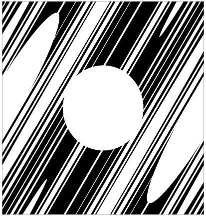

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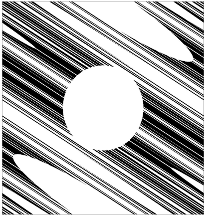
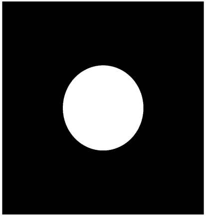
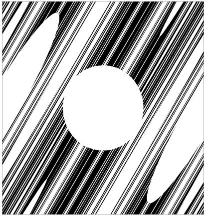


$\Gamma_-(N)$, $N = 5$



$\mathcal{U}$ (in white)



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Proving the entropy bound (Theorem 3)

Entropic Uncertainty Principle [Hirschman '57, Maassen–Uffink '88]

Assume that $f \in L^2(\mathbb{R})$, $\|f\|_{L^2} = 1$, and define the Shannon entropy

$$H(|f|^2) = -\int_{\mathbb{R}} |f(x)|^2 \log(|f(x)|^2)\, dx.$$

Then, defining $\widehat{f}(y) = \int_{\mathbb{R}} e^{-2\pi ixy} f(x)\, dx$,

$$H(|f|^2) + H(|\widehat{f}|^2) \geq 0.$$

- Write u_j as a superposition of stable wave packets and also unstable wave packets, with coefficients expressed as 2 functions $v_u, v_s \in L^2(\mathbb{R})$
- Using that u_j is an eigenfunction, show that v_u and v_s have roughly the same entropy and relate it to the entropy of μ
- Relate $\widehat{v_u}$ to v_s and use the Entropic Uncertainty Principle to get a lower bound on entropy of v_u, v_s and thus the entropy of μ

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- Write u_j as a superposition of stable wave packets and also unstable wave packets, with coefficients expressed as 2 functions $v_u, v_s \in L^2(\mathbb{R})$
- Using that u_j is an eigenfunction, show that v_u and v_s have roughly the same entropy and relate it to the entropy of μ
- Relate $\widehat{v}_u$ to v_s and use the Entropic Uncertainty Principle to get a lower bound on entropy of v_u, v_s and thus the entropy of μ

Thank you for your attention!

Fractal uncertainty principle

Definition

A set $X \subset [0,1]$ is ν-porous ($\nu > 0$) on scales h to 1 if for each interval I of size $h \leq |I| \leq 1$, there is an interval $J \subset I$ with $|J| = \nu|I|$ and $J \cap X = \emptyset$

Example: mid-third Cantor set $\mathcal{C} \subset [0,1]$ is $\frac{1}{6}$-porous on scales 0 to 1

Fractal uncertainty principle [Bourgain–D '18]

Assume that $X, Y \subset [0,1]$ are ν-porous on scales h to 1. Then there exist $\beta > 0$, C depending on ν but not on X, Y, h such that

$$f \in L^2(\mathbb{R}), \quad \mathrm{supp}(\mathcal{F}_h f) \subset X \quad \Longrightarrow \quad \|f\|_{L^2(Y)} \leq C h^{\beta} \|f\|_{L^2(\mathbb{R})}.$$

Here $\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the semiclassical Fourier transform:

$$\mathcal{F}_h f(\xi) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ix\xi/h} f(x)\, dx$$

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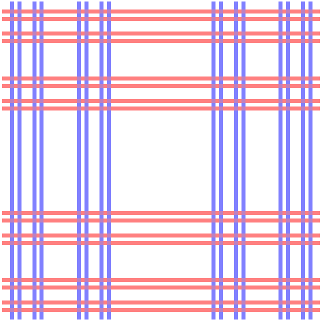
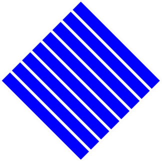
Interpretation: no quantum state can be localized on a porous set in both position and frequency



Stable/unstable packet heuristic

- Write a Laplace eigenfunction u as a superposition of stable wave packets e_s^y, each of which is microlocalized h-close to a weak stable leaf indexed by $y \in \mathbb{R}$:

$$u(x) = \int_{\mathbb{R}} e_s^y(x) v_s(y)\, dy \quad \text{for some } v_s \in L^2(\mathbb{R})$$



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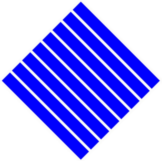

- Write u as a superposition of unstable wave packets e_u^z:

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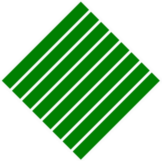

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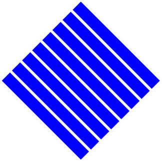

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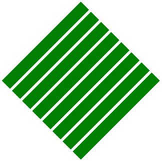

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- The coefficients v_s, v_u are related by semiclassical Fourier transform:

$$v_s(y) = \mathcal{F}_h v_u(y) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-iyz/h} v_u(z)\, dz$$

Proof of Theorem 2 and stable/unstable packets

- Write the eigenfunction u in stable and unstable bases:

$$u(x) = \int_{\mathbb{R}} e_s^y(x) v_s(y)\, dy = \int_{\mathbb{R}} e_u^z(x) v_u(z)\, dz, \qquad v_s = \mathcal{F}_h v_u$$

- Argue by contradiction: assume $\mathcal{U} \subset S^*M$ open nonempty and $\mu(\mathcal{U}) = 0$. Then u is microlocalized away from $\mathcal{U}$.
- Since u is a Laplace eigenfunction, its microlocalization is invariant under φ^t. So u is microlocalized on the sets

$$\Gamma_\pm(T) = \{\rho \in S^*M \mid \varphi^{\pm t}(\rho) \notin \mathcal{U},\ t = 0, \dots, T\}$$

- Fix $T = \log(1/h)$. Then Γ_+ is foliated by stable leaves and porous in the unstable direction. Same for Γ_-, switching stable/unstable. So $\operatorname{supp} v_s \subset X$, $\operatorname{supp} v_u \subset Y$ where $X, Y \subset \mathbb{R}$ are porous.
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