A QUICK INTRODUCTION TO B-CALCULUS

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Abstract. Some rough notes for a few lectures given at MSRI in Fall 2024. The goal of these lectures is to use simple examples to demonstrate the key ingredients needed to construct a full parametrix for an elliptic b-differential operator.

1. Prelude: the compact case

Let (M, g) be a compact Riemannian manifold without boundary, and Δ_g be the Laplace–Beltrami operator. A standard fact in the theory of elliptic PDE is that

$$\Delta_g : H^{s+2}(M) \to H^s(M) \quad \text{is a Fredholm operator.} \tag{1.1}$$

Here $H^s(M)$ is the Sobolev space of order $s \in \mathbb{R}$; for $s \in \mathbb{N}_0$ it is the space of functions whose derivatives of order $\leq s$ lie in $L^2(M)$.

A common way to prove (1.1) is to use an *elliptic estimate*: for any N

$$\|u\|_{H^{s+2}} \leq C\|\Delta_g u\|_{H^s} + C\|u\|_{H^{-N}}. \tag{1.2}$$

The implication (1.2) $\implies$ (1.1) goes through a functional analysis argument, where the key input is the fact that the inclusion $H^{s+2}(M) \hookrightarrow H^{-N}(M)$ is compact as soon as $s + 2 > -N$. It is here that the compactness of M is used.

One way to show the estimate (1.2) is via the construction of an *elliptic parametrix*, an operator Q on distributions on M with the following properties:

(1) Q is bounded $H^s(M) \to H^{s+2}(M)$;
(2) $Q\Delta_g = I + R$ where the operator R is smoothing, i.e. maps $H^{-N}(M) \to H^N(M)$ for all N, or equivalently R has a smooth integral kernel:

$$Rf(y) = \int_M \mathbf{R}(y, y')f(y')\,dy' \quad \text{where } \mathbf{R} \in C^\infty(M \times M). \tag{1.3}$$

Indeed, we can apply the identity $Q\Delta_g = I + R$ to a distribution $u \in H^{s+2}(M)$ and get

$$u = Q\Delta_g u - Ru,$$

which gives (1.2):

$$\|u\|_{H^{s+2}} \leq \|Q\Delta_g u\|_{H^{s+2}} + \|Ru\|_{H^{s+2}} \leq C\|\Delta_g u\|_{H^s} + C\|u\|_{H^{-N}}.$$

From now on, we will actually switch what we mean by a parametrix, replacing the equation $Q\Delta_g = I + R$ by

$$\Delta_g Q = I + R. \tag{1.4}$$

This makes the presentation below a bit easier and from here we can get the original parametrix by taking the adjoint.

1.1. **Elliptic parametrix: basic example.** To construct an elliptic parametrix Q, we construct its distributional kernel $\mathbf{Q} \in \mathcal{D}'(M \times M)$, where $\mathcal{D}'(M \times M)$ is the space of distributions on $M \times M$ (dual to the space of smooth functions) and Q and $\mathbf{Q}$ are related by the following formula (interpreted distributionally):

$$Qf(y) = \int_M \mathbf{Q}(y, y')f(y')\,dy'. \tag{1.5}$$

Note that to canonically define $\mathbf{Q}$ this way, we would need to fix the density dy'. A geometrically nice way around this is to work with sections of the line bundle $|\Omega|^{1/2}$ of *half-densities* (which can be defined e.g. using coordinate charts with transition functions given by the square root of the Jacobian). These half-densities can be expressed in coordinates as $f(y)|dy|^{1/2}$ where f is a function. Note that the L^2 norm of such an object is invariantly defined. Moreover, if $\mathbf{Q} \in \mathcal{D}'(M \times M; |\Omega|^{1/2})$ is a half-density on $M \times M$ then it canonically defines an operator Q on $L^2(M; |\Omega|^{1/2})$. Of course, since we have a metric we can identify functions with half-densities and that's what we will do here.

In terms of distributional kernels, the parametrix property (1.4) becomes

$$\Delta_y \mathbf{Q}(y, y') = \delta(y - y') + \mathbf{R}(y, y') \quad \text{where } \mathbf{R} \in C^\infty(M \times M). \tag{1.6}$$

Here $\delta(y - y')$ is the delta function on the diagonal $\{y = y'\} \subset M \times M$, which is the integral kernel of the identity operator, and Δ_y is the Laplace–Beltrami operator taken in the y variable.

Here is a basic example of an elliptic parametrix: on the circle $\mathbb{S}^1_y = \mathbb{R}/2\pi\mathbb{Z}$, consider the Laplace–Beltrami operator $\Delta_g = -\partial_y^2$ and define

$$\mathbf{Q}(y, y') = -\chi(y, y')\frac{|y - y'|}{2}$$

where $\chi \in C^\infty_{\mathrm{c}}(\mathbb{S}^1 \times \mathbb{S}^1)$ is a cutoff to a neighborhood of the diagonal $\{y = y'\}$ and $|y - y'|$ is just the distance on the circle (well-defined near the diagonal). Since $\partial_y^2|y - y'| = 2\delta(y - y')$, we get

$$-\partial_y^2 \mathbf{Q}(y, y') = \delta(y - y') + \mathbf{R}(y, y') \quad \text{where } \mathbf{R} \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1).$$

Here $\mathbf{R}$ comes from the derivatives hitting the cutoff. Note that we could replace $\frac{|y-y'|}{2}$ by, say, the Heaviside function $H(y - y')$ – this does not matter as it just changes Q by a smoothing operator.

Of course, we would also need to show boundedness on Sobolev spaces – this is definitely doable from the formula for $\mathbf{Q}$ but we do not do it here (or in the next subsection for that matter).

1.2. **Elliptic parametrix: general case.** For the general case, we construct $\mathbf{Q}(y, y')$ as a *conormal distribution* to the diagonal $\{y = y'\} \subset M \times M$. This means that away from $\{y = y'\}$, $\mathbf{Q}$ is smooth (we in fact do not lose anything by making it equal to 0 outside of a neighborhood of the diagonal), and near the diagonal in local coordinates $\mathbf{Q}$ has the following Fourier integral form (where $\dim M = n$):

$$\mathbf{Q}(y, y') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle y - y', \eta\rangle} q(y, \eta)\, d\eta. \tag{1.7}$$

Here $q(y, \eta)$ is a symbol: it is smooth in y and has expansions in homogeneous functions of η as $|\eta| \to \infty$. One can think of this in terms of Fourier transform: if

$$\check{q}(y, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle z, \eta\rangle} q(y, \eta)\, d\eta$$

is the inverse Fourier transform of q in the second variable, then

$$\mathbf{Q}(y, y') = \check{q}(y, y - y').$$

Since we only currently care about the singularities of $\mathbf{Q}$, it is the behavior as $|\eta| \to \infty$ that matters. In fact, if q is rapidly decaying as $|\eta| \to \infty$, then $\mathbf{Q}$ is smooth. Also, the integrals above may not converge in the usual sense, but we can use e.g. Fourier transform of tempered distributions to make sense of these as distributions in y, y'.

We now construct q to satisfy the parametrix equation (1.6). First, by the Fourier inversion formula we have the representation

$$\delta(y - y') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle y - y', \eta\rangle}\, d\eta.$$

Let us pretend for simplicity that g is Euclidean in our coordinates (the reader is encouraged to consider the general case which is similar but with a longer computation). Then $\Delta_y = -\sum_{j=1}^n \partial_{y_j}^2$ and we can compute

$$\Delta_y \mathbf{Q}(y, y') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle y - y', \eta\rangle} \big(|\eta|^2 q(y, \eta) - 2i\langle \eta, \partial_y q(y, \eta)\rangle + \Delta_y q(y, \eta)\big)\, d\eta.$$

Thus we need to solve the symbolic equation

$$|\eta|^2 q(y, \eta) - 2i\langle \eta, \partial_y q(y, \eta)\rangle + \Delta_y q(y, \eta) = 1. \tag{1.8}$$

Note that as $\eta \to \infty$, the dominant term is the first one, $|\eta|^2 q(y, \eta)$. We then construct q as an asymptotic series in η as $|\eta| \to \infty$:

$$q(y, \eta) \sim \sum_{\ell \geq 0} q_\ell(y, \eta), \quad q_\ell(y, \eta) = \mathcal{O}(|\eta|^{-2-\ell}).$$

To do this, we put the principal term to be

$$q_0(y,\eta) = |\eta|^{-2} \quad \text{for } |\eta| \geq 1$$

and solve for the other terms iteratively. (Actually if the metric is Euclidean, all the other terms are 0, but that's an artifact of the constant coefficients.) This gives the elliptic parametrix satisfying (1.6).

A geometer might wonder whether the class of conormal distributions defined by (1.7) is coordinate invariant. It is, and the symbol $q_0(y,\eta)$ makes sense invariantly as a function on the cotangent bundle T^*M. This can be seen using the method of stationary phase, but we do not give any details here.

For more details on the constructions in this section, see for example [Dya22, §14].

2. Fredholm property on manifolds with cylindrical ends

The goal of these lectures is to establish the Fredholm property for manifolds with cylindrical ends. A basic example is given by the exact cylinder with the product metric

$$M = \mathbb{R}_t \times Y_y, \quad g = dt^2 + h(y, dy) \tag{2.1}$$

where (Y, h) is a compact Riemannian manifold. More generally, one can consider a manifold M consisting of a compact core and an infinite end of the form $[0,\infty)_t \times Y_y$ with the product metric, where $t = 0$ is where we link with the compact core and $t \to \infty$ is the infinity. (This includes the case of several ends since we do not assume Y to be connected.)

Even more generally, we can allow the coefficients of the metric in the infinite end to depend on t in a mild way, that is being smooth in e^{-t}, y, giving a metric of the form

$$g = a(e^{-t}, y)dt^2 + b(e^{-t}, y)\, dtdy + c(e^{-t}, y, dy). \tag{2.2}$$

Of course, what matters most here is the behavior at $t = \infty$ which is encoded by the metric

$$g_0 = a(0, y)dt^2 + b(0, y)dtdy + c(0, y, dy).$$

2.1. Compactification and b-metrics.

We now rewrite the cylinder metric using a compactification of M. On an infinite end $[0,\infty)_t \times Y_y$, introduce the variable

$$x = e^{-t}.$$

This turns the infinite end and the product cylindrical metric (2.1) into

$$(0,1]_x \times Y_y, \quad g = \frac{dx^2}{x^2} + h(y, dy).$$

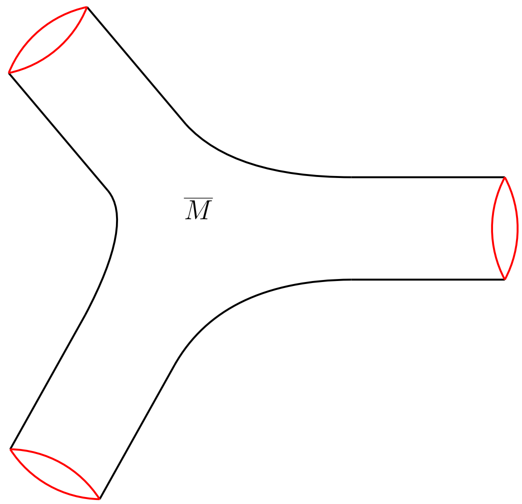

FIGURE 1. Example of a manifold with cylindrical ends with the ends compactified. Highlighted in red is the the boundary of the compactification $Y = \partial\overline{M}$

Note that now $x = 0$ is the infinite end and $x = 1$ is where we link with the compact core. The more general metric (2.2) becomes

$$g = g\Big(x, y, \frac{dx}{x}, dy\Big) = a(x, y)\frac{dx^2}{x^2} - b(x, y)\frac{dx}{x}dy + c(x, y, dy). \tag{2.3}$$

What we do now is to compactify the cylindrical end to $[0, 1]_x \times Y_y$, adding the boundary hypersurface $\{x = 0\} = \{t = \infty\}$. This yields a manifold with boundary, denoted by $\overline{M}$, and the boundary is $\partial\overline{M} = Y$. See figure 1. For example, the compactification of the basic cylinder $\mathbb{R} \times Y$ is diffeomorphic to $[-1, 1] \times Y$, with ± 1 corresponding to the two infinite ends $t = \pm\infty$. The function x on the infinite end can be extended to a *boundary defining function* $x : \overline{M} \to [0, \infty)$, that is x is smooth on $\overline{M}$, the interior is given by $M = \{x > 0\}$, the boundary is $\partial\overline{M} = \{x = 0\}$, and dx is nonvanishing on $\partial\overline{M}$.

To think geometrically about metrics of the form (2.3), let us now consider a general compact manifold with boundary $\overline{M}$ and define the Lie algebra of *b-vector fields*

$$\mathcal{V}_{\mathrm{b}}(\overline{M})$$

consisting of smooth vector fields on $\overline{M}$ tangent to the boundary. If we take local coordinates $(x, y_1, \ldots, y_{n-1})$ near a point of the boundary $\partial\overline{M}$, with x a boundary defining function, then $\mathcal{V}_{\mathrm{b}}(\overline{M})$ is spanned over $C^\infty(\overline{M})$ by vector fields

$$x\partial_x, \partial_{y_1}, \ldots, \partial_{y_{n-1}}.$$

We now say that a Riemannian metric g on M is a *b-metric* if the inverse tensor g^{-1} can locally be written as a positive definite quadratic form in b-vector fields. This gives exactly metrics of the form (2.3), where $Y = \partial\overline{M}$.

2.2. **b-Sobolev spaces and the Fredholm property.** Let us assume now that $\overline{M}$ is a compact manifold with boundary $Y = \partial\overline{M}$ and g is a b-metric on it (that is, (M, g) is an asymptotically cylindrical manifold). Let x be a boundary defining function.

The volume density

$$d\,\mathrm{vol}_g$$

of the metric g looks like $\frac{|dxdy|}{x}$ near the boundary in coordinates (2.3). It is natural to consider the L^2 space

$$L^2(M; d\,\mathrm{vol}_g).$$

Then we have *b-Sobolev spaces*

$$H^s_{\mathrm{b}}(M),\ \ s \in \mathbb{R}$$

which give the natural notion of Sobolev space adapted to the metric g (or really, to b-metrics as a class: different b-metrics give equivalent spaces). If $s \geq 0$ is an integer, then we can define them by testing by b-vector fields:

$$H^s_{\mathrm{b}}(M) = \{f \colon V_1 \dots V_m f \in L^2(M; d\,\mathrm{vol}_g) \text{ for all } V_1, \dots, V_m \in \mathcal{V}_{\mathrm{b}}(M),\ m \leq s\}.$$

For example, on the cylinder $\mathbb{R}_t \times \mathbb{S}^1_y$ with the metric $dt^2 + dy^2$ this just corresponds to applying the fields ∂_t, ∂_y no more than s times and getting a function in $L^2(\mathbb{R} \times \mathbb{S}^1; |dtdy|)$. On the line $\mathbb{R}_t$ with the metric dt^2, we just have $H^s_{\mathrm{b}}(\mathbb{R}) = H^s(\mathbb{R})$, the standard Sobolev space.

We cannot expect the Fredholm property of Δ_g to hold in general. This is already evident for the case of the line: the operator $-\partial_t^2$ is not Fredholm $H^2(\mathbb{R}) \to L^2(\mathbb{R})$. More precisely, the range $-\partial_t^2(H^2(\mathbb{R}))$ is not closed in $L^2(\mathbb{R})$. Let us show here the similar (but slightly shorter to prove) statement that $\partial_t(H^1(\mathbb{R}))$ is not closed in $L^2(\mathbb{R})$. By basic ODE analysis the space $\partial_t(H^1(\mathbb{R})) \cap C^\infty_{\mathrm{c}}(\mathbb{R}) = \partial_t(C^\infty_{\mathrm{c}}(\mathbb{R}))$ consists of functions $f \in C^\infty_{\mathrm{c}}(\mathbb{R})$ such that $\int_{\mathbb{R}} f(t)\,dt = 0$ (here C^∞_{c} denotes the space of smooth compactly supported functions). Now take $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R})$ such that $\int_{\mathbb{R}} \chi(t)\,dt = 1$ and consider the family of functions $f_\delta(t) := \chi(t) - \delta\chi(\delta t) \in C^\infty_{\mathrm{c}}(\mathbb{R})$ for $\delta > 0$. Then $\int_{\mathbb{R}} f_\delta(t)\,dt = 0$ and thus $f_\delta(t) \in \partial_t(H^1(\mathbb{R}))$ but as $\delta \to 0$ we have $f_\delta \to \chi$ in $L^2(\mathbb{R})$, and $\chi \notin \partial_t(H^1(\mathbb{R}))$.

However, we will get the Fredholm property in *weighted b-Sobolev spaces*

$$x^\alpha H^s_{\mathrm{b}}(M),$$

where the order $\alpha \in \mathbb{R}$ of the space has to satisfy the condition (2.7) below. We have the inclusion $x^\alpha H^s_{\mathrm{b}}(M) \subset x^{\alpha'} H^{s'}_{\mathrm{b}}(M)$ when $\alpha \geq \alpha'$ and $s \geq s'$. For example, on the line $\mathbb{R}_t$, if we forget that $|t|$ is not smooth at the origin, we can take $x = e^{-|t|}$ so that

$$x^\alpha H^s_{\mathrm{b}}(\mathbb{R}) = e^{-\alpha|t|} H^s(\mathbb{R}).$$

To describe the condition on α, we look for solutions to the equation $\Delta_g u = 0$ asymptotically in the infinite end which have the form

$$u(x,y) = x^\lambda v(y) = e^{-\lambda t} v(y). \tag{2.4}$$

Here $\lambda \in \mathbb{C}$ is a constant. The action of the operator Δ_g on such functions is described by its *indicial operator*, which is a λ-dependent family of second order elliptic partial differential operators $I(\lambda)$ on the boundary $Y = \partial\overline{M}$ defined as follows:

$$x^{-\lambda}\Delta_g(x^\lambda v(y)) = I(\lambda)v(y) + \mathcal{O}(x) \quad \text{as } x \to 0. \tag{2.5}$$

One can compute the indicial operator by replacing $x\partial_x = -\partial_t$ with λ in Δ_g and then setting $x = 0$, since $x^{-\lambda}(x\partial_x)x^\lambda = x\partial_x + \lambda$. For example, in case of the product metric $g = \frac{dx^2}{x^2} + h(y, dy)$, we have $\Delta_g = -(x\partial_x)^2 + \Delta_h$ and we compute

$$I(\lambda) = \Delta_h - \lambda^2. \tag{2.6}$$

We say λ is an *indicial root* of Δ_g if the operator $I(\lambda)$ is not invertible (say, as an operator $H^2(Y) \to L^2(Y)$). The set indicial roots is a discrete subset of $\mathbb{C}$. In the case of the product metric the indicial roots are solutions to $\lambda^2 \in \operatorname{Spec}(\Delta_h)$, where $\operatorname{Spec}(\Delta_h) \subset [0,\infty)$ is the L^2 spectrum of Δ_h on the compact manifold Y.

If an indicial root λ satisfies $\operatorname{Re}\lambda = \alpha$ then the function u from (2.4) gives an element of the kernel of Δ_g in the infinite end (or if the metric in the end depends on x, a good enough approximate solution) which barely misses being in the space $x^\alpha L^2(M; d\,\mathrm{vol}_g)$ (in the sense that it lies in $x^{\alpha-\varepsilon}L^2(M; d\,\mathrm{vol}_g)$ for any $\varepsilon > 0$) and this is what causes the lack of Fredholm property. As a basic example, for the operator $-\partial_t^2 - 1$ on $\mathbb{R}$, the indicial roots are $\pm i$ and the problematic solutions are $e^{\pm it}$ which barely miss being in $L^2(\mathbb{R})$. On the other hand, for the operator $-\partial_t^2 + 1$ the indicial roots are ± 1 and this operator is Fredholm $H^2(\mathbb{R}) \to L^2(\mathbb{R})$.

The main result presented in these notes is that the situation described in the above paragraph is the only obstruction to the Fredholm property:

Theorem 1. *Let $\overline{M}$ be a compact manifold with boundary $Y = \partial\overline{M}$ and g be a b-metric on $\overline{M}$. Let $\alpha \in \mathbb{R}$ and assume the condition*

$$\text{there are no indicial roots } \lambda \text{ of } \Delta_g \text{ on the line } \{\operatorname{Re}\lambda = \alpha\}. \tag{2.7}$$

Then for any $s \in \mathbb{R}$

$$\Delta_g : x^\alpha H_{\mathrm{b}}^{s+2}(M) \to x^\alpha H_{\mathrm{b}}^s(M) \quad \text{is a Fredholm operator.} \tag{2.8}$$

Note that in the product case $g = \frac{dx^2}{x^2} + h(y, dy)$, the condition (2.7) says that $\alpha^2 \notin \operatorname{Spec}(\Delta_h)$.

2.3. **The estimate and the parametrix.** As in the case of compact manifolds, the Fredholm property follows from an elliptic estimate. However, this time we need improvement in both Sobolev regularity and in behavior at infinity: for $x^\alpha H^s_{\mathrm{b}}(M)$ to embed compactly into $x^{\alpha'} H^{s'}_{\mathrm{b}}(M)$ we need both $s > s'$ and $\alpha > \alpha'$ (e.g. $H^1(\mathbb{R}_t)$ does not embed compactly into $L^2(\mathbb{R}_t)$ but $e^{-|t|}H^1(\mathbb{R})$ does). The elliptic estimate takes the following form: there exists $\alpha' < \alpha$ such that for all s and N

$$\|x^{-\alpha}u\|_{H^{s+2}_{\mathrm{b}}(M)} \leq C\|x^{-\alpha}\Delta_g u\|_{H^s_{\mathrm{b}}(M)} + C\|x^{-\alpha'}u\|_{H^{-N}_{\mathrm{b}}(M)}. \tag{2.9}$$

We remark that the estimate (2.9) with $\alpha' = \alpha$ can be obtained by generalizing the elliptic parametrix construction for compact manifolds (often called the *small calculus* in the context of b-calculus), but as mentioned above this is not enough to get the compact embedding needed for the Fredholm property. To get (2.9) with $\alpha' < \alpha$ requires much more work (see below) and this is also where the condition (2.7) is used. In fact, the proof will need α' and α to satisfy the following condition:

$$\text{there are no indicial roots } \lambda \text{ in the strip } \{\alpha' \leq \operatorname{Re}\lambda \leq \alpha\}. \tag{2.10}$$

To prove the estimate (2.9), we again construct an elliptic parametrix. This time it is a bounded operator

$$Q : x^\alpha H^s_{\mathrm{b}}(M) \to x^\alpha H^{s+2}_{\mathrm{b}}(M)$$

such that

$$\Delta_g Q = I + R \quad \text{where } R : x^{\alpha'} H^{-N}_{\mathrm{b}}(M) \to x^\alpha H^N_{\mathrm{b}}(M). \tag{2.11}$$

As before, we identify Q with its distributional kernel, $\mathbf{Q} \in \mathcal{D}'(M \times M)$, as follows:

$$Qf(z) = \int_M \mathbf{Q}(z, z') f(z')\, d\operatorname{vol}_g(z')$$

and construct $\mathbf{Q}$. Note that since $\overline{M}$ is a manifold with boundary, the product $\overline{M} \times \overline{M}$ is a manifold with corners.

2.4. **A basic example and a preview of the general construction.** To provide some inspiration for the general construction of $\mathbf{Q}$ below, here we look at the cylinder

$$M = \mathbb{R}_t \times \mathbb{S}^1_y, \quad \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}, \quad g = dt^2 + dy^2.$$

We would like to describe the inverse of $\Delta_g : H^2_{\mathrm{b}}(M) \to H^0_{\mathrm{b}}(M)$. This inverse does not exist and the condition (2.7) fails since 0 is an indicial root. We could fix this by instead looking at the operator $\Delta_g + 1$, but this makes the formulas longer. So we instead illegally ignore the zero Fourier mode in the presentation below.

Writing the Fourier mode expansion $f(t, y) = \sum_{k\in\mathbb{Z}} f_k(t)e^{iky}$, we see that $(\Delta_g f)_k = P_k f_k$ where P_k is the operator on $\mathbb{R}$ given by

$$P_k = -\partial_t^2 + k^2.$$

For $k \neq 0$, the inverse of $P_k : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ has the integral kernel

$$\mathbf{R}_k(t,t') = \frac{e^{-|k|\cdot|t-t'|}}{2|k|}.$$

(The reader can check that $(-\partial_t^2 + k^2)\mathbf{R}_k(t,t') = \delta(t-t')$.) Then we sum over Fourier modes $k \neq 0$ and get the 'inverse' to Δ_g with the following integral kernel:

$$\mathbf{Q}(t,t',y,y') = \sum_{k\in\mathbb{Z}\setminus 0} \mathbf{R}_k(t,t')e^{ik(y-y')}.$$

To make our computations simpler (avoiding the logs that come up otherwise), let us just compute the y-derivative of this kernel:

$$\begin{aligned}
\partial_y \mathbf{Q}(t,t',y,y') &= \frac{i}{2}\sum_{k\neq 0} \operatorname{sgn} k\, e^{-|k|\cdot|t-t'|+ik(y-y')} \\
&= \frac{i}{2}\sum_{k\geq 0} e^{-k|t-t'|}\big(e^{ik(y-y')} - e^{-ik(y-y')}\big) \\
&= \frac{i}{2}\left(\frac{1}{1-e^{-|t-t'|+i(y-y')}} - \frac{1}{1-e^{-|t-t'|-i(y-y')}}\right) \\
&= \frac{\sin(y'-y)}{2(\cosh(t-t') - \cos(y-y'))}.
\end{aligned} \tag{2.12}$$

This is a function of $t-t', y, y'$ (in fact, of $t-t'$ and $y-y'$ but this uses our particular choice of $Y = \mathbb{S}^1$):

$$\partial_y \mathbf{Q}(t,t',y,y') = \mathbf{Z}(t-t',y,y') \quad \text{where } \mathbf{Z}(\tau,y,y') = \frac{\sin(y'-y)}{2(\cosh\tau - \cos(y-y'))}.$$

Note that $\mathbf{Z}$ is smooth on $\mathbb{R}_\tau \times \mathbb{T}^2_{y,y'} \setminus \{\tau = 0,\ y = y'\}$, consistent with elliptic regularity. Near the submanifold $\{\tau = 0,\ y = y'\}$ we can use the standard approximations of sin and cos at 0 to get

$$\mathbf{Z}(\tau,y,y') \approx \frac{y'-y}{\tau^2 + (y-y')^2}.$$

This shows that $\mathbf{Z}$ is conormal to $\{\tau = 0,\ y = y'\}$ (as it is homogeneous in $\tau, y-y'$ and smooth except when $\tau = y-y' = 0$).

In the (x,x',y,y') variables, where $x = e^{-t}$ and $x' = e^{-t'}$, we have

$$\partial_y \mathbf{Q}(x,x',y,y') = \mathbf{Z}(\tau,y,y') \quad \text{where } \tau = \log\left(\frac{x'}{x}\right). \tag{2.13}$$

We see that this is not smooth on $\overline{M}\times\overline{M}$ because of the dependence on x'/x. Moreover, the singularities of $\partial_y\mathbf{Q}$ in the interior $M \times M$ are on the diagonal $\{x = x',\ y = y'\}$ which meets the boundary of $\overline{M}\times\overline{M}$ at the corner $Y \times Y = \{x = x' = 0\}$.

To construct the integral kernel of the elliptic parametrix $\mathbf{Q}$ in the next section we pass from $\overline{M}\times\overline{M}$ to a *blown-up space* $[\overline{M}\times\overline{M}; Y\times Y]$ on which x'/x is 'more smooth' and the diagonal meets the boundary of the blown-up space transversally at the front face, which is produced by the blow-up. We next construct $\mathbf{Q}$ in a form which is inspired by the example (2.13). This general construction has several steps:

(1) Construction of $\mathbf{Q}$ near the diagonal $\{x = x',\ y = y'\}$, similar to the elliptic parametrix construction in the compact case (this is known as the small calculus).
(2) Construction of $\mathbf{Q}$ near the front face, using the Fourier transform and the integral kernel of the inverse of the indicial operator $I(\lambda)^{-1}$. One has to make a choice of the contour in the Fourier transform and this is where the indicial roots play a crucial role.
(3) Analysis of $\mathbf{Q}$ near the corners where the front face meets the left and right boundary and extension of it to those two boundaries.

2.5. **The blown-up double space.** Let us first examine the geometry of $\overline{M} \times \overline{M}$ near the boundary. Recall that near the boundary $Y = \partial\overline{M}$, we have local coordinates (x, y) where x is a boundary defining function and $y \in Y$. Upon possibly rescaling, we may further assume that the coordinate patch in a neighborhood of Y is given by $\{x < 1\}$. Then in a neighborhood of the corner $Y \times Y = \{x = x' = 0\} \subset \overline{M} \times \overline{M}$, we can write the coordinates

$$(x, x', y, y') \quad \text{where} \quad (x, x') \subset [0, 1)^2, \quad (y, y') \in Y \times Y \tag{2.14}$$

The Laplacian Δ_g is roughly translation invariant (in the t variable) near the infinite ends of M, so in the compactified picture, Δ_g is *dilation invariant* (in the x variable) near the boundary $\partial\overline{M}$. Therefore, it is useful to introduce "polar coordinates" near the corner by

$$\theta = \frac{x'}{x + x'} \in [0, 1], \qquad r = x + x' \in [0, 1). \tag{2.15}$$

These coordinates are degenerate at $x = x' = 0$, but this can be resolved geometrically by introducing a blowup. We will not give a precise definition of the blowup. Instead, see Figure 2. Essentially, we replace the corner $Y \times Y$ with the hypersurface $[0, 1]_\theta \times Y_y \times Y_{y'}$, called the *front face* (denoted ff). A point in this front face captures the direction at which a point in the corner is approached. The part of the boundary at $\theta = 1$ (or equivalently $x = 0$, $x' > 0$) is called the *left boundary* (denoted lb). The part of the boundary at $\theta = 0$ (or equivalently $x' = 0$, $x > 0$) is called the *right boundary* (denoted rb). The blown up space is denoted by

$$[\overline{M} \times \overline{M}; Y \times Y]. \tag{2.16}$$

The polar coordinates (2.15) give us a description of the blown up space near ff, but in practice, they are a bit inconvenient for computations. Instead, we will consider a neighborhood of ff in three coordinate patches.

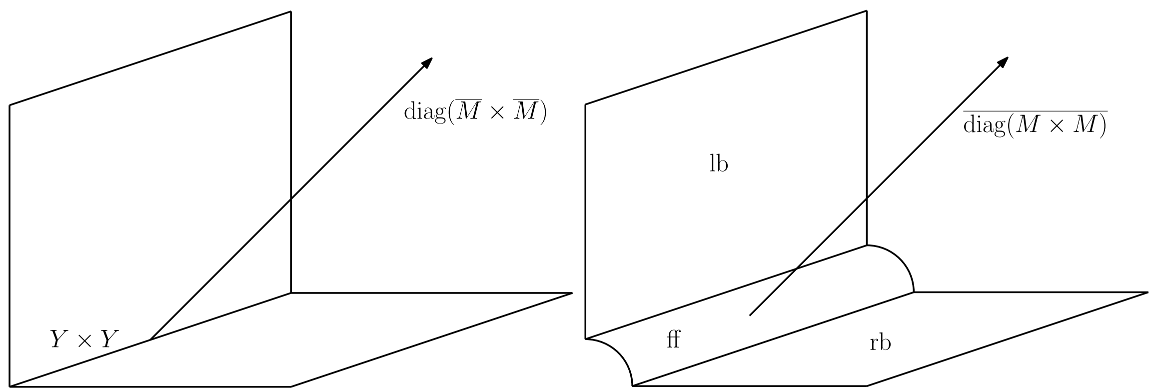

FIGURE 2. **Left:** the space $\overline{M}\times\overline{M}$ with the diagonal and the corner $Y\times Y$ labelled. **Right:** the blown up space $[\overline{M}\times\overline{M};Y\times Y]$ where the corner $Y\times Y$ is replaced by the hypersurface ff. The diagonal is again labelled. The blow up is constructed so that there exists a smooth *blow down map* $\beta:[\overline{M}\times\overline{M};Y\times Y]\to\overline{M}\times\overline{M}$ that acts as the identity on the interior.

Region I: Near $\overline{\mathrm{diag}(M\times M)}\cap \mathrm{ff}$ (where the closure is taken in $[\overline{M}\times\overline{M};Y\times Y]$):

$$\tau := \log\left(\frac{x'}{x}\right) = t-t' \in (-1,1), \quad r = x+x' \in [0,1), \quad y,y'\in Y. \tag{2.17}$$

Region II: Near $\mathrm{rb}\cap\mathrm{ff}$

$$s = e^{\tau} = x'/x \in [0,\frac{1}{2}), \quad x\in[0,1), \quad y,y'\in Y. \tag{2.18}$$

Region III: Near $\mathrm{lb}\cap\mathrm{ff}$:

$$s' = e^{-\tau} = x/x' \in [0,\frac{1}{2}), \quad x'\in[0,1), \quad y,y'\in Y. \tag{2.19}$$

See Figure 3 for an illustration of these coordinates.

Example. Recall the parametrix we computed in (2.12) for the Laplacian on the manifold $\mathbb{R}_t\times\mathbb{S}^1_y$. Let's write $\partial_y\mathbf{Q}$ in the three three coordinate patches near the front face to get an intuition for the behavior of the parametrix in the blown up geometry. First, in Region I, we see that

$$\partial_y\mathbf{Q}(\tau,r,y,y') = \frac{\sin(y'-y)}{2(\cosh(\tau)-\cos(y-y'))}. \tag{2.20}$$

In particular, near the diagonal, that is for $\tau, y-y'\ll 1$, we have

$$\partial_y\mathbf{Q}(\tau,r,y,y') \approx \frac{y'-y}{\tau^2+(y-y')^2},$$

Note that this is independent of r. Since the Laplacian is dilation invariant with respect to the compactified $x=e^{-t}$ coordinates, it is expected that the parametrix should also

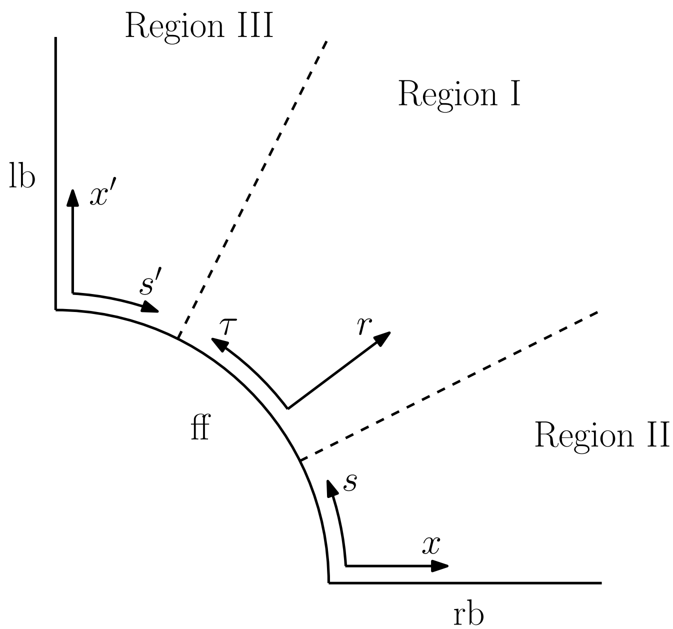

FIGURE 3. Diagram of the coordinate patches defined in (2.17), (2.18), and (2.19).

dilation invariant. More generally, it is crucial that $\mathbf{Q}(\tau, r, y, y')$ is smooth in r up to $r = 0$ as a family of conormal distributions in (τ, y, y') with respect to $\{y = y'\}$.

In Region II, we have

$$\begin{aligned} \partial_y \mathbf{Q}(\tau, r, y, y') &= \frac{\sin(y' - y)}{s + s^{-1} - 2\cos(y - y')} \\ &\sim \sin(y' - y)s + 2\sin(y' - y)\cos(y' - y)s^2 + \cdots \end{aligned} \tag{2.21}$$

as $s \to 0$. In particular, there exists an expansion in powers of s as we approach rb as a family of smooth functions in (x, y, y'), and if we were to continue the series, the remainder can be made to lie in $\mathcal{O}_{C^\infty([0,1)_x \times Y_y \times Y_{y'})}(s^N)$. We say that $\partial \mathbf{Q}$ is *polyhomogeneous* near ff $\cap$ rb with respect to rb. Similarly, in Region III, we can see that $\partial \mathbf{Q}$ is *polyhomogeneous* near ff $\cap$ lb with respect to lb.

Remark. In this example, $\mathbf{Q}$ was actually smooth up to all the faces. But in general Q is smooth up to ff and polyhomogeneous up the other 2 faces (i.e. has expansions in powers of s, but not necessarily with integer coefficients; the coefficients depend on the indicial roots, see the full parametrix construction in §3.3).

3. Construction of the parametrix

3.1. **Reduction to front face.** For simplicity, let us first work near an exact cylindrical end. This means that the Laplacian takes the simple form

$$\Delta_g = -\partial_t^2 + \Delta_{h,y} = -\partial_t^2 + \Delta_y.$$

Here, Δ_h is the Laplacian on (Y, h), but we will omit the the metric h in the notation and instead write Δ_y to emphasize that it hits only the y variable. In view of (2.11), we see that the Schwartz kernel $\mathbf{Q}$ of the full parametrix should satisfy

$$\Delta_g \mathbf{Q}(\tau, r, y, y') = \delta(\tau)\delta(y - y') + \mathbf{R}(\tau, r, y, y') \tag{3.1}$$

near the corner $Y \times Y \subset \overline{M} \times \overline{M}$, where $\mathbf{R}$ is the Schwartz kernel of the remainder term in (2.11).

In the cylindrical end, Δ_g is translation invariant with respect to t, thus dilation invariant with respect to $x = e^{-t}$, so we should try to look for a parametrix that is also dilation invariant. What this means for the Schwartz kernel $\mathbf{Q}$ near $Y \times Y$ is that it should only depend on (τ, y, y'). That is, it is killed by the vector field $-2r\partial_r$ which is same as $\partial_t + \partial_{t'}$. So we can actually just ignore the r variable near $\{r = 0\}$.

By the same reasoning, we can also expect the remainder $\mathbf{R}$ to be dilation invariant. Furthermore, the remainder term must improve in both decay and regularity. Therefore, in our exact cylindrical end case, we should require that $\mathbf{R}$ is smooth in the interior $M \times M \subset [\overline{M} \times \overline{M}; Y \times Y]$, and vanishes in a small neighborhood of the front face:

$$\mathbf{R} \in C^\infty(M \times M), \quad \mathbf{R}(\tau, r, y, y') = 0 \quad \text{for} \quad r \in [0, 1). \tag{3.2}$$

Then restricting (3.1) to a neighborhood of the front face, we arrive at the equation

$$(-\partial_\tau^2 + \Delta_y)\mathbf{Q}(\tau, y, y') = \delta(\tau)\delta(y - y'). \tag{3.3}$$

Remark. In more general cases, we might not have exact dilation invariance, so we cannot just ignore the r variable. However, we only need the remainder $\mathbf{R}$ to vanish *at* $r = 0$ to gain decay. Observe that

$$\begin{aligned}\Delta_g \mathbf{Q}(\tau, r, y, y') &= (-\partial_\tau^2 + \Delta_y)\mathbf{Q}(\tau, r, y, y') \\ &\quad + \left(\frac{2}{e^\tau + 1}\partial_\tau - \frac{e^\tau}{(e^\tau + 1)^2} - \frac{1}{(e^\tau + 1)^2} r\partial_r\right) r\partial_r \mathbf{Q}(\tau, r, y, y'),\end{aligned} \tag{3.4}$$

so the second term vanishes at $r = 0$. Therefore, if we restrict to $r = 0$, we in fact end up with the same equation as (3.3).

Geometrically, we have just reduced the problem of finding a parametrix that improves decay near infinity to solving an exact problem at the front face of the blowup. To motivate the formula we will write for $\mathbf{Q}$, let us try to formally solve the equation by taking the Fourier transform in τ. Let σ be the Fourier dual variable to τ, then the formal Fourier transform of (3.3) is given by

$$(\sigma^2 + \Delta_y)\widehat{\mathbf{Q}}(\sigma, y, y') = \delta(y - y').$$

Therefore, we are motivated to define

$$\mathbf{Q}_a(\tau, y, y') = \mathbf{Q}(\tau, y, y') := \frac{1}{2\pi}\int_{\mathbb{R}-ia} e^{i\sigma\tau}(\sigma^2 + \Delta_y)^{-1}\, d\sigma \tag{3.5}$$

for $a \in \mathbb{R}$, where by a slight abuse of notation, we write $(\sigma^2 + \Delta_y)^{-1}$ for both the operator and its Schwartz kernel. Note in particular that if $\sigma^2 + \Delta_y$ is invertible for all $\sigma \in \mathbb{R} - ia$, then the integral makes perfect sense, since it follows from the spectral theorem that

$$\|(\sigma^2 + \Delta_y)^{-1}\|_{L^2(Y)\to L^2(Y)} \leq C_a|\sigma|^{-2}, \quad \sigma \in \mathbb{R} - ia.$$

Furthermore, with $\mathbf{Q}$ defined by (3.5), one can make sense of the Fourier transform of $\mathbf{Q}$ and its derivatives, so $\mathbf{Q}$ will indeed satisfy (3.3) near the front face.

3.2. **A 1-D example.** In the end, we wish to understand the inverse Fourier transform in (3.5). Let us first work out a toy example in one dimension. Consider the operator $-\partial_t^2 + 1$ on $\mathbb{R}$. Then a Schwartz kernel of a parametrix near $t = t' = \infty$ in $\tau = t - t'$ coordinates is given by

$$\mathbf{Q}(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}-ia} e^{i\sigma\tau}(\sigma^2 + 1)^{-1}\, d\sigma.$$

This integral converges absolutely when $a \neq \pm 1$. Let us consider the behavior as $\tau \to \pm\infty$.

We will compute $\mathbf{Q}$ in both τ-coordinates and coordinates $s = e^\tau$ from (2.18). The s coordinates will illustrate the behavior of $\mathbf{Q}$ to rb (as well as to lb if we consider $s \to \infty$). Note that $\mathbf{Q}$ depends on the choice of a we make. We have three cases, which we will denote by $\mathbf{Q}_j$ for $j = -1, 0, 1$.

Case 1: $a > 1$**.** For $\tau > 0$, we can deform the contour $\mathbb{R} - ia$ to the contour $\mathbb{R} + iN$ as $N \to +\infty$ and find that

$$\mathbf{Q}_1(\tau) = \frac{e^{-\tau} - e^\tau}{2}, \qquad \tau > 0.$$

Deforming the contour to $\mathbb{R} - iN$ as $N \to -\infty$ for $\tau < 0$, we have

$$\mathbf{Q}_1(\tau) = 0, \qquad \tau < 0.$$

In s coordinates, we see that

$$\mathbf{Q}_1(s) = \begin{cases} \frac{1}{2}(e^{-\tau} - e^\tau), & s > 1 \\ 0, & 0 < s < 1. \end{cases}$$

Case 2: $a < -1$. By similar contour deformations, we find that

$$\mathbf{Q}_{-1}(\tau) = \begin{cases} 0, & \tau > 0, \\ \frac{1}{2}(e^{\tau} - e^{-\tau}), & \tau < 0, \end{cases}$$

and

$$\mathbf{Q}_{-1}(s) = \begin{cases} 0, & s > 1, \\ \frac{1}{2}(s - s^{-1}), & 0 < s < 1. \end{cases}$$

Case 3: $|a| < 1$. Finally, we have

$$\mathbf{Q}_0(\tau) = \frac{1}{2}e^{-|\tau|},$$

and

$$\mathbf{Q}_0(s) = \begin{cases} \frac{1}{2}s^{-1}, & s > 1, \\ \frac{1}{2}s, & s < 1. \end{cases}$$

Let us analyze the mapping properties of Q_j near $x = 0$ (that is $t = \infty$), where Q_j is the parametrix with Schwartz kernel $\mathbf{Q}_j$. We first work in the t, t' coordinates since Q_j are simply convolution operators in these coordinates. Let us apply Q_0 to functions of the form $u(t) = e^{-\ell t}$ for $t \geq 0$ and 0 for $t < 0$. Then

$$Q_0 u(t) = \int_0^\infty e^{-|t-t'|} e^{-\ell t} \, dt \tag{3.6}$$

In order for the integral to converge, note that we must have $\ell > -1$. If $\ell > 1$, then $e^{-\ell t}$ decays much faster than the convolution kernel, so then as $t \to \infty$, $Q_0 u(t) \simeq e^{-t}$. On the other hand, if $-1 < \ell < 1$, the convolution kernel decays much faster than $e^{-\ell t}$, so $Q_0 u(t) \simeq e^{-\ell t}$ as $t \to \infty$.

To emphasize the geometric aspects, we also consider the same mapping properties in the $x = e^{-t}$ coordinates. We consider how Q_j acts on functions that are locally invariant by the generator of dilations in x near $x = 0$. Fix a cutoff $\chi \in C_c^\infty([0, \infty))$ such that $\chi(x) = 1$ near 0, and consider $u(x) := \chi(x)x^\ell$. Then

$$Q_0 u(x) = \int_0^\infty \mathbf{Q}_0(x, x')\chi(x')x'^\ell \frac{dx'}{x'} = x^\ell \int_0^\infty \mathbf{Q}_0(s)\chi(xs)s^\ell \frac{ds}{s} \tag{3.7}$$

Since $\mathbf{Q}_0(s) = \frac{1}{2}s$, near 0, we see that in order for the integral to converge, we must have

$$\ell > -1. \tag{3.8}$$

In other words, Q_0 can only be applied to functions that decays faster than x^{-1} near 0. Now let's look at the decay of the output. If $\ell > -1$, then as $x \to 0$,

$$Q_0 u(x) \simeq x^\ell \int_0^{x^{-1}} \mathbf{Q}_0(s)s^\ell \frac{ds}{s} \simeq x^\ell + x^\ell \int_2^\infty s^{-1}s^\ell \frac{ds}{s} \simeq x^\ell + x. \tag{3.9}$$

Therefore, we see that if $-1 < \ell < 1$, then $Q_0 u(x) \simeq x^\ell$ near 0, and if $\ell \geq 1$, then $Q_0 u(x) \simeq x$ near 0.

The mapping properties we just found can be viewed in a more geometric way. We can view $\mathbf{Q}_0$ as a function on $[0,1)_x \times [0,1)_{x'}$. We can resolve the singularity at 0 by blowing up the corner. Then $\mathbf{Q}_0$ conormal to the diagonal smoothly up to ff, and vanishes to first order to both lb and rb. More generally, we have the following proposition.

Proposition 3.1 (rough statement). *Consider $\mathbf{Q} \in C^\infty((0,1)^2)$ such that in the blown up space $[[0,1)^2; \{(0,0)\}]$, $\mathbf{Q}$ is smooth up to ff. Furthermore, assume that near ff $\cap$ lb (using Region II (2.18) coordinates), there exists a discrete set $\mathcal{E}_{\mathrm{rb}} \subset \mathbb{R}$ that is bounded below such that*

$$\mathbf{Q}(x,s) \sim \sum_{\lambda \in \mathcal{E}_{\mathrm{rb}}} s^\lambda v_s(x), \quad v_s \in C^\infty([0,1)), \tag{3.10}$$

and near ff $\cap$ lb (using Region III (2.19) coordinates), there exists a discrete set $\mathcal{E}_{\mathrm{lb}} \subset \mathbb{R}$ that is bounded below such that

$$\mathbf{Q}(x,s) \sim \sum_{\lambda \in \mathcal{E}_{\mathrm{lb}}} (s')^\lambda w_s(x'), \quad w_s \in C^\infty([0,1)). \tag{3.11}$$

Let $E_\bullet := \min \mathcal{E}_\bullet$ where $\bullet = \mathrm{lb}, \mathrm{rb}$. Then the operator Q with Schwartz kernel $\mathbf{Q}$ has the mapping property

$$Q : x^\alpha H^s_{\mathrm{b}}([0,1)) \to x^\beta H^{s+N}_{\mathrm{b}}([0,1)) \tag{3.12}$$

if $\beta \leq \alpha$, $\alpha > -E_{\mathrm{rb}}$, and $\beta < E_{\mathrm{lb}}$, for all $s, N \in \mathbb{R}$.

Remark. With asymptotic expansions (3.10) and (3.11), we are requiring that $\mathbf{Q}$ is polyhomogeneous conormal to rb and lb. More general polyhomogeneous conormal distributions may contain logs and complex powers in their expansion, and we will inevitably encounter these. However, the proposition remains true with little change, so for the sake of exposition, we keep the expansions simple and refer the reader to [Mel93] for a more precise treatement. E_{rb} and E_{lb} are the order of vanishing of $\mathbf{Q}$ to rb and lb respectively. We see from the above proposition that the permissible range of weights for the mapping property is determined by the order of vanishing of the Schwartz kernel to the left and right boundary of the blown up space. For simplicity, we assumed above that $\mathbf{Q}$ is smooth on the interior of $[0,1)^2$, so from a differential point of view, these operators are smoothing. However, we can also allow $\mathbf{Q}(r,\tau)$ to be an r-dependent family of conormal distributions in τ with respect to $\{\tau = 0\}$ in the Region I (2.17) coordinates. In this case Proposition 3.1 holds with the expected adjustments to the differential order in the Sobolev spaces. A consequence of the Proposition 3.1 applied to the operators Q_j computed in this subsection is that

$$Q_j : x^\alpha H^s_{\mathrm{b}}([0,1)) \to x^\alpha H^{s+2}_{\mathrm{b}}([0,1)) \quad \text{for } \begin{cases} |\alpha| < 1,\ j = 0 \\ \alpha > 1,\ j = 1 \\ \alpha < -1,\ j = -1. \end{cases} \tag{3.13}$$

In anticipation of the full parametrix remainder, we also consider the Schwartz kernels that vanish to infinite order up to the front face. We have the following mapping property, which follows from similar analysis.

Proposition 3.2 (rough statement)**.** *Let* $\mathbf{Q}$ *be as in Proposition 3.1. Further assume that* $\mathbf{Q}$ *vanishes to infinite order up to the front face. More explicitely, this means that the in Regions II and III, the asymptotic expansions* (3.10) *and* (3.11) *further satisfy*

$$v_\lambda(x), w_\lambda(x) = \mathcal{O}(x^\infty), \tag{3.14}$$

and in Region I, $\mathcal{Q}(r,\tau) = \mathcal{O}_{C^\infty((-1,1)_\tau)}(r^\infty)$*. Then the operator* Q *with Schwartz kernel* $\mathbf{Q}$ *has the mapping property*

$$Q : x^\alpha H^s_{\mathrm{b}}([0,1)) \to x^\beta H^{s+N}_{\mathrm{b}}([0,1)) \tag{3.15}$$

if $\alpha > -E_{\mathrm{rb}}$*, and* $\beta < E_{\mathrm{lb}}$*, for all* $s, N \in \mathbb{R}$*.*

In particular, note that if $\mathbf{Q}$ vanishes to infinite order up to ff, the condition $\beta \le \alpha$ is no longer required, which means that the operator Q can *improve* decay.

3.3. **Full parametrix asymptotics.** Now we return to (3.5). First, recall that $(\sigma^2 + \Delta_y)^{-1}$ is a meromorphic family of operators. define

$$\mathrm{Spec}_{\mathrm{b}}(\Delta_g) := \{\sigma \in \mathbb{C} \mid \sigma^2 + \Delta_y \text{ is not invertible}\}.$$

Recall that we defined indicial roots as the roots of the indicial family (2.6). It is no coincidence that $\sigma \in \mathrm{Spec}_{\mathrm{b}}(\Delta_g)$ if and only if $\lambda = i\sigma$ is an indicial root. Note that by taking the Fourier transform of $\Delta_g \mathbf{Q}$ in (3.5), what we are essentially doing is studying how Δ_g acts on functions of the form $e^{-i\sigma t} = x^{i\sigma}$, which is precisely how we arrived at the indicial family (2.6).

Remark. In most of the literature (see, for instance, [Mel93]), the indicial roots of Δ_g refer to the set $\mathrm{Spec}_{\mathrm{b}}(\Delta_g)$. This convention is designed so that the indicial family is simply given by the Fourier transform of the Schwartz kernel restricted to ff using the τ coordinates. For the sake of exposition, we defined the indicial family as (2.6) to emphasize the dilation invariant structure of b-operators.

Clearly, $z \in \mathbb{C}$ is a pole of $(\sigma^2 + \Delta_y)^{-1}$ if and only if $z \in \mathrm{Spec}_{\mathrm{b}}(\Delta_g)$, and for σ in a sufficiently small neighborhood of $z \in \mathrm{Spec}_{\mathrm{b}}(\Delta_g)$, we have

$$(\sigma^2 + \Delta_y)^{-1} = \frac{\Pi_{-z^2}}{\sigma^2 - z^2} + R_{\mathrm{hol}}(\sigma) \tag{3.16}$$

where R_{hol} is a holomorphic family of bounded operators on L^2, and Π_{-z^2} is the L^2-orthogonal projection onto the $(-z^2)$-eigenspace of Δ_y. We see that in the case of an exact cylindrical end, the poles are in fact simple except at $\sigma = 0$, which has order 2. Furthermore, it follows from elliptic regularity for compact manifolds that Π_{-z^2} is a smoothing operator, so the Schwartz kernel is smooth. Most importantly, all the poles lie on the imaginary axis, which means that there are only finitely many poles in any finite strip $\mathbb{R} + i[\alpha, \beta]$, $-\infty < \alpha < \beta < \infty$.

We first consider the behavior of $\mathbf{Q}$ in Region I of Figure 3. For $\tau \neq 0$, we can integrate by parts using $\frac{1}{i\tau}\partial_\sigma$ and find that

$$\mathbf{Q}(\tau, y, y') = \frac{1}{2\pi}\int_{\mathbb{R}-ia} e^{i\sigma\tau}(\sigma^2 + \Delta_y)^{-1}\, d\sigma = \frac{1}{2\pi}\int_{\mathbb{R}-ia} \frac{1}{\tau} e^{i\sigma\tau} D_\sigma(\sigma^2 + \Delta_y)^{-1}\, d\sigma. \quad (3.17)$$

Differentiating the resolvent in σ, we see that

$$D_\sigma(\sigma^2 + 1)^{-1} = 2i\sigma(\sigma^2 + \Delta_y)^{-1}(\sigma^2 + \Delta_y)^{-1}. \quad (3.18)$$

For $\sigma \in \mathbb{R} + ia$, there exists $C > 0$ such that

$$\|(\sigma^2 + \Delta_y)^{-1}\|_{L^2 \to L^2} \le C|\sigma|^{-2}, \qquad \|(\sigma^2 + \Delta_y)^{-1}\|_{H^s \to H^{s+2}} \le C. \quad (3.19)$$

Therefore, it follows from (3.17) and (3.18) that for $\tau \neq 0$,

$$\|\mathbf{Q}(\tau, \bullet, \bullet)\|_{L^2(Y) \to H^2(Y)} < \infty.$$

Iteratively integrating by parts, we find that $\mathbf{Q}(\tau, \bullet, \bullet)$ is in fact the Schwartz kernel of a smoothing operator on Y for $\tau \neq 0$, depending smoothly in τ for $\tau \neq 0$. Therefore $\mathbf{Q}(\tau, y, y')$ is in fact smooth away from $\tau = 0$.

Since the Schwartz kernel of $(\sigma^2 + \Delta_y)^{-1}$ is smooth away from $\{y = y'\}$, we also see that $\mathbf{Q}(\tau, y, y')$ is smooth away from $y = y'$. Therefore, the only singularity is on the diagonal $\{y = y',\ \tau = 0\}$. We state without proof that $\mathbf{Q}(\tau, y, y')$ is conormal to the diagonal. Roughly speaking, this is because $(\sigma^2 + \Delta_y)^{-1}$ is a symbol in σ, valued in distributions in (y, y') conormal to $\{y = y'\}$. Hence the inverse Fourier transform is conormal to $\{\tau = 0,\ y = y'\}$.

To analyze the behavior of $\mathbf{Q}$ near rb, it is again convenient to use the coordinates $s = e^\tau$ so that

$$\mathbf{Q}(s) = \frac{1}{2\pi}\int_{\mathbb{R}-ia} s^{i\sigma}(\sigma^2 + \Delta_y)^{-1}\, d\sigma. \quad (3.20)$$

This is Region II in Figure 3, and we are interested in the behavior as $s \to 0$. By just taking the integral, we see that $\mathbf{Q}(s)$ is a pseudodifferential operator on Y and satisfies the bound

$$\|\mathbf{Q}(s)\|_{H^\ell(Y) \to H^{\ell+2}(Y)} \lesssim s^a \quad \text{as} \quad s \to 0 \quad (3.21)$$

Deforming the contour down, we see that

$$\mathbf{Q}(s) = \sum_{\substack{z\in \mathrm{Spec}_{\mathrm{b}}(\Delta_g)\\ -a'<\operatorname{Im} z<-a\\ k\leq \mathrm{ord}(z)}} s^{iz}(\log s)^k A_{z,k} + \int_{\mathbb{R}-ia'} s^{i\sigma}(\sigma^2+\Delta_y)^{-1}\,d\sigma, \quad s<1 \tag{3.22}$$

where $A_{z,k}$ are smoothing operators (since they are a multiple of projection operators onto an eigenspace following (3.16)), and $\mathrm{ord}(z)$ denotes the order the pole of $(\sigma^2+\Delta_y)^{-1}$ at $\sigma = z$. In our case, $\mathrm{ord}(z) = 1$ when $z\neq 0$. Furthermore, the remainder term satisfies the bound

$$\left\|\int_{\mathbb{R}-ia'} s^{i\sigma}(\sigma^2+\Delta_Y)^{-1}\,d\sigma\right\|_{H^\ell(Y)\to H^{\ell+2}(Y)} \lesssim s^{a'}, \tag{3.23}$$

which in particular decays faster than s^a as $s\to 0$ since $a'>a$. This expansion therefore gives precise asymptotics towards rb.

Similarly, for the behavior as of $\mathbf{Q}$ near lb, we can use the coordinate $s' = e^{-\tau}$. This is Region III in Figure 3. As $s'\to 0$, we can deform the contour up and see that

$$\mathbf{Q}(s') = \sum_{\substack{z\in \mathrm{Spec}_{\mathrm{b}}(\Delta_g)\\ -a<\operatorname{Im} z<-a'\\ k\leq \mathrm{ord}(z)}} (s')^{-iz}(\log s')^k A_{z,k} + \int_{\mathbb{R}-ia'} (s')^{-i\sigma}(\sigma^2+\Delta_Y)^{-1}\,d\sigma \quad s'<1, \tag{3.24}$$

and the remainder satisfies the bound

$$\left\|\int_{\mathbb{R}-ia'} s^{i\sigma}(\sigma^2+\Delta_Y)^{-1}\,d\sigma\right\|_{H^\ell(Y)\to H^{\ell+2}(Y)} \lesssim (s')^{-a'}, \tag{3.25}$$

as $s'\to 0$. Since $a'<a$, this remainder decays faster than (3.21) as $s'\to 0$. The expansion (3.24) is therefore giving us precise asymptotics towards lb.

Summarizing our analysis of the full parametrix and applying Proposition 3.1, we arrive at the following result for the parametrix cutoff near ff.

Proposition 3.3. *Let $a\in\mathbb{R}$ be such that $(\mathbb{R}+ia)\cap \mathrm{Spec}_{\mathrm{b}}(\Delta_g)=\emptyset$. Let*

$$E_{\mathrm{rb}} := -\max\{\operatorname{Im} z : z\in \mathrm{Spec}_{\mathrm{b}}(\Delta_g),\ \operatorname{Im} z<-a\},$$
$$E_{\mathrm{lb}} := \min\{\operatorname{Im} z : z\in \mathrm{Spec}_{\mathrm{b}}(\Delta_g),\ \operatorname{Im} z>-a\}.$$

Fix $\chi\in C_{\mathrm{c}}^\infty([0,1))$ be such that $\chi(r)=1$ near $r=0$ and $\chi(r)$ as a function on M is supported in the exact cylindrical end. Define the cutoff parametrix

$$\widetilde{\mathbf{Q}}(\tau,r,y,y') := \frac{1}{2\pi}\chi(r)\int_{\mathbb{R}-ia} e^{i\sigma\tau}(\sigma^2+\Delta_y)^{-1}\,d\sigma.$$

Then

$$\Delta_g\widetilde{\mathbf{Q}}(\tau,r,y,y') = \delta(\tau)\delta(y-y') \quad \text{in a small neighborhood of ff},$$

and

$$\widetilde{Q} : x^\alpha H_{\mathrm{b}}^\ell(M)\to x^\beta H_{\mathrm{b}}^{\ell+2}(M) \tag{3.26}$$

is bounded for all $\beta \le \alpha$, $\alpha > -E_{\mathrm{rb}}$ and $\beta < E_{\mathrm{lb}}$, where $\widetilde{Q}$ is the operator with kernel $\widetilde{\mathbf{Q}}$.

Remark. Note that E_{rb} and E_{lb} are the order of vanishing to rb and lb respectively, meaning they are the leading order in the polyhomogeneous expansions (3.22) and (3.24) respectively. Note that

$$E_{\mathrm{lb}} > -E_{\mathrm{rb}},$$

so a simple case of the mapping property (3.26) is by taking $\alpha = \beta = a$.

Now a global parametrix Q can be constructed by gluing $\widetilde{Q}$ together with a small parametrix Q_s:

$$\mathbf{Q} := \widetilde{\mathbf{Q}} + (1-\chi)\mathbf{Q}_s \tag{3.27}$$

where $\chi = \chi(r)$ is the same cutoff as in Proposition 3.3. The small parametrix is constructed so that $\mathbf{Q}_s$ vanishes to infinite order near lb and rb, and can also be constructed so that

$$\widetilde{\mathbf{Q}}(\tau, r, y, y') = \mathbf{Q}_s(\tau, r, y, y') \quad \text{for } r \in \operatorname{supp}\chi',\ r \text{ near } 0, \text{ and } y \text{ and } y'. \tag{3.28}$$

The one can check that

$$\Delta_g Q = \mathrm{Id} + R \tag{3.29}$$

where the remainder R is has Schwartz kernel $R \in C^\infty(M \times M)$ that vanishes to infinite order near ff, and has order of vanishing E_{lb} and E_{rb} to lb and rb respectively. The vanishing to ff is precisely what gives us the desired Fredholm property. In view of Proposition 3.2, one finds that

$$R : x^\alpha H_{\mathrm{b}}^{\ell}(M) \to x^\beta H_{\mathrm{b}}^{\ell+N}(M) \tag{3.30}$$

for any $\alpha > -E_{\mathrm{rb}}$ and $\beta < E_{\mathrm{lb}}$. Again, we emphasize that that we obtain extra decay here since $\mathbf{R}$ vanishes to infinite order to ff, so we do not need to require $\beta \le \alpha$ unlike the mapping properties for $\widetilde{Q}$ from Proposition 3.3.

4. Notes

Here is a brief overview of some literature related to the topic of these notes:

- For the case of a compact manifold, there are many treatments of elliptic parametrix and the Fredholm property, such as Hörmander [Hör07, Theorem 19.2.1]. Operators with distributional kernels conormal to the diagonal are commonly known as *pseudodifferential operators*; this is a class which includes differential operators, singular integral operators, and elliptic parametrices. Our presentation in §1 largely follows the first author's lecture notes [Dya22].

- b-calculus was developed by Richard Melrose in the 1980s. The earliest source is the unpublished MSRI preprint of Melrose–Mendoza [MM83]. The classical reference is the 'green book' by Melrose [Mel93] which applies b-calculus to index theory. In particular, our Theorem 1 is a special case of [MM83, Theorem 6.17] and [Mel93, Theorem 5.40].
- b-calculus is a particular case of *geometric scattering theory* which uses tools such as conormal distributions, manifolds with corners, and blow-ups to construct parametrices for a wide range of geometries at infinity (including manifolds with cylindrical, asymptotically hyperbolic, or asymptotically Euclidean ends). Roughly speaking, the idea is to make the geometry complicated and the analysis easier. The notes by Grieser [Gri01] contain a lot of the geometric motivation and tools in b-calculus, as do the earlier notes by Melrose [Mel96]. More recent applications of geometric scattering theory to problems in geometry include [HHM04], [GH09], [KR24b], [KR24a], [DM18].
- The notes by Hintz [Hin23] are another great introduction to geometric scattering theory. In particular, [Hin23, §3] gives a proof of the elliptic estimate (2.9) without explicitly constructing an elliptic parametrix, but instead combines a b-regularity estimate with indicial analysis.

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