Fractal Uncertainty Principle and Quantum Chaos

Semyon Dyatlov (MIT)

July 12, 2022
This talk presents several recent results in quantum chaos, including
- lower bounds on mass of eigenfunctions and semiclassical measures
- observability for Schrödinger equations
- spectral gaps and exponential wave decay for open systems

The proofs are based on the following ideas:
- Use the classical/quantum correspondence to its limit
- Apply the fractal uncertainty principle (FUP): No function can be localized in both position and frequency near a fractal set
- General FUP is only known in dimension 1, and most (but not all) results are in the setting of negatively curved surfaces
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Special thanks to mentors and collaborators

B. Delarue  G. Dyatlov  V. Dyatlov  F. Faure  J. Galkowski  S. Ghosh
C. Guillarmou  M. Jézéquel  L. Jin  R. Melrose  S. Nonnenmacher  G. Paternain
J. Wang  R. Ward  A. Waters  T. Weich  J. Zahl  M. Zworski
Control of eigenfunctions

- \((M, g)\) compact negatively curved surface
- Geodesic flow on \(M\) is a standard model of classical chaos
- Eigenfunctions of the Laplacian \(-\Delta_g\) studied by quantum chaos

\[
(-\Delta_g - \lambda^2) u = 0, \quad \|u\|_{L^2} = 1
\]

**Theorem 1**

Let \(\Omega \subset M\) be an arbitrary nonempty open set. Then

\[
\|u\|_{L^2(\Omega)} \geq c > 0
\]

where \(c\) depends on \(M, \Omega\) but not on \(\lambda\)

Constant curvature: D–Jin '18, using D–Zahl '16 and Bourgain–D '18
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For bounded $\lambda$ the estimate follows from unique continuation principle

The new result is in the high frequency limit $\lambda \to \infty$
An illustration

Picture on the right courtesy of Alex Strohmaier, using *Strohmaier–Uski ’12*

Disk (Dirichlet b.c.)
Whitespace in the middle

Hyperbolic surface
No whitespace
Applications to PDE

**Theorem 2 [Jin ’18, D–Jin–Nonnenmacher ’22]**

Let \((M, g)\) be a compact negatively curved surface and \(\Omega \subset M\) nonempty open. Then \(\forall T > 0 \exists C > 0\): any \(u(t, x)\) solving the Schrödinger equation

\[
(i\partial_t + \Delta_g)u(t, x) = 0, \quad u(0, x) = u_0(x)
\]

satisfies the observability estimate

\[
\|u_0\|_{L^2(M)}^2 \leq C \int_0^T \int_{\Omega} |u(t, x)|^2 \, d\text{vol}_g(x) \, dt.
\]

Previously known only for flat tori: Jaffard ’90, Haraux ’89, Komornik ’92, Anantharaman–Macià ’10, Burq–Zworski ’12,’17, Bourgain–B–Z ’13

Another application is to exponential energy decay for solutions to the damped wave equation: Jin ’20, D–Jin–Nonnenmacher ’22
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Semiclassical measures I

- Stronger version of Theorem 1: localization in position and frequency
- Use semiclassical quantization $\text{Op}_h(a) = a(x, -ih\partial_x)$ where $a(x, \xi) \in C_c^\infty(T^*M)$ and $h = \lambda^{-1}$ (here $(-\Delta_g - \lambda^2)u = 0$)
- If $(-\Delta_g - \lambda_j^2)u_j = 0$ and $\lambda_j \to \infty$, we say $u_j$ converges semiclassically to a measure $\mu$ on the cotangent bundle $T^*M$ if
  $$\langle \text{Op}_{h_j}(a)u_j, u_j \rangle_{L^2(M)} \to \int_{T^*M} a \, d\mu \quad \text{for all} \quad a \in C_c^\infty(T^*M)$$
- The pushforward $\pi_*\mu$, $\pi : T^*M \to M$, is the weak limit of the probability measures $|u_j|^2 \, d\operatorname{vol}_g$

Properties of semiclassical measures

- $\mu$ is a probability measure
- $\operatorname{supp} \mu \subset S^*M = \{(x, \xi) \in T^*M : |\xi|_g = 1\}$
- $\mu$ is invariant under the geodesic flow $\varphi^t : S^*M \to S^*M$
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Theorem 3 [D–Jin ’18, D–Jin–Nonnenmacher ’22]

Let \((M, g)\) be a compact negatively curved surface and \(\mu\) be a semiclassical measure associated to a sequence of eigenfunctions. Then \(\text{supp } \mu = S^*M\).

Previous results

- **Quantum Ergodicity (QE):** if \(\varphi^t\) is ergodic then a density 1 sequence of \(u_j\)'s converges to the Liouville measure \(\mu_L\). [Shnirelman ’74, Zelditch ’87, Colin de Verdière ’85, Zelditch–Zworski ’96]

- **CdV ’85:** conjecture that in \(K < 0\) (negative sectional curvature), \(\mu\) cannot be the delta measure on a closed geodesic

- **Rudnick–Sarnak ’94:** QUE conjecture that in \(K < 0\), \(\mu = \mu_L\)

- **Lindenstrauss ’06:** proved QUE for arithmetic hyperbolic surfaces

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Main tool: fractal uncertainty principle (FUP)

No function can be localized in both position and frequency near a fractal set

Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is $\nu$-porous up to scale $h$ if for each interval $I \subset \mathbb{R}$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu |I|$, $J \cap X = \emptyset$.

Theorem 4 [Bourgain–D ‘18]

Assume that $X, Y \subset \mathbb{R}$ are $\nu$-porous up to scale $h \ll 1$. Then $\exists \beta, C > 0$ depending only on $\nu$ such that for all $f \in L^2(\mathbb{R})$

$$\text{supp} \hat{f} \subset h^{-1} Y \quad \Rightarrow \quad \|f\|_{L^2(X)} \leq Ch^{\beta} \|f\|_{L^2(\mathbb{R})}.$$
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Example: mid-third Cantor set $C \subset [0, 1]$ is $\frac{1}{6}$-porous on scales 0 to 1.

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A bit about proof of Theorem 1

- Assume that \((-\Delta_g - \lambda^2)u = 0, \|u\|_{L^2(M)} = 1, \lambda \gg 1,\) and \(\|u\|_{L^2(\Omega)} \ll 1\) for some fixed nonempty open \(\Omega \subset M\)

- Using semiclassical quantization, can study ‘localization’ of \(u\) in the position-frequency space \(T^*M\) (up to a limit given by uncertainty principle)

- Using microlocal analysis, we see that this ‘localization’ is invariant under the geodesic flow \(\varphi^t\) (again, up to a certain point)

- From here we see that \(u\) is localized close to each of the two sets

\[\Gamma_{\pm} := \{(x, \xi) \in S^*M \mid \forall t \geq 0, \varphi^{\mp t}(x, \xi) \notin \Omega\}\]

of geodesics which do not pass \(\Omega\) in past/future time

- The sets \(\Gamma_{\pm}\) have porous structure in certain directions (see next slide)

- Fractal uncertainty principle (Theorem 4) implies that no function \(u\) can be localized close to both \(\Gamma_+\) and \(\Gamma_-\), giving a contradiction
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of geodesics which do not pass \(\Omega\) in past/future time.
- The sets \(\Gamma_{\pm}\) have porous structure in certain directions (see next slide).
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Illustration: Arnold cat map

Simpler model than the geodesic flow: an Arnold cat map on $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$

\[ \varphi : \mathbb{T}^2 \to \mathbb{T}^2, \quad \varphi(x_1, x_2) = (2x_1 + x_2, x_1 + x_2) \mod \mathbb{Z}^2 \]

Define $\Gamma_{\pm}(N) = \{ x \in \mathbb{T}^2 \mid \forall j = 0, \ldots, N, \varphi^{\mp j}(x) \notin \Omega \}$

We see that $\Gamma_{\pm}(N)$ have porous structure in the stable/unstable directions

Schwartz '21: analog of Theorem 3 for quantum cat maps
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$\Gamma_-(N), \ N = 2$

$\Omega$ (in white)

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Open quantum chaos and resonances

\((M, g)\) noncompact convex co-compact hyperbolic \((K = -1)\) surface

Resonances: poles of the scattering resolvent

\[ R(\lambda) = \left( -\Delta_g - \frac{1}{4} - \lambda^2 \right)^{-1} : \begin{cases} L^2(M) \to L^2(M), & \text{Im} \lambda > 0 \\ L^2_{\text{comp}}(M) \to L^2_{\text{loc}}(M), & \text{Im} \lambda \leq 0 \end{cases} \]

Existence of meromorphic continuation: Patterson '75,'76, Perry '87,'89, Mazzeo–Melrose '87, Guillopé–Zworski '95, Guillarmou '05, Vasy '13
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Existence of meromorphic continuation: Patterson ’75,’76, Perry ’87,’89, Mazzeo–Melrose ’87, Guillopé–Zworski ’95, Guillarmou ’05, Vasy ’13
Open quantum chaos and resonances

\((M, g)\) noncompact convex co-compact hyperbolic \((K = -1)\) surface

Resonances: poles of the scattering resolvent

Also correspond to zeroes of the Selberg zeta function

\[ Z_M(s) = \prod_{T \in \mathcal{L}_M} \prod_{k \geq 0} (1 - e^{-(s+k)T}), \quad s = \frac{1}{2} - i\lambda \]

where \(\mathcal{L}_M\) consists of lengths of primitive closed geodesics
**Resonances:** poles of the scattering resolvent $R(\lambda)$

Featured in resonance expansions of waves:

$$\text{Re} \, \lambda = \text{rate of oscillation}, \quad -\text{Im} \, \lambda = \text{rate of decay}$$

**Borthwick ’13, Borthwick–Weich ’14:** numerics for resonances
Resonances: poles of the scattering resolvent $R(\lambda)$

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Borthwick ’13, Borthwick–Weich ’14: numerics for resonances

Pictures courtesy of David Borthwick
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Spectral gap

Theorem 5 [D–Zahl ’16, Bourgain–D ’18, D–Zworski ’20]

Let \((M, g)\) be a convex co-compact hyperbolic surface. Then it has an essential spectral gap: there exists \(\beta > 0\) such that there are only finitely many resonances \(\lambda\) with \(\text{Im} \, \lambda > -\beta\).

- Gives \(O(e^{-\beta t})\) local energy decay for linear waves (at high frequency)
- Also implies Strichartz estimates: Wang ’19
- Follows a long history of study of spectral gaps in this and other similar settings (e.g. obstacle scattering):
  - Lax–Phillips ’67, Patterson ’76, Sullivan ’79, Ikawa ’88, Gaspard–Rice ’89, Naud ’05, Nonnenmacher–Zworski ’09, Petkov–Stoyanov ’10, Stoyanov ’11 ...
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  - Petkov–Stoyanov ’10, Stoyanov ’11 ...
A physically relevant setting: scattering by several convex obstacles in $\mathbb{R}^n$.

**Resonances:** poles of the meromorphic continuation of $(-\Delta_g - \lambda^2)^{-1}$.

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**Theorem 5 [Vacossin ’22, using Bourgain–D ’18]**

Let $M$ be the exterior of several convex obstacles in $\mathbb{R}^2$, which satisfy the no-eclipse condition (no line intersects 3 obstacles). Then there exists $\beta > 0$ such that there are only finitely many resonances in $\{\text{Im} \lambda > -\beta\}$.

**Observed experimentally:** Barkhofen–Weich–Potzuweit–Stöckmann–Kuhl–Zworski ’13
Higher dimensional FUP?

- The results above applied to surfaces (dim = 2)
- To make them work for general manifolds of dim > 2, we need a fractal uncertainty principle for subsets of $\mathbb{R}^n, n \geq 2$
- **Counterexample:** $X, Y \subset \mathbb{R}^2$ are two orthogonal lines. Then $\hat{\delta_X} = 2\pi \delta_Y$ and FUP fails

Here is what is known to date:

- **Han–Schlag ’20:** FUP if $X$ is a product of porous subsets of $\mathbb{R}$
- **D–Jézéquel ’21:** Theorem 1 for certain higher dimensional quantum cat maps, still using 1D FUP
- **D–Zhang ’22?** FUP in 2D if $X$ is a curve
- **Cohen ’22?:** FUP for arithmetic Cantor sets that don’t contain orthogonal lines
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Thank you for your attention!