NOTES ON HYPERBOLIC DYNAMICS

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Abstract. This expository note presents a proof of the Stable/Unstable Manifold Theorem (also known as the Hadamard–Perron Theorem), partially following [KaHa97]. It also gives examples of hyperbolic dynamics: geodesic flows on surfaces of negative curvature and dispersing billiards.

> Примерно каждые пять лет, если не чаще, кто-нибудь заново “открывает” теорему Адамара — Перрона, доказывая ее либо по схеме доказательства Адамара, либо по схеме Перрона. Я сам в этом повинен...
>
> *Every five years or so, if not more often, someone “discovers” again the Hadamard–Perron Theorem, proving it using either Hadamard’s or Perron’s method. I have been guilty of this myself...*
>
> Dmitri Anosov [An67, p. 23]

These expository notes are intended as an introduction to some aspects of hyperbolic dynamics, with emphasis on the Stable/Unstable Manifold Theorem (also known as the Hadamard–Perron Theorem). They are structured as follows:

- In §2 we present the Stable/Unstable Manifold Theorem in a simple setting, capturing the essential components of the proof without some of the technical and notational complications.
- In §3 we give a proof of the general Stable/Unstable Manifold Theorem for sequences of transformations on $\mathbb{R}^n$ with canonical stable/unstable spaces at the origin, building on the special case in §2.
- In §4 we use the results of §3 to prove the Stable/Unstable Manifold Theorem for general hyperbolic maps and flows.
- In §5 we give two important examples of hyperbolic systems: geodesic flows on surfaces of negative curvature (§5.1) and dispersing billiard ball maps (§5.2).

For the (long and rich) history of hyperbolic dynamics we refer the reader to [KaHa97].

2. Stable/unstable manifolds in a simple setting

In this section we present the Stable/Unstable Manifold Theorem (broken into two parts, Theorem 1 in §2.1 and Theorem 2 in §2.4) under several simplifying assumptions:

- we study iterates of a single map φ, defined on a neighborhood of 0 in $\mathbb{R}^2$;
- $\varphi(0) = 0$ and $d\varphi(0)$ is a hyperbolic matrix, with eigenvalues 2 and $\frac{1}{2}$;
- the map φ is close to the linearized map $x \mapsto d\varphi(0) \cdot x$ in the C^{N+1} norm.

These assumptions are made to make the notation below simpler, however they do not impact the substance of the proof. As explained below in §§3–4, the arguments of this section can be adapted to the setting of general hyperbolic maps and flows.

2.1. Existence of stable/unstable manifolds.

Throughout this section we use the following notation for ℓ_∞ balls in $\mathbb{R}^2$:

$$\overline{B}_\infty(0, r) := \{(x_1, x_2) \in \mathbb{R}^2 \colon \max(|x_1|, |x_2|) \le r\}.$$

We assume that $U_\varphi, V_\varphi \subset \mathbb{R}^2$ are open sets with $\overline{B}_\infty(0, 1) \subset U_\varphi \cap V_\varphi$ and

$$\varphi : U_\varphi \to V_\varphi$$

is a C^{N+1} map (here $N \ge 1$ is fixed) which satisfies the following assumptions:

(1) $\varphi(0) = 0$;
(2) The differential $d\varphi(0)$ is equal to

$$d\varphi(0) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}; \tag{2.1}$$

(3) for a small constant $\delta > 0$ (chosen later in Theorems 1 and 2) and all multi-indices α with $2 \le |\alpha| \le N + 1$, we have

$$\sup_{U_\varphi} |\partial^\alpha \varphi| \le \delta; \tag{2.2}$$

(4) φ is a diffeomorphism onto its image.

We remark that assumptions (3) and (4) above can be arranged to hold locally by zooming in to a small neighborhood of 0, see §3.1 below.

It follows from (2.1) that the space $E_u(0) := \mathbb{R}\partial_{x_1}$ is preserved by the linearized map $x \mapsto d\varphi(0) \cdot x$ and vectors in this space are expanded exponentially by the powers of $d\varphi(0)$. Similarly the space $E_s(0) := \mathbb{R}\partial_{x_2}$ is invariant and contracted exponentially by the powers of $d\varphi(0)$. We call $E_u(0)$ the *unstable space* and $E_s(0)$ the *stable space* of φ at 0.

The main results of this section are the nonlinear versions of the above observations: namely there exist one-dimensional *unstable/stable submanifolds* $W_u, W_s \subset \mathbb{R}^2$ which

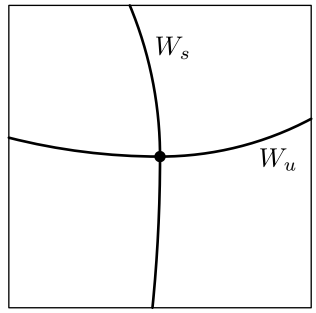

FIGURE 1. The manifolds W_u, W_s. The square is $\overline{B}_\infty(0,1)$, the horizontal direction is x_1, and the vertical direction is x_2.

are (locally) invariant under the map φ; the iterates of φ are exponentially expanding on the unstable manifold and exponentially contracting on the stable one.

We construct the unstable/stable manifolds as graphs of C^N functions. For a function $F : [-1,1] \to \mathbb{R}$ we define its unstable/stable graphs

$$\mathcal{G}_u(F) := \{x_2 = F(x_1),\ |x_1| \le 1\}, \quad \mathcal{G}_s(F) := \{x_1 = F(x_2),\ |x_2| \le 1\} \tag{2.3}$$

which are subsets of $\mathbb{R}^2$.

Theorem 1 below asserts existence of unstable/stable manifolds. The fact that φ is expanding on W_u and contracting on W_s is proved later in Theorem 2.

Theorem 1. *Assume that δ is small enough (depending only on N) and assumptions (1)–(4) above hold. Then there exist C^N functions*

$$F_u, F_s : [-1,1] \to [-1,1], \quad F_u(0) = F_s(0) = 0, \quad \partial_{x_1} F_u(0) = \partial_{x_2} F_s(0) = 0$$

such that, denoting the graphs (see Figure 1)

$$W_u := \mathcal{G}_u(F_u), \quad W_s := \mathcal{G}_s(F_s), \tag{2.4}$$

we have

$$\varphi(W_u) \cap \overline{B}_\infty(0,1) = W_u, \quad \varphi^{-1}(W_s) \cap \overline{B}_\infty(0,1) = W_s. \tag{2.5}$$

Moreover, $W_u \cap W_s = \{0\}$.

The proof of the theorem, given in §§2.2–2.3 below, is partially based on the proof of the more general Hadamard–Perron theorem in [KaHa97, Theorem 6.2.8]. The main idea is to show that the action of φ on unstable graphs is a contraction mapping with respect to an appropriately chosen metric (and same with the action of φ^{-1} on stable graphs). There are however two points in which our proof differs from the one in [KaHa97]:

- We run the contraction mapping argument on the metric space of C^N functions whose N-th derivative has Lipschitz norm bounded by 1, with the C^N metric.

This is slightly different from the space used in [KaHa97, §6.2.d, Step 3] and it requires having $N+1$ derivatives of the map φ to obtain C^N regularity for invariant graphs (rather than N derivatives as in [KaHa97]). The upshot is that we do not need separate arguments for establishing regularity of the manifolds W_u, W_s [KaHa97, §6.2.d, Steps 1–2, 4–5].

- We only consider the action of φ on the ball $\overline{B}_\infty(0,1)$ rather than extending it to the entire $\mathbb{R}^2$ as in [KaHa97, Lemma 6.2.7]. Because of this parts (3)–(4) of Theorem 2 below have a somewhat different proof than the corresponding statement [KaHa97, Theorem 6.2.8(iii)].

Notation: In the remainder of this section we denote by C constants which depend only on N (in particular they do not depend on δ), and write $R = \mathcal{O}(\delta)$ if $|R| \leq C\delta$. We assume that $\delta > 0$ is chosen small (depending only on N).

2.2. **Action on graphs and derivative bounds.** We start the proof of Theorem 1 by considering the action of φ on unstable graphs. (The action of φ^{-1} on stable graphs is handled similarly.) This action, denoted by Φ_u and called the *graph transform*, is defined by

Lemma 2.1. *Let $F : [-1,1] \to \mathbb{R}$ satisfy*

$$F(0) = 0, \quad \sup |\partial_{x_1} F| \leq 1. \tag{2.6}$$

Then there exists a function

$$\Phi_u F : [-1,1] \to \mathbb{R}, \quad \Phi_u F(0) = 0$$

such that (see Figure 2)

$$\varphi(\mathcal{G}_u(F)) \cap \{|x_1| \leq 1\} = \mathcal{G}_u(\Phi_u F). \tag{2.7}$$

Proof. Define the components $\varphi_1, \varphi_2 : \overline{B}_\infty(0,1) \to \mathbb{R}$ of φ by

$$\varphi(x) = (\varphi_1(x), \varphi_2(x)), \quad x \in \overline{B}_\infty(0,1). \tag{2.8}$$

Next, define the functions $G_1, G_2 : [-1,1] \to \mathbb{R}$ by

$$G_1(x_1) := \varphi_1(x_1, F(x_1)), \quad G_2(x_1) := \varphi_2(x_1, F(x_1)), \tag{2.9}$$

so that the manifold $\varphi(\mathcal{G}_u(F))$ has the form

$$\varphi(\mathcal{G}_u(F)) = \{(G_1(x_1), G_2(x_1)) \colon |x_1| \leq 1\}.$$

To write $\varphi(\mathcal{G}_u(F))$ as a graph, we need to show that G_1 is invertible. Note that $G_1(0) = 0$. We next compute

$$\partial_{x_1} G_1(x_1) = \partial_{x_1}\varphi_1(x_1, F(x_1)) + \partial_{x_2}\varphi_1(x_1, F(x_1)) \cdot \partial_{x_1} F(x_1).$$

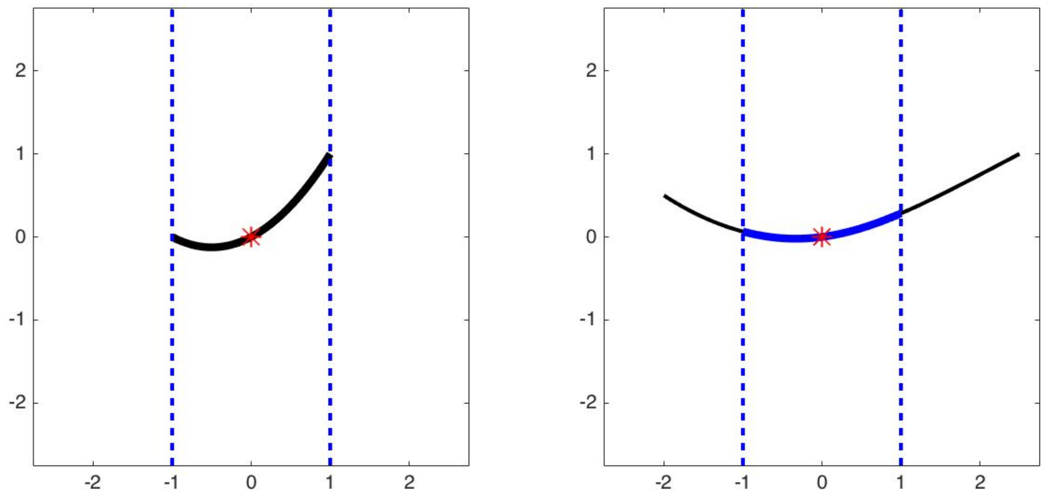

FIGURE 2. Left: the graph $\mathcal{G}_u(F)$ for some function F satisfying (2.6). Right: the image of $\mathcal{G}_u(F)$ under φ. The solid blue part is the graph $\mathcal{G}_u(\Phi_u F)$. Figures 2–7 are plotted numerically using the map $\varphi(x_1, x_2) = (2x_1 + \frac{1}{2}x_2^2, \frac{1}{2}x_2 + \frac{1}{2}x_1^2)$.

Together (2.1) and (2.2) imply for all $(x_1, x_2) \in \overline{B}_\infty(0, 1)$

$$\partial_{x_1}\varphi_1(x_1, x_2) = 2 + \mathcal{O}(\delta), \quad \partial_{x_2}\varphi_1(x_1, x_2) = \mathcal{O}(\delta),$$

so for δ small enough

$$\partial_{x_1}G_1(x_1) = 2 + \mathcal{O}(\delta) \geq \frac{3}{2} \quad \text{for all } x_1 \in [-1, 1]. \tag{2.10}$$

Therefore G_1 is a diffeomorphism and its image contains $[-1, 1]$. It follows that $\varphi(\mathcal{G}_u(F)) \cap \{|x_1| \leq 1\} = \mathcal{G}_u(\Phi_u F)$ for the function $\Phi_u F$ defined by

$$\Phi_u F(y_1) = G_2(G_1^{-1}(y_1)), \quad y_1 \in [-1, 1] \tag{2.11}$$

where

$$G_1^{-1} : [-1, 1] \to [-1, 1] \tag{2.12}$$

is the inverse of G_1. □

We now want to estimate the function $\Phi_u F$ in terms of F, ultimately showing that Φ_u is a contraction with respect to a certain norm. For that we use the following formula for the derivatives of $\Phi_u F$:

Lemma 2.2. *Let $1 \leq k \leq N$. Assume that $F \in C^k([-1, 1]; \mathbb{R})$ satisfies*

$$F(0) = 0, \quad \sup |\partial_{x_1}^j F| \leq 1 \quad \textit{for all } j = 1, \dots, k. \tag{2.13}$$

Then we have for all $y_1 \in [-1,1]$ and G_1^{-1} defined in (2.12)

$$\partial_{x_1}^k(\Phi_u F)(y_1) = L_k(x_1, F(x_1), \partial_{x_1}F(x_1), \dots, \partial_{x_1}^k F(x_1)), \quad x_1 := G_1^{-1}(y_1) \tag{2.14}$$

where the function $L_k(x_1, \tau_0, \dots, \tau_k)$, depending on φ but not on F, is continuous on the cube $Q_k := [-1,1]^{k+2}$. Moreover $L_k(x_1, \tau_0, \dots, \tau_k) = 2^{-k-1}\tau_k + \mathcal{O}(\delta)$, with the remainder satisfying the derivative bounds

$$\begin{gathered}\sup_{Q_k} \left|\partial_{x_1}^{\alpha}\partial_{\tau_0}^{\beta_0}\dots\partial_{\tau_k}^{\beta_k}\big(L_k(x_1,\tau_0,\dots,\tau_k) - 2^{-k-1}\tau_k\big)\right| \le C_{\alpha\beta}\,\delta \\ \text{for all} \quad \alpha, \beta_0, \dots, \beta_k \quad \text{such that} \quad \alpha + \beta_0 + k \le N+1.\end{gathered} \tag{2.15}$$

Remark. In the linear case $\varphi(x) = d\varphi(0)\cdot x$ we have $G_1(x_1) = 2x_1$, $G_2(x_1) = \frac{1}{2}F(x_1)$, therefore $\Phi_u F(y_1) = \frac{1}{2}F(\frac{1}{2}y_1)$. The meaning of (2.15) is that the action of Φ_u on the derivatives of F for nonlinear φ is $\mathcal{O}(\delta)$-close to the linear case.

Proof. For $x \in \overline{B}_\infty(0,1)$ define the matrix

$$A(x) := d\varphi(x) = (A_{jk}(x)), \quad A_{jk}(x) := \partial_{x_k}\varphi_j(x).$$

By (2.1) and (2.2) we have

$$A(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} + \mathcal{O}(\delta), \tag{2.16}$$

and the remainder in (2.16) is $\mathcal{O}(\delta)$ with derivatives of order $\le N$.

We argue by induction on k. For $k=1$, from the definition (2.11) of $\Phi_u F$ we have

$$\partial_{x_1}(\Phi_u F)(y_1) = \frac{\partial_{x_1}G_2(x_1)}{\partial_{x_1}G_1(x_1)}, \quad x_1 := G_1^{-1}(y_1)$$

so (2.14) holds with

$$L_1(x_1, \tau_0, \tau_1) = \frac{A_{21}(x_1,\tau_0) + A_{22}(x_1,\tau_0)\tau_1}{A_{11}(x_1,\tau_0) + A_{12}(x_1,\tau_0)\tau_1}. \tag{2.17}$$

From (2.16) we see that $L_1(x_1,\tau_0,\tau_1) = \frac{1}{4}\tau_1 + \mathcal{O}(\delta)$ and the stronger remainder estimate (2.15) holds.

Now assume that $2 \le k \le N$ and (2.14), (2.15) hold for $k-1$. Then by the chain rule (2.14) holds for k with

$$L_k(x_1,\tau_0,\dots,\tau_k) := \frac{\partial_{x_1}L_{k-1}(x_1,\tau_0,\dots,\tau_{k-1}) + \sum_{j=0}^{k-1}\partial_{\tau_j}L_{k-1}(x_1,\tau_0,\dots,\tau_{k-1})\tau_{j+1}}{A_{11}(x_1,\tau_0) + A_{12}(x_1,\tau_0)\tau_1}.$$

It is straightforward to check that $L_k(x_1,\tau_0,\dots,\tau_k) = 2^{-k-1}\tau_k + \mathcal{O}(\delta)$ and the stronger remainder estimate (2.15) holds. □

Armed with Lemma 2.2 we estimate the derivatives of $\Phi_u F$ in terms of the derivatives of F. Let $1 \leq k \leq N$. We use the following seminorm on $C^k([-1,1];\mathbb{R})$:

$$\|F\|_{C^k} := \max_{1\leq j\leq k} \sup |\partial_{x_1}^j F|. \tag{2.18}$$

We will work with functions satisfying $F(0)=0$, on which $\|\bullet\|_{C^k}$ is a norm:

$$F(0)=0 \quad \Longrightarrow \quad \sup|F| \leq \|F\|_{C^1}. \tag{2.19}$$

To establish the contraction property (see the remark following the proof of Lemma 2.4) we also need the space of functions $C^{k,1}([-1,1];\mathbb{R})$ with Lipschitz continuous k-th derivative, endowed with the seminorm

$$\|F\|_{C^{k,1}} := \max\left(\|F\|_{C^k}, \sup_{x_1\neq\tilde{x}_1} \frac{|\partial_{x_1}^k F(x_1) - \partial_{x_1}^k F(\tilde{x}_1)|}{|x_1-\tilde{x}_1|}\right). \tag{2.20}$$

Note that $\|F\|_{C^k} \leq \|F\|_{C^{k,1}} \leq \|F\|_{C^{k+1}}$.

Our first estimate implies that Φ_u maps the unit balls in C^k, $C^{k,1}$ into themselves:

Lemma 2.3. *Let $1 \leq k \leq N$ and assume that $F(0)=0$ and $\|F\|_{C^k} \leq 1$. Then*

$$\|\Phi_u F\|_{C^k} \leq \frac{1}{4}\|F\|_{C^k} + C\delta. \tag{2.21}$$

If additionally $\|F\|_{C^{k,1}} \leq 1$ then

$$\|\Phi_u F\|_{C^{k,1}} \leq \frac{1}{4}\|F\|_{C^{k,1}} + C\delta. \tag{2.22}$$

Proof. Let $y_1 \in [-1,1]$ and $x_1 := G_1^{-1}(y_1) \in [-1,1]$. By Lemma 2.2 we have for all $j=1,\dots,k$

$$|\partial_{x_1}^j(\Phi_u F)(y_1)| \leq 2^{-j-1}|\partial_{x_1}^j F(x_1)| + C\delta \leq \frac{1}{4}\|F\|_{C^k} + C\delta$$

which implies (2.21).

Next, assume that $\|F\|_{C^{k,1}} \leq 1$. Take $y_1, \tilde{y}_1 \in [-1,1]$ such that $y_1 \neq \tilde{y}_1$. Put $x_1 := G_1^{-1}(y_1)$, $\tilde{x}_1 := G_1^{-1}(\tilde{y}_1)$. Then by Lemma 2.2

$$\begin{aligned}|\partial_{x_1}^k(\Phi_u F)(y_1) - \partial_{x_1}^k(\Phi_u F)(\tilde{y}_1)| &\leq 2^{-k-1}|\partial_{x_1}^k F(x_1) - \partial_{x_1}^k F(\tilde{x}_1)| + C\delta|x_1-\tilde{x}_1| \\ &\quad + C\delta \max_{0\leq j\leq k} |\partial_{x_1}^j F(x_1) - \partial_{x_1}^j F(\tilde{x}_1)| \\ &\leq \left(\frac{1}{4}\|F\|_{C^{k,1}} + C\delta + C\delta\|F\|_{C^{k,1}}\right)|x_1-\tilde{x}_1|.\end{aligned}$$

Since $|x_1-\tilde{x}_1| \leq |y_1-\tilde{y}_1|$ by (2.10), this implies (2.22). □

The next estimate gives the contraction property of Φ_u in C^k norm:

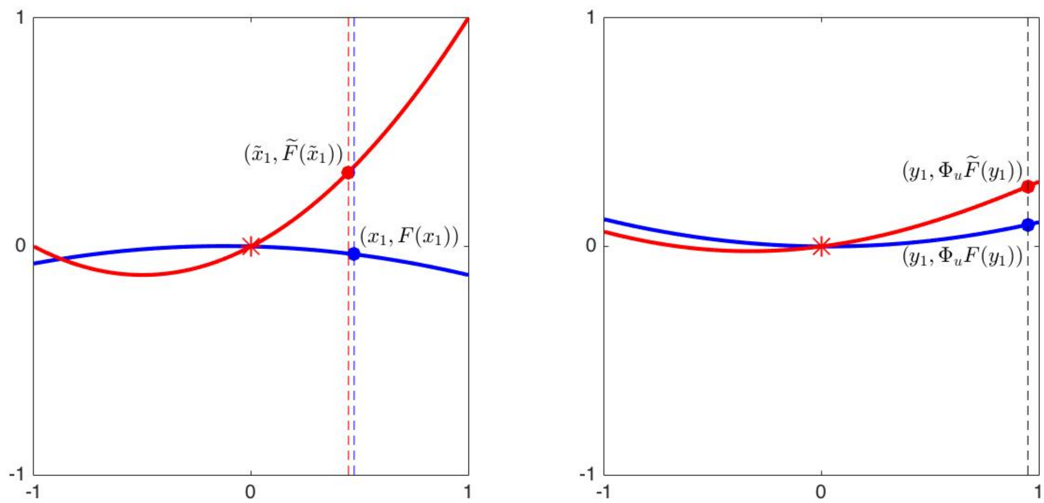

FIGURE 3. Left: the points $(x_1, F(x_1))$, $(\tilde x_1, \widetilde F(\tilde x_1))$. The blue curve is the graph of F and the red curve is the graph of $\widetilde F$. Right: the image of the picture on the left by φ.

Lemma 2.4. *Let $1 \le k \le N$. Assume that $F, \widetilde F \in C^{k,1}$ satisfy $F(0) = \widetilde F(0) = 0$ and $\|F\|_{C^{k,1}}, \|\widetilde F\|_{C^{k,1}} \le 1$. Then*

$$\|\Phi_u F - \Phi_u \widetilde F\|_{C^k} \le \Big(\frac{1}{4} + C\delta\Big)\|F - \widetilde F\|_{C^k}. \tag{2.23}$$

Proof. Let G_1 and $\widetilde G_1$ be defined by (2.9) using the functions F and $\widetilde F$ respectively. Take $y_1 \in [-1,1]$ and put $x_1 := G_1^{-1}(y_1)$, $\tilde x_1 := \widetilde G_1^{-1}(y_1)$, see Figure 3. We first estimate the difference between the two inverses x_1, $\tilde x_1$:

$$|x_1 - \tilde x_1| \le C\delta \|F - \widetilde F\|_{C^1}. \tag{2.24}$$

To show (2.24), we write

$$\begin{aligned}|x_1 - \tilde x_1| &\le |\widetilde G_1(x_1) - \widetilde G_1(\tilde x_1)| = |\widetilde G_1(x_1) - G_1(x_1)| \\ &= |\varphi_1(x_1, \widetilde F(x_1)) - \varphi_1(x_1, F(x_1))| \le C\delta\|F - \widetilde F\|_{C^1}\end{aligned}$$

where the first inequality follows from (2.10) and the last inequality uses that $\partial_{x_2}\varphi_1 = \mathcal O(\delta)$ by (2.16) and $|\widetilde F(x_1) - F(x_1)| \le \|F - \widetilde F\|_{C^1}$ by (2.19).

Next we have for all $j = 0, \dots, k$

$$|\partial_{x_1}^j F(x_1) - \partial_{x_1}^j \widetilde F(\tilde x_1)| \le (1 + C\delta)\|F - \widetilde F\|_{C^k}. \tag{2.25}$$

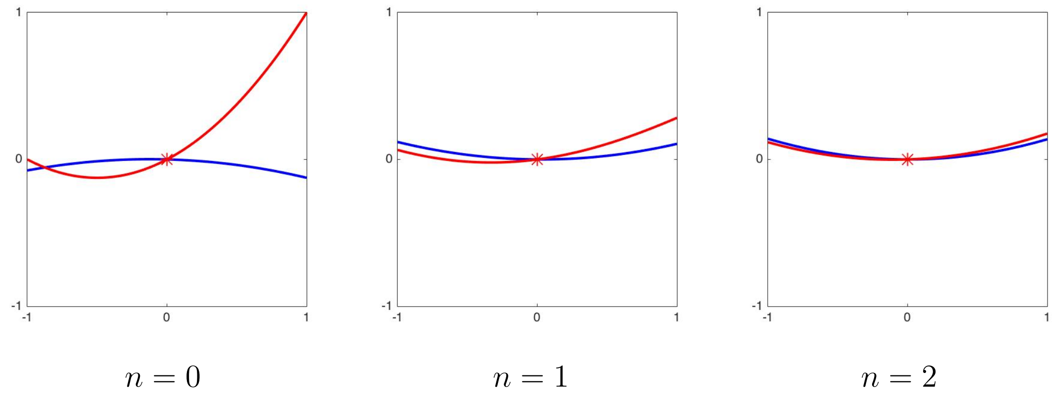

FIGURE 4. The iterations $\varphi^n(\mathcal{G}_{F_0})$ for two different choices of F_0. Both converge to W_u, illustrating (2.28).

Indeed,

$$\begin{aligned}|\partial_{x_1}^j F(x_1) - \partial_{x_1}^j \widetilde{F}(\tilde{x}_1)| &\leq |\partial_{x_1}^j F(x_1) - \partial_{x_1}^j F(\tilde{x}_1)| + |\partial_{x_1}^j F(\tilde{x}_1) - \partial_{x_1}^j \widetilde{F}(\tilde{x}_1)| \\ &\leq |x_1 - \tilde{x}_1| + \|F - \widetilde{F}\|_{C^k} \leq (1 + C\delta)\|F - \widetilde{F}\|_{C^k}\end{aligned}$$

where the second inequality uses the fact that $\|F\|_{C^{k,1}} \leq 1$ and the last inequality used (2.24).

Finally, by Lemma 2.2 we estimate for all $j = 1, \dots, k$

$$\begin{aligned}|\partial_{x_1}^j(\Phi_u F)(y_1) - \partial_{x_1}^j(\Phi_u \widetilde{F})(y_1)| &\leq 2^{-j-1}|\partial_{x_1}^j F(x_1) - \partial_{x_1}^j \widetilde{F}(\tilde{x}_1)| \\ &\quad + C\delta|x_1 - \tilde{x}_1| + C\delta \max_{0\leq \ell\leq j} |\partial_{x_1}^\ell F(x_1) - \partial_{x_1}^\ell \widetilde{F}(\tilde{x}_1)| \\ &\leq \Big(\frac{1}{4} + C\delta\Big)\|F - \widetilde{F}\|_{C^k}\end{aligned}$$

where the second inequality uses (2.24) and (2.25). This implies (2.23). □

Remark. The a priori bound $\|F\|_{C^{k,1}} \leq 1$ was used in the proof of (2.25). Without it we would not be able to estimate the difference $|\partial_{x_1}^k F(x_1) - \partial_{x_1}^k \widetilde{F}(\tilde{x}_1)|$ since the functions $\partial_{x_1}^k F$, $\partial_{x_1}^k \widetilde{F}$ are evaluated at two different values of x_1. We do not use the stronger a priori bound $\|F\|_{C^{k+1}} \leq 1$ because it would make it harder to set up a complete metric space for the contraction mapping argument in (2.26) below.

2.3. **Contraction mapping argument.**

We now give the proof of Theorem 1 using the estimates from the previous section. We show existence of the function F_u; the function F_s is constructed similarly, replacing φ by φ^{-1} and switching the roles of x_1 and x_2.

Consider the metric space $(\mathcal{X}_N, d_N)$ defined using the seminorms (2.18), (2.20):

$$\begin{aligned}\mathcal{X}_N &:= \{F \in C^{N,1}([-1,1];\mathbb{R}) \colon F(0) = 0,\ \|F\|_{C^{N,1}} \leq 1\},\\ d_N(F, \widetilde{F}) &:= \|F - \widetilde{F}\|_{C^N}.\end{aligned} \tag{2.26}$$

Then $(\mathcal{X}_N, d_N)$ is a complete metric space. Indeed, it is the subset of the closed unit ball in C^N defined using the closed conditions

$$F(0) = 0, \quad |\partial_{x_1}^N F(x_1) - \partial_{x_1}^N F(\tilde{x}_1)| \leq |x_1 - \tilde{x}_1| \quad \text{for all } x_1, \tilde{x}_1 \in [-1,1].$$

By Lemmas 2.1 and 2.3, for δ small enough the graph transform defines a map

$$\Phi_u : \mathcal{X}_N \to \mathcal{X}_N.$$

By Lemma 2.4, for δ small enough this map is a contraction, specifically

$$d_N(\Phi_u F, \Phi_u \widetilde{F}) \leq \frac{1}{3} d_N(F, \widetilde{F}) \quad \text{for all } F, \widetilde{F} \in \mathcal{X}_N. \tag{2.27}$$

Therefore by the Contraction Mapping Principle the map Φ_u has a unique fixed point

$$F_u \in \mathcal{X}_N, \quad \Phi_u F_u = F_u.$$

In fact for each fixed $F_0 \in \mathcal{X}_N$ we have (see Figure 4)

$$(\Phi_u)^n F_0 \to F_u \quad \text{in } C^N \quad \text{as } n \to \infty. \tag{2.28}$$

Let $W_u := \mathcal{G}_u(F_u) \subset \overline{B}_\infty(0,1)$ be the unstable graph of F_u. Recalling the definition (2.7) of $\Phi_u F$, we have

$$\varphi(W_u) \cap \{|x_1| \leq 1\} = \mathcal{G}_u(\Phi_u F_u) = W_u.$$

It follows that $\varphi(W_u) \cap \overline{B}_\infty(0,1) = W_u$, giving (2.5).

Next, we see from (2.17) that $F(0) = \partial_{x_1} F(0) = 0$ implies $\partial_{x_1}(\Phi_u F)(0) = 0$. Using (2.28) with $F_0 \equiv 0$, we get $\partial_{x_1} F_u(0) = 0$.

By Lemma 2.3 we have the following derivative bounds on F_u, F_s:

$$\|F_u\|_{C^{N,1}} \leq C\delta, \quad \|F_s\|_{C^{N,1}} \leq C\delta. \tag{2.29}$$

This implies that $W_u \cap W_s = \{0\}$. Indeed, if $(x_1, x_2) \in W_u \cap W_s$, then $x_2 = F_u(x_1)$ and $x_1 = F_s(x_2)$. Therefore, $x_1 = F_s(F_u(x_1))$. However (2.29) implies that

$$\sup_{[-1,1]} |\partial_{x_1}(F_s \circ F_u)| \leq C\delta. \tag{2.30}$$

Therefore $F_s \circ F_u : [-1,1] \to [-1,1]$ is a contraction and the equation $x_1 = F_s(F_u(x_1))$ has only one solution, $x_1 = 0$.

Remark. The above proof shows that $F_u \in C^{N,1}$. One can prove in fact that $F_u \in C^{N+1}$, see [KaHa97, §6.2.d, Step 5].

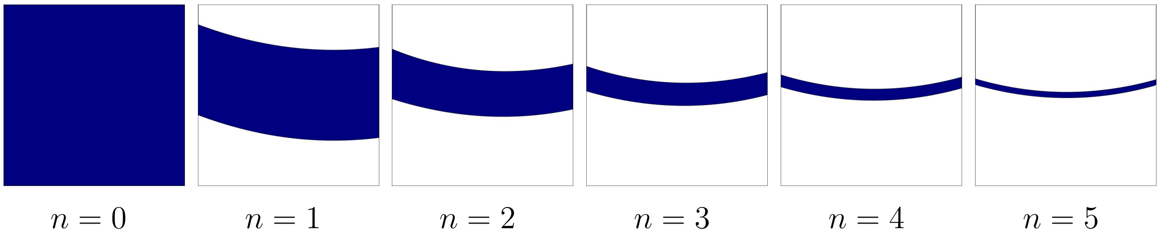

FIGURE 5. The sets of points w such that $w, \varphi^{-1}(w), \dots, \varphi^{(-n)}(w) \in \overline{B}_\infty(0,1)$. By (2.33), in the limit $n \to \infty$ we obtain W_u.

2.4. **Further properties.** In this section we prove the following theorem which relates the manifolds W_u, W_s to the behavior of large iterates φ^n of the map φ:

Theorem 2. *Let φ be as in Theorem 1 and δ be small enough depending only on N. Let W_u, W_s be defined in (2.4). Then:*

(1) *If $w \in W_u$ then $\varphi^{-n}(w) \to 0$ as $n \to \infty$, more precisely*
$$|\varphi^{-n}(w)| \leq \Big(\frac{1}{2} + C\delta\Big)^n |w| \quad \text{for all } n \geq 0. \tag{2.31}$$
(2) *If $w \in W_s$ then $\varphi^n(w) \to 0$ as $n \to \infty$, more precisely*
$$|\varphi^{n}(w)| \leq \Big(\frac{1}{2} + C\delta\Big)^n |w| \quad \text{for all } n \geq 0. \tag{2.32}$$
(3) *If $w \in \overline{B}_\infty(0,1)$ satisfies $\varphi^{-n}(w) \in \overline{B}_\infty(0,1)$ for all $n \geq 0$, then $w \in W_u$.*
(4) *If $w \in \overline{B}_\infty(0,1)$ satisfies $\varphi^{n}(w) \in \overline{B}_\infty(0,1)$ for all $n \geq 0$, then $w \in W_s$.*

Remark. Theorem 2 implies the following dynamical characterization of the unstable/stable manifolds W_u, W_s:
$$\begin{aligned} w \in W_u \quad &\Longleftrightarrow \quad \varphi^{-n}(w) \in \overline{B}_\infty(0,1) \quad \text{for all } n \geq 0; \\ w \in W_s \quad &\Longleftrightarrow \quad \varphi^{n}(w) \in \overline{B}_\infty(0,1) \quad \text{for all } n \geq 0. \end{aligned} \tag{2.33}$$
See Figure 5.

We only give the proof of parts (1) and (3) in Theorem 2. Parts (2) and (4), characterizing the stable manifold, are proved similarly, replacing φ by φ^{-1} and switching the roles of x_1 and x_2.

Part (1) of Theorem 2 follows by iteration (putting $y := w$, $\tilde{y} := 0$) from

Lemma 2.5. *Let $y, \tilde{y} \in W_u$ and put $x := \varphi^{-1}(y)$, $\tilde{x} := \varphi^{-1}(\tilde{y})$. Then (see Figure 6)*
$$|x - \tilde{x}| \leq \Big(\frac{1}{2} + C\delta\Big)|y - \tilde{y}|. \tag{2.34}$$

Proof. By (2.5) we have $x, \tilde{x} \in W_u \subset \overline{B}_\infty(0,1)$. We write
$$x = (x_1, F_u(x_1)), \quad \tilde{x} = (\tilde{x}_1, F_u(\tilde{x}_1)), \quad y = (y_1, F_u(y_1)), \quad \tilde{y} = (\tilde{y}_1, F_u(\tilde{y}_1)).$$

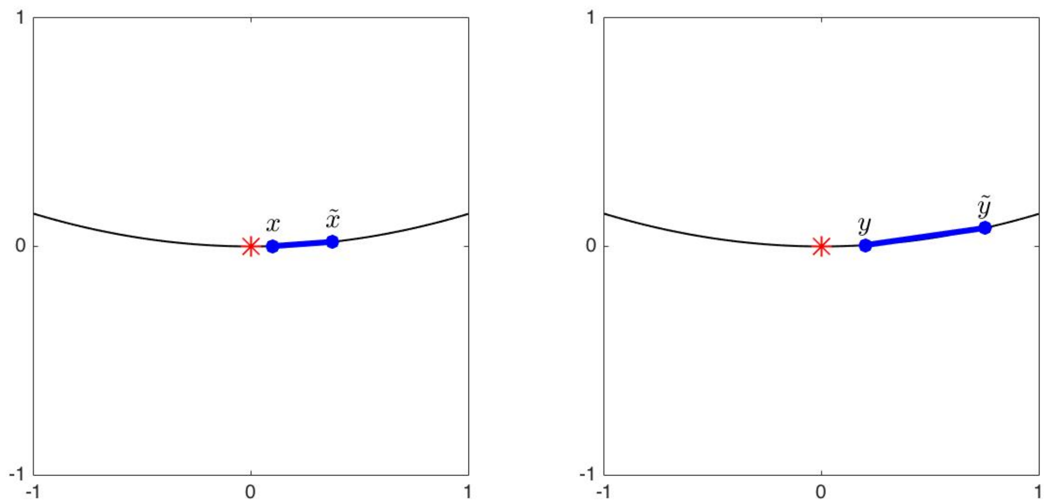

FIGURE 6. The points $x, \tilde{x}, y, \tilde{y}$ from Lemma 2.5. The curve is W_u.

Since $\|F_u\|_{C^1} \leq C\delta$ by (2.29), we have

$$|x - \tilde{x}| \leq (1 + C\delta)|x_1 - \tilde{x}_1|, \tag{2.35}$$

$$|y - \tilde{y}| \geq (1 - C\delta)|y_1 - \tilde{y}_1|. \tag{2.36}$$

We have $y_1 = \varphi_1(x_1, F_u(x_1))$ and $\tilde{y}_1 = \varphi_1(\tilde{x}_1, F_u(\tilde{x}_1))$. Thus by (2.16)

$$y_1 - \tilde{y}_1 = 2(x_1 - \tilde{x}_1) + \mathcal{O}(\delta)|x_1 - \tilde{x}_1|. \tag{2.37}$$

Together (2.35)–(2.37) give (2.34). □

It remains to show part (3) of Theorem 2. For a point $w = (w_1, w_2) \in \overline{B}_\infty(0, 1)$, define the distance from it to the unstable manifold by

$$d(w, W_u) := |w_2 - F_u(w_1)|. \tag{2.38}$$

The key component of the proof is

Lemma 2.6. *Assume that $w \in \overline{B}_\infty(0, 1)$ and $\varphi(w) \in \overline{B}_\infty(0, 1)$. Then (see Figure 7)*

$$d(\varphi(w), W_u) \leq \Big(\frac{1}{2} + C\delta\Big) d(w, W_u). \tag{2.39}$$

Proof. We write

$$w = (w_1, w_2), \quad z := \varphi(w) = (z_1, z_2).$$

Define

$$x := (w_1, F_u(w_1)), \quad y := \varphi(x) = (y_1, F_u(y_1)).$$

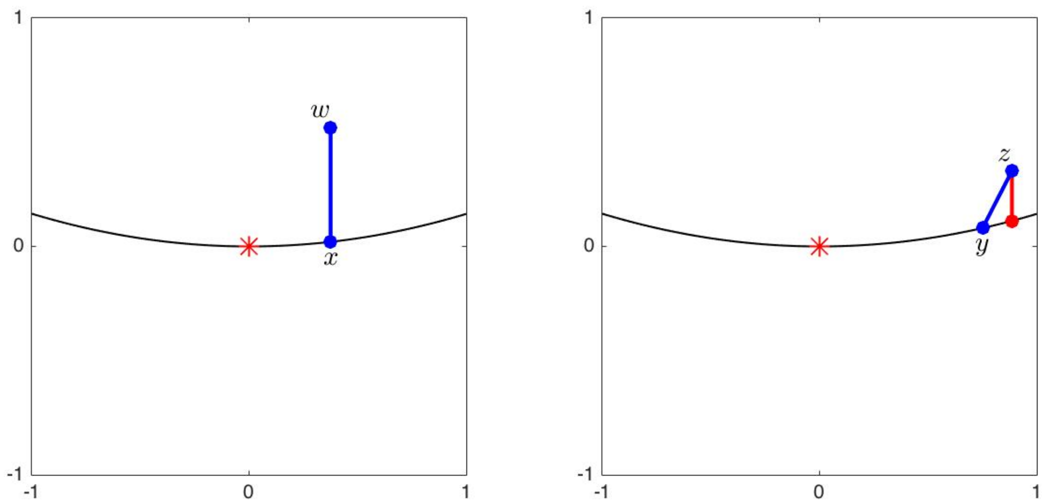

FIGURE 7. An illustration of Lemma 2.6. The curve is W_u and the picture on the right is the image of the picture on the left under φ. The blue segment on the left has length $d(w, W_u)$ and the red segment on the right has length $d(z, W_u)$.

(Since it might happen that $y_1 \notin [-1, 1]$, strictly speaking we extend the function F_u to a larger interval by making $\varphi(W_u)$ the graph of the extended F_u. The resulting function still satisfies the bound (2.29).)

By (2.16) we have

$$z - y = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} (w - x) + \mathcal{O}(\delta)|w - x|.$$

Since $|w - x| = d(w, W_u)$ this implies

$$z_1 - y_1 = \mathcal{O}(\delta) d(w, W_u), \tag{2.40}$$

$$z_2 - F_u(y_1) = \frac{1}{2}(w_2 - F_u(w_1)) + \mathcal{O}(\delta) d(w, W_u). \tag{2.41}$$

It follows from (2.29) and (2.40) that

$$F_u(z_1) - F_u(y_1) = \mathcal{O}(\delta) d(w, W_u).$$

From here and (2.41) we obtain

$$z_2 - F_u(z_1) = \frac{1}{2}(w_2 - F_u(w_1)) + \mathcal{O}(\delta) d(w, W_u).$$

Since $d(w, W_u) = |w_2 - F_u(w_1)|$ and $d(z, W_u) = |z_2 - F_u(z_1)|$ this implies (2.39). □

We now finish the proof of part (3) of Theorem 2. Assume that $w \in \overline{B}_\infty(0,1)$ and

$$w^{(n)} := \varphi^{-n}(w) \in \overline{B}_\infty(0,1) \quad \text{for all } n \geq 0.$$

Since $w^{(n-1)} = \varphi(w^{(n)})$, by Lemma 2.6 for small enough δ we have

$$d(w^{(n-1)}, W_u) \leq \frac{2}{3} d(w^{(n)}, W_u). \tag{2.42}$$

Since $d(w^{(n)}, W_u) \leq 2$ for all $n \geq 0$, we iterate this to get

$$d(w, W_u) \leq 2 \cdot \Big(\frac{2}{3}\Big)^n d(w^{(n)}, W_u) \leq \Big(\frac{2}{3}\Big)^n \quad \text{for all } n \geq 0 \tag{2.43}$$

which implies $d(w, W_u) = 0$ and thus $w \in W_u$.

Remark. The above proof in fact gives stronger versions of parts (3) and (4) of Theorem 2: if $0 \leq \sigma \leq 1$ and $n \geq 0$, then

$$w, \varphi^{-1}(w), \dots, \varphi^{-n}(w) \in \overline{B}_\infty(0,\sigma) \quad \Longrightarrow \quad d(w, W_u) \leq (2/3)^n \cdot 2\sigma, \tag{2.44}$$

$$w, \varphi(w), \dots, \varphi^{n}(w) \in \overline{B}_\infty(0,\sigma) \quad \Longrightarrow \quad d(w, W_s) \leq (2/3)^n \cdot 2\sigma. \tag{2.45}$$

Here we define $d(w, W_s) := |w_1 - F_s(w_2)|$ similarly to (2.38).

Another version of (2.44), (2.45) is available using the following estimate:

$$|w| \leq 4\big(d(w, W_u) + d(w, W_s)\big) \quad \text{for all} \quad w \in \overline{B}_\infty(0,1). \tag{2.46}$$

To prove (2.46) we use (2.29) and (2.30):

$$\begin{aligned} \frac{1}{2}|w_1| &\leq |w_1 - F_s(F_u(w_1))| \leq |w_1 - F_s(w_2)| + |F_s(w_2) - F_s(F_u(w_1))| \\ &\leq |w_1 - F_s(w_2)| + |w_2 - F_u(w_1)| = d(w, W_s) + d(w, W_u) \end{aligned}$$

and $|w_2|$ is estimated similarly.

Combining (2.44)–(2.46) we get the following bound: for $n, r \geq 0$ and $0 \leq \sigma \leq 1$

$$\begin{gathered} \varphi^{-n}(w), \dots, \varphi^{-1}(w), w, \varphi(w), \dots, \varphi^{r}(w) \in \overline{B}_\infty(0,\sigma) \\ \Longrightarrow \quad |w| \leq \big((2/3)^n + (2/3)^r\big) \cdot 8\sigma. \end{gathered} \tag{2.47}$$

The bound (2.47) can be interpreted as 'long time strict convexity': if the trajectory of a point w stays in the ball $\overline{B}_\infty(0,1)$ for long positive and negative times, then w is close to 0.

3. The general setting

In this section we explain how to extend the proofs of Theorems 1 and 2 to general families of hyperbolic transformations, yielding the general Theorem 3 stated in §3.5. Rather than give a complete formal proof we explain below several generalizations of Theorems 1 and 2 which together give Theorem 3. We refer the reader to [KaHa97, Theorem 6.2.8] for a detailed proof.

3.1. **Making δ small by rescaling.** We first show how to arrange for the assumption (3) in §2.1 (that is, φ being close to its linearization $d\varphi(0)$) to hold by a rescaling argument. Assume that $\varphi : U_\varphi \to V_\varphi$ is a C^{N+1} map and it satisfies assumptions (1)–(2) in §2.1, namely

$$\varphi(0) = 0, \quad d\varphi(0) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}. \tag{3.1}$$

Fix small $\delta_1 > 0$ and consider the rescaling map

$$T : \overline{B}_\infty(0,1) \to \overline{B}_\infty(0,\delta_1), \quad T(x) = \delta_1 x.$$

We conjugate φ by T to get the map

$$\widetilde{\varphi} := T^{-1} \circ \varphi \circ T : \overline{B}_\infty(0,1) \to \mathbb{R}^2.$$

The map $\widetilde{\varphi}$ still satisfies (3.1), and its higher derivatives are given by

$$\partial^\alpha \widetilde{\varphi}(x) = \delta_1^{|\alpha|-1} \partial^\alpha \varphi(\delta_1 x).$$

Therefore, $\widetilde{\varphi}$ satisfies the assumption (3) with $\delta = C\delta_1$, where C depends on φ. By the Inverse Mapping Theorem, $\widetilde{\varphi}$ also satisfies the assumption (4); that is, it is a diffeomorphism $\widetilde{U}_\varphi \to \widetilde{V}_\varphi$ for some open sets $\widetilde{U}_\varphi, \widetilde{V}_\varphi$ containing $\overline{B}_\infty(0,1)$.

It follows that for δ_1 small enough (depending on φ) Theorems 1 and 2 apply to $\widetilde{\varphi}$, giving the unstable/stable manifolds $\widetilde{W}_u, \widetilde{W}_s$. The manifolds

$$W_{u,\delta_1} := T(\widetilde{W}_u), \quad W_{s,\delta_1} := T(\widetilde{W}_s)$$

satisfy the conclusions of Theorems 1 and 2 for φ with the ball $\overline{B}_\infty(0,1)$ replaced by $\overline{B}_\infty(0,\delta_1)$. We call these the ($\delta_1$-)*local unstable/stable manifolds* of φ at 0.

3.2. **General expansion/contraction rates.** We next explain why Theorems 1–2 hold if the condition (3.1) is replaced by

$$\varphi(0) = 0, \quad d\varphi(0) = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{where } 0 < \lambda < 1 < \mu \text{ are fixed.} \tag{3.2}$$

Note that the value of δ for which Theorems 1 and 2 apply will depend on λ, μ, in particular it will go to 0 if $\lambda \to 1$ or $\mu \to 1$.

The proofs in §2 apply with the following changes:

- in the proof of Lemma 2.1, (2.10) is replaced by
$$\partial_{x_1} G_1(x_1) = \mu + \mathcal{O}(\delta) > 1 \quad \text{for all } x_1 \in [-1,1];$$
- in Lemma 2.2, we have
$$L_k(x_1, \tau_0, \dots, \tau_k) = \lambda\mu^{-k}\tau_k + \mathcal{O}(\delta)$$
and the estimate (2.15) is changed accordingly;
- in the estimates (2.21), (2.22), and (2.23) the constant $\frac{1}{4}$ is replaced by $\frac{\lambda}{\mu}$;

- in the contraction property (2.27) the constant $\frac{1}{3}$ is replaced by any fixed number in the interval $(\frac{\lambda}{\mu},1)$;
- in the estimate (2.31) in Theorem 2, as well as in Lemma 2.5, the constant $\frac{1}{2}$ is replaced by μ^{-1};
- in the estimate (2.32) in Theorem 2 the constant $\frac{1}{2}$ is replaced by λ;
- in Lemma 2.6 the constant $\frac{1}{2}$ is replaced by λ;
- in the estimates (2.42) and (2.43), as well as in (2.44), the constant $\frac{2}{3}$ is replaced by any fixed number in the interval $(\lambda,1)$;
- in (2.45), the constant $\frac{2}{3}$ is replaced by any fixed number in the interval $(\mu^{-1},1)$;
- in (2.47), the conclusion becomes $|w|\le(\tilde\lambda^n+\tilde\mu^{-r})\cdot 8\sigma$ where $\tilde\lambda\in(\lambda,1)$ and $\tilde\mu\in(1,\mu)$ are fixed.

3.3. **Higher dimensions.** We now generalize Theorems 1 and 2 to the case of higher dimensions. More precisely, consider a diffeomorphism

$$\varphi:U_\varphi\to V_\varphi,\quad U_\varphi,V_\varphi\subset\mathbb{R}^d,\quad \overline{B}_\infty(0,1)\subset U_\varphi\cap V_\varphi,$$

where $d=d_u+d_s$, we write elements of $\mathbb{R}^d$ as (x_1,x_2) with $x_1\in\mathbb{R}^{d_u}$, $x_2\in\mathbb{R}^{d_s}$, and (with $|\bullet|$ denoting the Euclidean norm)

$$\overline{B}_\infty(0,r):=\{(x_1,x_2)\in\mathbb{R}^d\colon\ \max(|x_1|,|x_2|)\le r\}. \tag{3.3}$$

The condition (3.1) is replaced by

$$\varphi(0)=0,\quad d\varphi(0)=\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}, \tag{3.4}$$

where $A_1:\mathbb{R}^{d_u}\to\mathbb{R}^{d_u}$, $A_2:\mathbb{R}^{d_s}\to\mathbb{R}^{d_s}$ are linear isomorphisms satisfying

$$\|A_1^{-1}\|\le\mu^{-1},\quad \|A_2\|\le\lambda,\quad \max(\|A_1\|,\|A_2^{-1}\|)\le C_0 \tag{3.5}$$

for some fixed constants λ,μ,C_0 such that $0<\lambda<1<\mu$. We assume that the bounds (2.2) on higher derivatives still hold for some small $\delta>0$.

The definitions (2.3) of the unstable/stable graphs still apply with the following adjustments. Define the unstable/stable balls

$$\overline{B}_u(0,1):=\{x_1\in\mathbb{R}^{d_u}\colon |x_1|\le 1\},\quad \overline{B}_s(0,1):=\{x_2\in\mathbb{R}^{d_s}\colon |x_2|\le 1\}. \tag{3.6}$$

Then for a C^N function $F:\overline{B}_u(0,1)\to\mathbb{R}^{d_s}$, its unstable graph $\mathcal{G}_u(F)$ is a d_u-dimensional submanifold (with boundary) of $\mathbb{R}^d$. If instead $F:\overline{B}_s(0,1)\to\mathbb{R}^{d_u}$, then the stable graph $\mathcal{G}_s(F)$ is a d_s-dimensional submanifold of $\mathbb{R}^d$.

Theorems 1 and 2 still hold for the map φ, with the constant δ now depending on d,λ,μ,C_0 and the constants in (2.31) and (2.32) modified as in §3.2. The proofs in §2 need to be modified as follows (in addition to the changes described in §3.2):

- in Lemma 2.1, the invertibility of G_1 and the fact that the image of G_1 contains $\overline{B}_u(0,1)$ follow from the standard (contraction mapping principle) proof of the Inverse Mapping Theorem, with the estimate (2.10) replaced by

$$\|\partial_{x_1} G_1(x_1) - A_1\| = \mathcal{O}(\delta) \quad \text{for all } x_1 \in \overline{B}_u(0,1);$$

- we use the notation $\mathbf{D}^k_{x_1} F(x_1) = (\partial^\alpha_{x_1} F(x_1))_{|\alpha|=k}$, giving a vector in a finite-dimensional space $\mathcal{V}^k$ which is identified with the space of homogeneous polynomials in d_u variables with values in $\mathbb{R}^{d_s}$ using the operation

$$\mathbf{D}^k_{x_1} F(x_1) \cdot v := \sum_{|\alpha|=k} \binom{k}{\alpha} \partial^\alpha_{x_1} F(x_1) v^\alpha = \partial_t|_{t=0}(F(x_1 + tv)), \quad v \in \mathbb{R}^{d_u};$$

- we use the following norms on $\mathcal{V}^k$:

$$\|\mathbf{D}^k_{x_1} F(x_1)\| := \sup \big\{ |\mathbf{D}^k_{x_1} F(x_1) \cdot v| \colon v \in \mathbb{R}^{d_u},\ |v| = 1 \big\}; \tag{3.7}$$

- in Lemma 2.2, the derivative bounds in (2.13) are now on $\|\mathbf{D}^j_{x_1} F(x_1)\|$ for $j = 1, \dots, k$;
- in Lemma 2.2, the derivative formula (2.14) is replaced by

$$\mathbf{D}^k_{x_1}(\Phi_u F)(y_1) = L_k(x_1, F(x_1), \mathbf{D}^1_{x_1} F(x_1), \dots, \mathbf{D}^k_{x_1} F(x_1)), \quad x_1 := G_1^{-1}(y_1)$$

  where $L_k(x_1, \tau_0, \tau_1, \dots, \tau_k) \in \mathcal{V}^k$ is defined for $x_1 \in \overline{B}_{\mathbb{R}^{d_u}}(0,1)$, $\tau_0 \in \overline{B}_{\mathbb{R}^{d_s}}(0,1)$ and $\tau_j \in \mathcal{V}^j$, $\|\tau_j\| \leq 1$ for $j = 1, \dots, k$;
- in Lemma 2.2, the approximation for L_k is changed to the following:

$$L_k(x_1, \tau_0, \tau_1, \dots, \tau_k) \cdot v = A_2\big(\tau_k \cdot (A_1^{-1} v)\big) + \mathcal{O}(\delta) \quad \text{for all } v \in \mathbb{R}^{d_u},\ |v| = 1$$

  and the derivative estimates (2.15) generalize naturally to the case of higher dimensions;
- in the proof of Lemma 2.2, the formula (2.17) is replaced by

$$L_1(x_1, \tau_0, \tau_1) = (A_{21}(x_1, \tau_0) + A_{22}(x_1, \tau_0)\tau_1)(A_{11}(x_1, \tau_0) + A_{12}(x_1, \tau_0)\tau_1)^{-1}$$

  where $A_{jk}(x_1, \tau_0)$ and τ_1 are now matrices;
- in the definitions (2.18) and (2.20) of $\|F\|_{C^k}$ and $\|F\|_{C^{k,1}}$ we use the norms (3.7) for the derivatives $\mathbf{D}^j_{x_1} F$, and we have the revised inequality $\|F\|_{C^{k,1}} \leq C\|F\|_{C^{k+1}}$;
- in the proof of Lemma 2.5, the equation (2.37) is replaced by

$$y_1 - \tilde{y}_1 = A_1(x_1 - \tilde{x}_1) + \mathcal{O}(\delta)|x_1 - \tilde{x}_1|;$$

- in the proof of Lemma 2.6, the equation (2.41) is replaced by

$$z_2 - F_u(y_1) = A_2(w_2 - F_u(w_1)) + \mathcal{O}(\delta) d(w, W_u).$$

3.4. **Iterating different transformations.** We next discuss a generalization of Theorems 1 and 2 from the case of a single map φ to a $\mathbb{Z}$-indexed family of maps. More precisely, we assume that

$$\varphi_m : U_\varphi \to V_\varphi, \quad m \in \mathbb{Z}$$

is a family of maps each of which satisfies the assumptions in §3.3 uniformly in m, that is the constants $\lambda, \mu, C_0, \delta$ are independent of m. The linear maps A_1, A_2 in (3.4) are allowed to depend on m.

We explain how the construction of the unstable manifold in Theorem 1 generalizes to the case of a family of transformations. The case of stable manifolds is handled similarly, and Theorem 2 generalizes naturally to this setting, see §3.5 below.

Instead of a single function F_u we construct a family of functions

$$F^u_m : \overline{B}_u(0,1) \to \overline{B}_s(0,1), \quad m \in \mathbb{Z}; \quad F^u_m(0) = 0, \quad dF^u_m(0) = 0.$$

Denote the graphs $W^u_m := \mathcal{G}_u(F^u_m)$. Then the invariance property (2.5) generalizes to

$$\varphi_m(W^u_m) \cap \overline{B}_\infty(0,1) = W^u_{m+1}. \tag{3.8}$$

To construct F^u_m we use the graph transform Φ^u_m of the map φ_m defined in Lemma 2.1. This transform satisfies the derivative bounds of Lemmas 2.3–2.4 uniformly in m.

As in §2.3, to show (3.8) it suffices to construct F^u_m such that for all $m \in \mathbb{Z}$,

$$F^u_m \in \mathcal{X}_N, \quad \Phi^u_m F^u_m = F^u_{m+1}. \tag{3.9}$$

To do this we modify the argument of §2.3 as follows: consider the space

$$\mathcal{X}_N^{\mathbb{Z}} := \{(F_m)_{m\in\mathbb{Z}} \mid F_m \in \mathcal{X}_N \text{ for all } m \in \mathbb{Z}\}$$

with the metric

$$d_N^{\mathbb{Z}}\big((F_m), (\widetilde{F}_m)\big) := \sup_{m\in\mathbb{Z}} d_N(F_m, \widetilde{F}_m).$$

Then $(\mathcal{X}_N^{\mathbb{Z}}, d_N^{\mathbb{Z}})$ is a complete metric space. Consider the map on $\mathcal{X}_N^{\mathbb{Z}}$

$$\Phi_u^{\mathbb{Z}} : (F_m) \mapsto (\widehat{F}_m), \quad \widehat{F}_{m+1} := \Phi^u_m F_m.$$

It follows from Lemma 2.4 that $\Phi_u^{\mathbb{Z}}$ is a contracting map on $(\mathcal{X}_N^{\mathbb{Z}}, d_N^{\mathbb{Z}})$. Applying the Contraction Mapping Principle, we obtain a fixed point

$$(F^u_m) \in \mathcal{X}_N^{\mathbb{Z}}, \quad \Phi_u^{\mathbb{Z}}(F^u_m) = (F^u_m)$$

which satisfies (3.9), finishing the proof.

3.5. **The general Stable/Unstable Manifold Theorem.** We finally combine the generalizations in §§3.2–3.4 and state the general version of the Stable/Unstable Manifold Theorem. We assume that:

(1) $d = d_u + d_s$, $d_u, d_s \geq 0$, elements of $\mathbb{R}^d$ are written as (x_1, x_2) where $x_1 \in \mathbb{R}^{d_u}$, $x_2 \in \mathbb{R}^{d_s}$, we use the Euclidean norm on $\mathbb{R}^d$, and $\overline{B}_\infty(0,1)$, $\overline{B}_u(0,1)$, $\overline{B}_s(0,1)$ are defined by (3.3), (3.6);

(2) we are given a family of C^{N+1} diffeomorphisms (here $N \geq 1$ is fixed)
$$\varphi_m : U_\varphi \to V_\varphi, \quad m \in \mathbb{Z}; \quad U_\varphi, V_\varphi \subset \mathbb{R}^d, \quad \overline{B}_\infty(0,1) \subset U_\varphi \cap V_\varphi; \tag{3.10}$$

(3) we have for all $m \in \mathbb{Z}$
$$\varphi_m(0) = 0, \quad d\varphi_m(0)(x_1, x_2) = (A_{1,m}x_1, A_{2,m}x_2) \tag{3.11}$$
where the linear maps $A_{1,m} : \mathbb{R}^{d_u} \to \mathbb{R}^{d_u}$, $A_{2,m} : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$ satisfy
$$\|A_{1,m}^{-1}\| \leq \mu^{-1}, \quad \|A_{2,m}\| \leq \lambda, \quad \max(\|A_{1,m}\|, \|A_{2,m}^{-1}\|) \leq C_0 \tag{3.12}$$
for some constants $0 < \lambda < 1 < \mu$, $C_0 > 0$;

(4) we have the derivative bounds for some $\delta > 0$
$$\sup_{U_\varphi} |\partial^\alpha \varphi_m| \leq \delta \quad \text{for all} \quad m \in \mathbb{Z}, \quad 2 \leq |\alpha| \leq N+1. \tag{3.13}$$

Note that the stable/unstable spaces at 0 are now given by
$$E_u(0) := \{(x_1, 0) \mid x_1 \in \mathbb{R}^{d_u}\}, \quad E_s(0) := \{(0, x_2) \mid x_2 \in \mathbb{R}^{d_s}\}.$$
The general form of Theorems 1 and 2 is then

Theorem 3. *There exists $\delta > 0$ depending only on d, N, λ, μ, C_0 such that the following holds. Assume that φ_m satisfy assumptions (1)–(4) of the present section. Then there exist families of C^N functions*
$$\begin{aligned} F_m^u &: \overline{B}_u(0,1) \to \overline{B}_s(0,1), \quad F_m^s : \overline{B}_s(0,1) \to \overline{B}_u(0,1), \\ F_m^u(0) &= 0, \quad dF_m^u(0) = 0, \quad F_m^s(0) = 0, \quad dF_m^s(0) = 0 \end{aligned} \tag{3.14}$$
bounded in C^N uniformly in m and such that the graphs
$$\begin{aligned} W_m^u &:= \{(x_1, x_2) \colon |x_1| \leq 1,\ x_2 = F_m^u(x_1)\}, \\ W_m^s &:= \{(x_1, x_2) \colon |x_2| \leq 1,\ x_1 = F_m^s(x_2)\} \end{aligned} \tag{3.15}$$
have the following properties for all $m \in \mathbb{Z}$:

(1) *$\varphi_m^{-1}(W_{m+1}^u) \subset W_m^u$ and $\varphi_m(W_m^s) \subset W_{m+1}^s$, more precisely*
$$\varphi_m(W_m^u) \cap \overline{B}_\infty(0,1) = W_{m+1}^u, \quad \varphi_m^{-1}(W_{m+1}^s) \cap \overline{B}_\infty(0,1) = W_m^s; \tag{3.16}$$

(2) *$W_m^u \cap W_m^s = \{0\}$;*

(3) *for each $x \in W_m^u$ we have $\varphi_{m-n}^{-1} \cdots \varphi_{m-1}^{-1}(x) \to 0$ as $n \to \infty$;*

(4) *for each $x \in W_m^s$ we have $\varphi_{m+n-1} \cdots \varphi_m(x) \to 0$ as $n \to \infty$;*

(5) *if* $\varphi_{m-n}^{-1}\cdots\varphi_{m-1}^{-1}(x)\in\overline{B}_\infty(0,1)$ *for all* $n\geq 0$*, then* $x\in W_m^u$*;*
(6) *if* $\varphi_{m+n-1}\cdots\varphi_m(x)\in\overline{B}_\infty(0,1)$ *for all* $n\geq 0$*, then* $x\in W_m^s$*.*

Remarks. 1. Similarly to (2.33), we obtain the following dynamical definition of the manifolds W_m^u, W_m^s:

$$\begin{aligned} x\in W_m^u &\iff \varphi_{m-n}^{-1}\cdots\varphi_{m-1}^{-1}(x)\in\overline{B}_\infty(0,1)\quad\text{for all } n\geq 0;\\ x\in W_m^s &\iff \varphi_{m+n-1}\cdots\varphi_m(x)\in\overline{B}_\infty(0,1)\quad\text{for all } n\geq 0.\end{aligned}\tag{3.17}$$

2. Similarly to Theorem 2 and Lemma 2.5, parts (3) and (4) of Theorem 3 can be made quantitative as follows. Fix $\tilde\lambda,\tilde\mu$ such that

$$0<\lambda<\tilde\lambda<1<\tilde\mu<\mu.\tag{3.18}$$

Then for δ small enough depending on $d,\lambda,\tilde\lambda,\mu,\tilde\mu,C_0$ and all $m\in\mathbb{Z}$, $n\geq 0$ we have

$$\begin{aligned} x,\tilde x\in W_m^u &\implies |\varphi_{m-n}^{-1}\cdots\varphi_{m-1}^{-1}(x)-\varphi_{m-n}^{-1}\cdots\varphi_{m-1}^{-1}(\tilde x)|\leq\tilde\mu^{-n}|x-\tilde x|,\\ x,\tilde x\in W_m^s &\implies |\varphi_{m+n-1}\cdots\varphi_m(x)-\varphi_{m+n-1}\cdots\varphi_m(\tilde x)|\leq\tilde\lambda^{n}|x-\tilde x|.\end{aligned}\tag{3.19}$$

3. Similarly to the remark at the end of §2.4, there is a quantitative version of parts (5) and (6) of Theorem 3 as well. Namely, if $\tilde\lambda,\tilde\mu$ satisfy (3.18) and δ is small enough depending on $d,\lambda,\tilde\lambda,\mu,\tilde\mu,C_0$, then for all $0\leq\sigma\leq 1$ and $n\geq 0$

$$\begin{aligned} \varphi_{m-\ell}^{-1}\cdots\varphi_{m-1}^{-1}(x)\in\overline{B}_\infty(0,\sigma),\quad \ell=0,1,\dots,n &\implies d(x,W_u)\leq\tilde\lambda^n\cdot 2\sigma;\\ \varphi_{m+\ell-1}\cdots\varphi_m(x)\in\overline{B}_\infty(0,\sigma),\quad \ell=0,1,\dots,n &\implies d(x,W_s)\leq\tilde\mu^{-n}\cdot 2\sigma.\end{aligned}\tag{3.20}$$

The following analog of the estimate (2.47) holds: for all $n,r\geq 0$ and $0\leq\sigma\leq 1$

$$\begin{aligned} \text{if}\quad &\varphi_{m-\ell}^{-1}\cdots\varphi_{m-1}^{-1}(x)\in\overline{B}_\infty(0,\sigma),\quad \ell=0,1,\dots,n\\ \text{and}\quad &\varphi_{m+\ell-1}\cdots\varphi_m(x)\in\overline{B}_\infty(0,\sigma),\quad \ell=0,1,\dots,r\\ \text{then}\quad &|x|\leq\big(\tilde\lambda^n+\tilde\mu^{-r}\big)\cdot 8\sigma.\end{aligned}\tag{3.21}$$

4. The rescaling argument of §3.1 applies to the setting of Theorem 3. That is, if φ_m satisfy (3.10)–(3.12), then one can make (3.13) hold by zooming in to a sufficiently small neighborhood of the origin.

3.6. **The case of expansion/contraction rate 1.** For applications to hyperbolic flows (which have a neutral direction) we also need to discuss which parts of Theorem 3 still hold when either λ or μ is equal to 1. Specifically, we replace the condition $\lambda<1<\mu$ with

$$\lambda=1<\mu.\tag{3.22}$$

That is, there is still expansion in the unstable directions but there does not have to be (strict) contraction in the stable directions.

The construction of the unstable manifolds W^u_m applies to the case (3.22) without any changes, and the resulting manifolds W^u_m satisfy conclusions (1) and (3) of Theorem 3. (In fact, this would work under an even weaker condition $\mu > \max(1, \lambda)$.) The estimate (3.19) still holds for W^u_m, assuming that $\tilde\mu$ satisfies $1 < \tilde\mu < \mu$.

However, we cannot construct the stable manifolds W^s_m under the assumption (3.22). The problem is in the proof of Lemma 2.1: the projection of $\varphi^{-1}(\mathcal{G}_s(F))$ onto the x_2 variable might no longer cover the unit ball, so the function $\Phi_s F$ cannot be defined. Moreover, conclusion (5) of Theorem 3 (as well as (3.20)) no longer holds, so the dynamical characterization (3.17) of the unstable manifolds W^u_m is no longer valid. The 'strict convexity' property (3.21) no longer holds.

Similarly, one can still construct the stable manifolds W^s_m and establish conclusions (1) and (4) of Theorem 3 for them if we replace the condition $\lambda < 1 < \mu$ with

$$\lambda < 1 = \mu. \tag{3.23}$$

4. Hyperbolic maps and flows

In this section we apply Theorem 3 to obtain the Stable/Unstable Manifold Theorem for hyperbolic maps (Theorem 4 in §4.1) and for hyperbolic flows (Theorem 5 in §4.6). This involves constructing an adapted metric (Lemma 4.4) and taking adapted coordinates to bring the map/flow into the model case handled by Theorem 3.

4.1. Hyperbolic maps.

Assume that M is a d-dimensional manifold without boundary and $d = d_u + d_s$ where $d_u, d_s \geq 0$. Let $\varphi : M \to M$ be a C^{N+1} diffeomorphism (here $N \geq 1$ is fixed) and assume that φ is hyperbolic on some compact φ-invariant set $K \subset M$ in the following sense:

Definition 4.1. *Let $K \subset M$ be a compact set such that $\varphi(K) = K$. We say that φ is **hyperbolic** on K if there exists a splitting*

$$T_xM = E_u(x) \oplus E_s(x), \quad x \in K \tag{4.1}$$

where $E_u(x), E_s(x) \subset T_xM$ are subspaces of dimensions d_u, d_s and:

- *E_u, E_s are invariant under $d\varphi$, namely*
$$d\varphi(x)E_u(x) = E_u(\varphi(x)), \quad d\varphi(x)E_s(x) = E_s(\varphi(x)) \quad \text{for all } x \in K; \tag{4.2}$$
- *large negative iterates of φ are contracting on E_u, namely there exist constants $C > 0$, $0 < \lambda < 1$ such that for some Riemannian metric $|\bullet|$ on M we have*
$$|d\varphi^{-n}(x)v| \leq C\lambda^n|v| \quad \text{for all} \quad v \in E_u(x),\ x \in K,\ n \geq 0; \tag{4.3}$$
- *large positive iterates of φ are contracting on E_s, namely*
$$|d\varphi^n(x)v| \leq C\lambda^n|v| \quad \text{for all} \quad v \in E_s(x),\ x \in K,\ n \geq 0. \tag{4.4}$$

Remarks. 1. The contraction properties (4.3), (4.4) do not depend on the choice of the metric on M, though the constant C (but not λ) will depend on the metric. Later in Lemma 4.4 we construct metrics which give (4.3), (4.4) with $C = 1$.

2. We do not assume a priori that the maps $x \mapsto E_u(x), E_s(x)$ are continuous. However we show in §4.2 below that Definition 4.1 implies that these maps are in fact Hölder continuous.

3. The basic example of a hyperbolic set is $K = \{x_0\}$ where $x_0 \in M$ is a fixed point of φ which is hyperbolic, namely $d\varphi(x_0)$ has no eigenvalues on the unit circle. More generally one can take as K a hyperbolic closed trajectory of φ. The opposite situation is when $K = M$; in this case φ is called an *Anosov diffeomorphism.*

4. Similarly to §3.5 we could take two different constants in (4.3), (4.4), corresponding to different minimal contraction rates in the unstable and the stable directions. We do not do this here to simplify notation, and since the examples in this note have time-reversal symmetry and thus equal stable/unstable contraction rates.

Fix a Riemannian metric on M which induces a distance function $d(\bullet, \bullet)$; denote for $x \in M$ and $r \geq 0$

$$\overline{B}_d(x, r) := \{y \in M \mid d(x, y) \leq r\}. \tag{4.5}$$

We now state the Stable/Unstable Manifold Theorem for hypebolic maps:

Theorem 4. *Assume that φ is hyperbolic on $K \subset M$. Then for each $x \in K$ there exist **local unstable/stable manifolds***

$$W_u(x), W_s(x) \subset M$$

which have the following properties for some $\varepsilon_0 > 0$ depending only on φ, K:

1. *$W_u(x), W_s(x)$ are C^N embedded disks of dimensions d_u, d_s, that is images of closed balls in $\mathbb{R}^{d_u}$, $\mathbb{R}^{d_s}$ under C^N embeddings, and the C^N norms of these embeddings are bounded uniformly in x;*
2. *$W_u(x) \cap W_s(x) = \{x\}$ and $T_xW_u(x) = E_u(x)$, $T_xW_s(x) = E_s(x)$;*
3. *the boundaries of $W_u(x), W_s(x)$ do not intersect $\overline{B}_d(x, \varepsilon_0)$;*
4. *$\varphi^{-1}(W_u(x)) \subset W_u(\varphi^{-1}(x))$ and $\varphi(W_s(x)) \subset W_s(\varphi(x))$;*
5. *for each $y \in W_u(x)$, we have $d(\varphi^{-n}(y), \varphi^{-n}(x)) \to 0$ as $n \to \infty$;*
6. *for each $y \in W_s(x)$, we have $d(\varphi^{n}(y), \varphi^{n}(x)) \to 0$ as $n \to \infty$;*
7. *if $y \in M$ and $d(\varphi^{-n}(y), \varphi^{-n}(x)) \leq \varepsilon_0$ for all $n \geq 0$, then $y \in W_u(x)$;*
8. *if $y \in M$ and $d(\varphi^{n}(y), \varphi^{n}(x)) \leq \varepsilon_0$ for all $n \geq 0$, then $y \in W_s(x)$;*
9. *if $x, y \in K$ and $d(x, y) \leq \varepsilon_0$ then $W_s(x) \cap W_u(y)$ consists of exactly one point.*

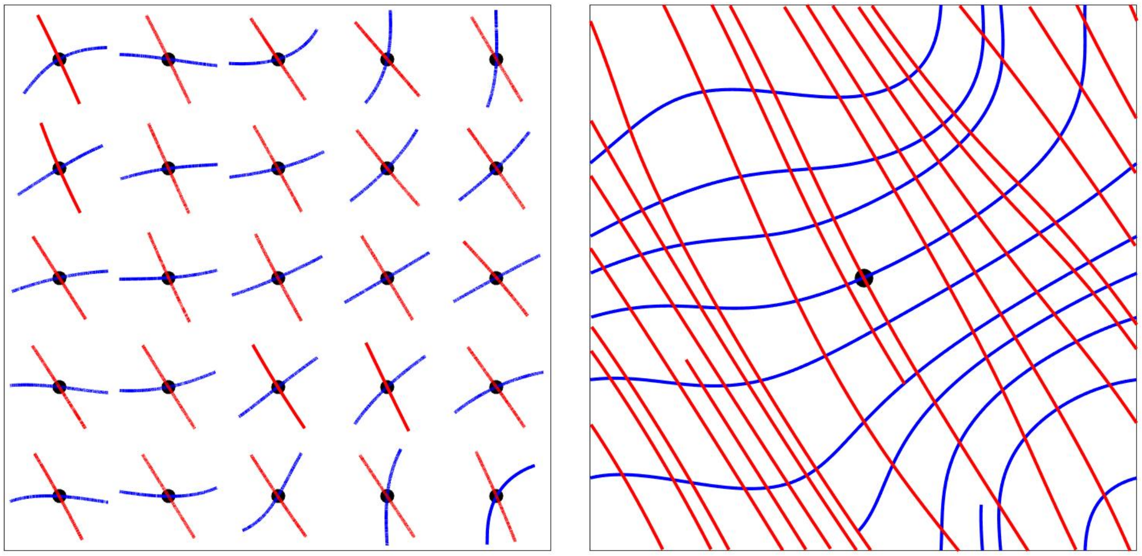

FIGURE 8. The unstable (blue) and stable (red) manifolds for a perturbed Arnold cat map on the torus $\mathbb{R}^2/\mathbb{Z}^2$. On the left are the local stable/unstable manifolds $W_u(x), W_s(x)$ for several choices of x. On the right are the manifolds $W_u^{(k)}(x), W_s^{(k)}(x)$ for $x = (0.5, 0.5)$ and $k = 4$.

Remarks. 1. Similarly to (3.19) there are quantitative versions of the statements (5) and (6): if we fix $\tilde\lambda$ such that $\lambda < \tilde\lambda < 1$ then for all $n \geq 0$ and $x \in K$

$$\begin{aligned} y, \tilde y \in W_u(x) \quad &\Longrightarrow \quad d(\varphi^{-n}(y), \varphi^{-n}(\tilde y)) \leq C\tilde\lambda^n d(y, \tilde y); \\ y, \tilde y \in W_s(x) \quad &\Longrightarrow \quad d(\varphi^{n}(y), \varphi^{n}(\tilde y)) \leq C\tilde\lambda^n d(y, \tilde y). \end{aligned} \tag{4.6}$$

where C is a constant depending only on $\varphi, K, \tilde\lambda$. Here and in Remark 2 below the manifolds W_u, W_s and the constant ε_0 depend on $\tilde\lambda$, in particular when $\tilde\lambda \to \lambda$ the stable/unstable manifolds might degenerate to a point and ε_0 might go to 0. (One can get rid of this dependence, but it is rather tedious and typically unnecessary.)

2. Similarly to (3.20) there are quantitative versions of the statements (7) and (8) as well: if we fix $\tilde\lambda$ as before then for all $n \geq 0$, $0 \leq \sigma \leq \varepsilon_0$, $x \in K$, and $y \in M$

$$\begin{aligned} d(\varphi^{-\ell}(y), \varphi^{-\ell}(x)) \leq \sigma \text{ for all } \ell = 0, \dots, n \quad &\Longrightarrow \quad d(y, W_u(x)) \leq C\tilde\lambda^n \sigma, \\ d(\varphi^{\ell}(y), \varphi^{\ell}(x)) \leq \sigma \text{ for all } \ell = 0, \dots, n \quad &\Longrightarrow \quad d(y, W_s(x)) \leq C\tilde\lambda^n \sigma \end{aligned} \tag{4.7}$$

where C is a constant depending only on $\varphi, K, \tilde\lambda$. We also have the following analog of the 'strict convexity' property (3.21): for all $n \geq 0$, $0 \leq \sigma \leq \varepsilon_0$, $x \in K$, and $y \in M$

$$d(\varphi^{\ell}(y), \varphi^{\ell}(x)) \leq \sigma \quad \text{for all} \quad |\ell| \leq n \quad \Longrightarrow \quad d(y, x) \leq C\tilde\lambda^n \sigma. \tag{4.8}$$

3. The manifolds $W_u(x)$ are not defined canonically since they have a somewhat arbitrarily determined boundary. For the same reason, if $W_u(x) \cap W_u(y) \neq \emptyset$, this does

not imply that $W_u(x) = W_u(y)$. However, we can show that in this case $W_u(x)$ and $W_u(y)$ are subsets of the same d_u-dimensional manifold, see (4.10) below. For $k \geq 0$ and $x \in K$ define

$$W_u^{(k)}(x) := \varphi^k(W_u(\varphi^{-k}(x))). \tag{4.9}$$

This is still a d_u-dimensional embedded disk in M. Moreover, by statement (4) in Theorem 4 we have

$$W_u(x) = W_u^{(0)}(x) \subset W_u^{(1)}(x) \subset \cdots \subset W_u^{(k)}(x) \subset W_u^{(k+1)}(x) \subset \dots$$

There exists $k_0 \geq 0$ such that for all $x, y \in K$

$$W_u(x) \cap W_u(y) \neq \emptyset \quad \Longrightarrow \quad W_u(x) \cup W_u(y) \subset W_u^{(k_0)}(x). \tag{4.10}$$

Indeed, assume that $z \in W_u(x) \cap W_u(y)$. By (4.6) if k_0 is large enough then

$$d(\varphi^{-k}(x), \varphi^{-k}(y)) \leq d(\varphi^{-k}(z), \varphi^{-k}(x)) + d(\varphi^{-k}(z), \varphi^{-k}(y)) \leq \frac{\varepsilon_0}{2} \quad \text{for all } k \geq k_0.$$

Let $w \in W_u(x) \cup W_u(y)$. Then for k_0 large enough we get from (4.6)

$$d(\varphi^{-k}(w), \varphi^{-k}(x)) \leq \varepsilon_0 \quad \text{for all } k \geq k_0. \tag{4.11}$$

It follows from statement (7) in Theorem 4 that $\varphi^{-k_0}(w) \in W_u(\varphi^{-k_0}(x))$ and thus $w \in W_u^{(k_0)}(x)$, proving (4.10).

Note that (4.10) implies that the tangent spaces to $W_u(x)$ are the unstable spaces:

$$y \in W_u(x) \cap K \quad \Longrightarrow \quad T_y(W_u(x)) = E_u(y). \tag{4.12}$$

The above discussion applies to stable manifolds where we define

$$W_s^{(k)}(x) := \varphi^{-k}(W_s(\varphi^k(x))). \tag{4.13}$$

4. Here is another version of 'local uniqueness' of stable/unstable manifolds: there exists $\varepsilon_1 > 0$ such that for all $x, y \in K$ we have

$$W_u(x) \cap W_u(y) \neq \emptyset \quad \Longrightarrow \quad W_u(y) \cap \overline{B}_d(x, \varepsilon_1) \subset W_u(x), \tag{4.14}$$

$$W_s(x) \cap W_s(y) \neq \emptyset \quad \Longrightarrow \quad W_s(y) \cap \overline{B}_d(x, \varepsilon_1) \subset W_s(x). \tag{4.15}$$

We show (4.14), with (4.15) proved similarly. Let $w \in W_u(y) \cap \overline{B}_d(x, \varepsilon_1)$. By (4.11) we have $d(\varphi^{-k}(w), \varphi^{-k}(x)) \leq \varepsilon_0$ for all $k \geq k_0$. On the other hand

$$d(\varphi^{-k}(w), \varphi^{-k}(x)) \leq C d(w, x) \leq C\varepsilon_1 \quad \text{for} \quad 0 \leq k < k_0.$$

Choosing ε_1 small enough we get $d(\varphi^{-k}(w), \varphi^{-k}(x)) \leq \varepsilon_0$ for all $k \geq 0$, which by statement (7) in Theorem 4 gives $w \in W_u(x)$ as needed.

5. One can take the unions of the manifolds (4.9), (4.13) to obtain *global unstable/stable manifolds*: for $x \in K$,

$$W_u^{(\infty)}(x) := \bigcup_{k \geq 0} W_u^{(k)}(x), \quad W_s^{(\infty)}(x) := \bigcup_{k \geq 0} W_s^{(k)}(x). \tag{4.16}$$

By statements (5)–(8) in Theorem 4 we can characterize these dynamically as follows:

$$\begin{aligned} y \in W_u^{(\infty)}(x) &\iff d(\varphi^{-n}(y), \varphi^{-n}(x)) \to 0 \quad \text{as } n \to \infty; \\ y \in W_s^{(\infty)}(x) &\iff d(\varphi^{n}(y), \varphi^{n}(x)) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.17}$$

Therefore the global stable/unstable manifolds are disjoint: if $W_u^{(\infty)}(x) \cap W_u^{(\infty)}(y) \neq \emptyset$, then $W_u^{(\infty)}(x) = W_u^{(\infty)}(y)$, and same is true for $W_s^{(\infty)}$.

The sets $W_u^{(\infty)}(x)$ and $W_s^{(\infty)}(x)$ are d_u and d_s-dimensional immersed submanifolds without boundary in M, however they are typically not embedded. In fact, in many cases these submanifolds are dense in M. See Figure 8.

4.2. **Continuity of the stable/unstable spaces.** In this section we study the regularity of the maps $x \mapsto E_u(x), E_s(x)$. To talk about these, it is convenient to introduce the Grassmanians

$$\begin{aligned} \mathscr{G}_u &:= \{(x, E) \colon x \in M,\ E \subset T_xM \text{ is a } d_u\text{-dimensional subspace}\}, \\ \mathscr{G}_s &:= \{(x, E) \colon x \in M,\ E \subset T_xM \text{ is a } d_s\text{-dimensional subspace}\}. \end{aligned}$$

These are smooth manifolds fibering over M. If φ is hyperbolic on K, then we have the maps

$$E_u : K \to \mathscr{G}_u, \quad E_s : K \to \mathscr{G}_s. \tag{4.18}$$

We first show that $E_u(x), E_s(x)$ depend continuously on x:

Lemma 4.2. *Assume that φ is hyperbolic on K. Then the maps (4.18) are continuous.*

Proof. We show continuity of E_s, the continuity of E_u is proved similarly. It suffices to show that if

$$x_k \in K, \quad x_k \to x_\infty, \quad v_k \in E_s(x_k), \quad |v_k| = 1, \quad v_k \to v_\infty \in T_{x_\infty}M$$

then $v_\infty \in E_s(x_\infty)$.

By (4.4) we have for all k and all $n \geq 0$

$$|d\varphi^n(x_k)v_k| \leq C\lambda^n.$$

Passing to the limit $k \to \infty$, we get for all $n \geq 0$

$$|d\varphi^n(x_\infty)v_\infty| \leq C\lambda^n.$$

Given (4.1) and (4.3) this implies that $v_\infty \in E_s(x_\infty)$, finishing the proof. □

Lemma 4.2 and the fact that $E_u(x), E_s(x)$ are transverse for each x implies that $E_u(x)$, $E_s(x)$ are uniformly transverse, namely there exists a constant C such that

$$\max\big(|v_u|, |v_s|\big) \leq C|v_u + v_s| \quad \text{for all} \quad v_u \in E_u(x),\ v_s \in E_s(x),\ x \in K. \tag{4.19}$$

A quantitative version of the proof of Lemma 4.2 shows that in fact $E_u(x), E_s(x)$ are Hölder continuous in x. (This statement is not used in the rest of these notes.)

Lemma 4.3. *Fix some smooth metrics on $\mathscr{G}_u, \mathscr{G}_s$. Assume that φ is hyperbolic on K. Then there exists $\gamma > 0$ such that the maps (4.18) have C^γ regularity.*

Proof. In this proof we denote by C constants which only depend on φ, K.

We show Hölder continuity of E_s, with the case of E_u handled similarly. Let $\overline{T}M$ be the fiber-radial compactification of TM, which is a manifold with interior TM and boundary diffeomorphic to the sphere bundle SM, with the boundary defining function $|v|^{-1}$. Denote by Φ the action of $d\varphi$ on TM:

$$\Phi(x, v) := (\varphi(x), d\varphi(x)v), \quad (x, v) \in TM.$$

Note that $\Phi^n(x, v) = (\varphi^n(x), d\varphi^n(x)v)$. Moreover, since Φ is homogeneous with respect to dilations in the fibers of TM, it extends to a smooth map on $\overline{T}M$.

Fix a Riemannian metric on $\overline{T}M$ (smooth up to the boundary) and denote by $d_{\overline{T}M}$ the corresponding distance function. Then there exists a constant $\Lambda \geq 1$ depending only on φ, K such that

$$d_{\overline{T}M}\big(\Phi(x, v), \Phi(y, w)\big) \leq \Lambda \cdot d_{\overline{T}M}\big((x, v), (y, w)\big) \quad \text{for} \quad x, y \in K. \tag{4.20}$$

We will show that the map E_s is Hölder continuous with exponent

$$\gamma := -\frac{2 \log \lambda}{\log(\Lambda/\lambda)} > 0.$$

To do this, assume that

$$x, y \in K, \quad v \in E_s(x), \quad w \in T_yM, \quad |v| = 1, \quad d_{\overline{T}M}\big((x, v), (y, w)\big) \leq Cd(x, y).$$

We write

$$w = w_u + w_s, \quad w_u \in E_u(y), \quad w_s \in E_s(y).$$

Then it suffices to prove that

$$|w_u| \leq Cd(x, y)^\gamma. \tag{4.21}$$

To show (4.21), we use the following bound true for all $n \geq 0$:

$$\begin{aligned} d_{\overline{T}M}\big(\Phi^n(y, w), (\varphi^n(x), 0)\big) &\leq C\Lambda^n d(x, y) + d_{\overline{T}M}\big(\Phi^n(x, v), (\varphi^n(x), 0)\big) \\ &\leq C\Lambda^n d(x, y) + C\lambda^n \end{aligned} \tag{4.22}$$

where the first inequality uses (4.20) iterated n times and the second one uses (4.4).

Assume that $d(x, y)$ is small and choose

$$n := \Big\lfloor -\frac{\log d(x, y)}{\log(\Lambda/\lambda)} \Big\rfloor \geq 0.$$

Then $\Lambda^n d(x, y) \leq \lambda^n$, so (4.22) implies

$$d_{\overline{T}M}\big(\Phi^n(y, w), (\varphi^n(x), 0)\big) \leq C\lambda^n.$$

If $d(x,y)$ is small, then n is large, thus $\Phi^n(y,w)$ is close to the zero section. Therefore

$$|d\varphi^n(y)w| \le C\lambda^n. \tag{4.23}$$

Recalling the decomposition $w = w_u + w_s$ and using the bounds (4.3), (4.4), and (4.19), we see that (4.23) implies

$$|w_u| \le C\lambda^n |d\varphi^n(y)w_u| \le C\lambda^n |d\varphi^n(y)w| + C\lambda^{2n} \le C\lambda^{2n} \le Cd(x,y)^\gamma.$$

This gives (4.21), finishing the proof. □

4.3. **Adapted metrics.** In preparation for the proof of Theorem 4, we show that there exist Riemannian metrics on M which are adapted to the map φ:

Lemma 4.4. *Assume that φ is hyperbolic on K. Fix $\tilde\lambda$ such that $\lambda < \tilde\lambda < 1$ where λ is given in Definition 4.1. Then there exist C^N Riemannian metrics $|\bullet|_u, |\bullet|_s$ on M such that*

$$|d\varphi^{-1}(x)v|_u \le \tilde\lambda |v|_u \quad \text{for all} \quad v \in E_u(x),\ x \in K; \tag{4.24}$$

$$|d\varphi(x)v|_s \le \tilde\lambda |v|_s \quad \text{for all} \quad v \in E_s(x),\ x \in K. \tag{4.25}$$

Remark. Iterating (4.24), (4.25) we get analogs of (4.3), (4.4) with $C = 1$.

Proof. We first construct the metric $|\bullet|_s$. Let $|\bullet|$ be a Riemannian metric on M. Take large fixed m to be chosen later and define the Riemannian metric $|\bullet|_s$ by

$$|v|_s^2 := \sum_{n=0}^{m-1} \tilde\lambda^{-2n} |d\varphi^n(x)v|^2, \quad x \in M, \quad v \in T_xM.$$

Assume that $x \in K$ and $v \in E_s(x)$. Then

$$\begin{aligned}
|d\varphi(x)v|_s^2 &= \sum_{n=0}^{m-1} \tilde\lambda^{-2n} |d\varphi^{n+1}(x)v|^2 = \sum_{n=1}^{m} \tilde\lambda^{2-2n} |d\varphi^n(x)v|^2 \\
&= \tilde\lambda^2 \big(|v|_s^2 - |v|^2 + \tilde\lambda^{-2m} |d\varphi^m(x)v|^2\big) \\
&\le \tilde\lambda^2 \big(|v|_s^2 - |v|^2 + C\lambda^{2m}\tilde\lambda^{-2m} |v|^2\big)
\end{aligned}$$

where in the last inequality we used (4.4) and C is a constant depending only on φ, K. Since $\tilde\lambda > \lambda$, choosing m large enough we can guarantee that $C\lambda^{2m}\tilde\lambda^{-2m} \le 1$, thus (4.25) holds.

The inequality (4.24) is proved similarly, using the metric

$$|v|_u^2 := \sum_{n=0}^{m-1} \tilde\lambda^{-2n} |d\varphi^{-n}(x)v|^2, \quad x \in M, \quad v \in T_xM. \quad \square$$

4.4. **Adapted charts.** To reduce Theorem 4 to Theorem 3, we introduce charts on M which are adapted to the map φ. We assume that φ is hyperbolic on $K \subset M$, fix $\tilde{\lambda} \in (\lambda, 1)$ and let $|\bullet|_u$, $|\bullet|_s$ be the Riemannian metrics on M constructed in Lemma 4.4. We write the elements of $\mathbb{R}^d$ as (x_1, x_2) where $x_1 \in \mathbb{R}^{d_u}$, $x_2 \in \mathbb{R}^{d_s}$. We use the canonical stable/unstable subspaces of $\mathbb{R}^d$

$$E_u(0) := \{(v_1, 0) \mid v_1 \in \mathbb{R}^{d_u}\}, \quad E_s(0) := \{(0, v_2) \mid v_2 \in \mathbb{R}^{d_s}\}. \tag{4.26}$$

Recall from (3.3) the notation

$$\overline{B}_\infty(0, r) = \{(x_1, x_2) \in \mathbb{R}^d \colon \max(|x_1|, |x_2|) \leq r\}.$$

Definition 4.5. *Let $x_0 \in K$. A diffeomorphism*

$$\varkappa : U_\varkappa \to V_\varkappa, \quad x_0 \in U_\varkappa \subset M, \quad 0 \in V_\varkappa \subset \mathbb{R}^d$$

is called an ***adapted chart*** *for φ centered at x_0, if:*

1. *$\varkappa(x_0) = 0$;*
2. *$d\varkappa(x_0) E_u(x_0) = E_u(0)$ and the restriction of $d\varkappa(x_0)$ to $E_u(x_0)$ is an isometry from the metric $|\bullet|_u$ to the Euclidean metric;*
3. *$d\varkappa(x_0) E_s(x_0) = E_s(0)$ and the restriction of $d\varkappa(x_0)$ to $E_s(x_0)$ is an isometry from the metric $|\bullet|_s$ to the Euclidean metric.*

For each $x_0 \in K$, there exists an adapted chart for φ centered at x_0. Moreover, it follows from uniform transversality (4.19) of E_u, E_s that we can select for each $x_0 \in K$ an adapted chart for φ centered at x_0

$$\varkappa_{x_0} : U_{x_0} \to V_{x_0}, \quad x_0 \in U_{x_0} \subset M, \quad 0 \in V_{x_0} \subset \mathbb{R}^d \tag{4.27}$$

such that the set $\{\varkappa_{x_0} \mid x_0 \in K\}$ is bounded in the class of C^{N+1} charts, more precisely:

1. there exists $\delta_0 > 0$ such that $\overline{B}_\infty(0, \delta_0) \subset V_{x_0}$ for all $x_0 \in K$;
2. all order $\leq N+1$ derivatives of $\varkappa_{x_0}$ and $\varkappa_{x_0}^{-1}$ are bounded uniformly in x_0.

Note that we do not require continuous dependence of $\varkappa_{x_0}$ on x_0, in fact in many cases such dependence is impossible because the bundles E_u, E_s are not topologically trivial.

We now study the action of the map φ in adapted charts. For each $x_0 \in K$, define the diffeomorphism ψ_{x_0} of neighborhoods of 0 in $\mathbb{R}^d$ by

$$\psi_{x_0} := \varkappa_{\varphi(x_0)} \circ \varphi \circ \varkappa_{x_0}^{-1}. \tag{4.28}$$

Note that the set $\{\psi_{x_0} \mid x_0 \in K\}$ is bounded in the class of C^{N+1} diffeomorphisms.

From the definition of adapted charts and the fact that $E_u(x), E_s(x)$ are φ-invariant we see that

$$\psi_{x_0}(0) = 0, \quad d\psi_{x_0}(0) = \begin{pmatrix} A_{1,x_0} & 0 \\ 0 & A_{2,x_0} \end{pmatrix} \tag{4.29}$$

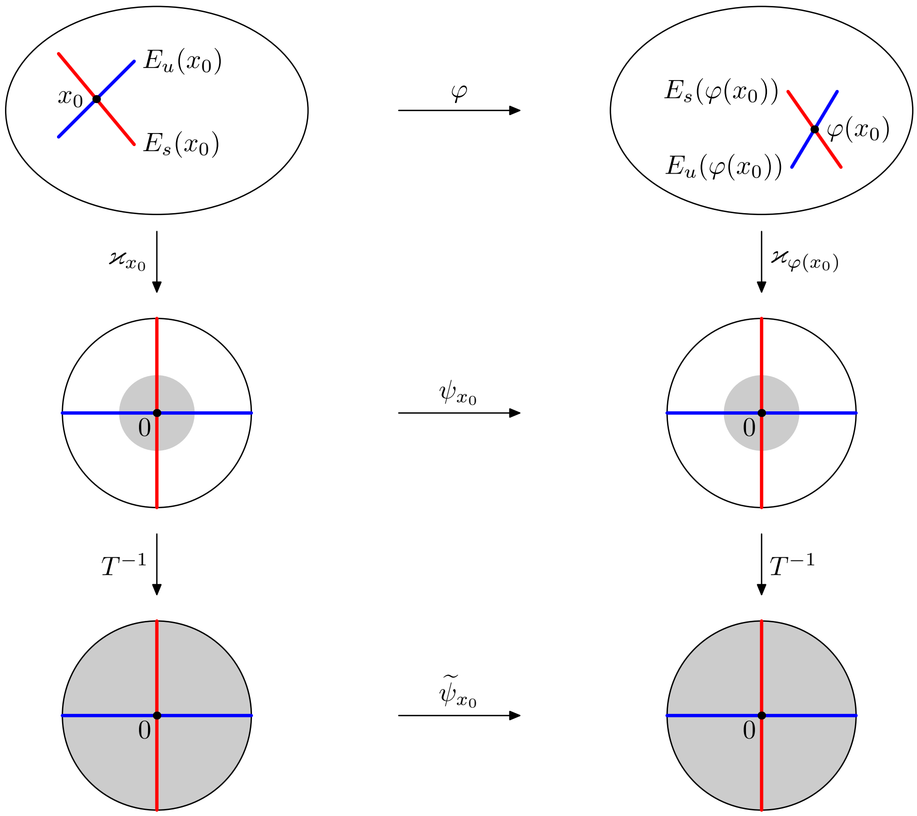

FIGURE 9. An illustration of the commutative diagram (4.33). The blue/red lines are the unstable/stable subspaces of the tangent spaces.

where $A_{1,x_0} : \mathbb{R}^{d_u} \to \mathbb{R}^{d_u}$, $A_{2,x_0} : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$ are linear isomorphisms. Moreover, from the properties (4.24), (4.25) of the adapted metrics $|\bullet|_u$, $|\bullet|_s$ we get

$$\|A_{1,x_0}^{-1}\| \leq \tilde{\lambda}, \quad \|A_{2,x_0}\| \leq \tilde{\lambda}, \quad \max(\|A_{1,x_0}\|, \|A_{2,x_0}^{-1}\|) \leq C_0 \quad \text{for all } x_0 \in K \tag{4.30}$$

for some fixed C_0, where $\|\bullet\|$ is the operator norm with respect to the Euclidean norm.

To make the maps ψ_{x_0} close to their linearizations $d\psi_{x_0}(0)$, we use the rescaling procedure introduced in §3.1. Fix small $\delta_1 > 0$ to be chosen later (when we apply Theorem 3) and consider the rescaling map

$$T : \mathbb{R}^d \to \mathbb{R}^d, \quad T(x) = \delta_1 x.$$

Define the rescaled charts

$$\widetilde{\varkappa}_{x_0} := T^{-1} \circ \varkappa_{x_0}, \quad x_0 \in K \tag{4.31}$$

and the corresponding maps

$$\widetilde{\psi}_{x_0} := \widetilde{\varkappa}_{\varphi(x_0)} \circ \varphi \circ \widetilde{\varkappa}_{x_0}^{-1} = T^{-1} \circ \psi_{x_0} \circ T. \tag{4.32}$$

That is, we have the commutative diagram (see also Figure 9)

$$
\begin{array}{ccccc}
 & M & \xrightarrow{\ \varphi\ } & M & \\
 & \downarrow{\scriptstyle \varkappa_{x_0}} & & \downarrow{\scriptstyle \varkappa_{\varphi(x_0)}} & \\
\widetilde{\varkappa}_{x_0}\Big\downarrow & \mathbb{R}^d & \xrightarrow{\ \psi_{x_0}\ } & \mathbb{R}^d & \Big\downarrow\widetilde{\varkappa}_{\varphi(x_0)} \\
 & \downarrow{\scriptstyle T^{-1}} & & \downarrow{\scriptstyle T^{-1}} & \\
 & \mathbb{R}^d & \xrightarrow{\ \widetilde{\psi}_{x_0}\ } & \mathbb{R}^d &
\end{array}
\quad . \tag{4.33}
$$

We choose δ_1 small enough so that $\overline{B}_\infty(0,1)$ is contained in both the domain and the range of each $\widetilde{\psi}_{x_0}$. The map $\widetilde{\psi}_{x_0}$ still satisfies (4.29):

$$
\widetilde{\psi}_{x_0}(0) = 0, \quad d\widetilde{\psi}_{x_0}(0) = \begin{pmatrix} A_{1,x_0} & 0 \\ 0 & A_{2,x_0} \end{pmatrix} \tag{4.34}
$$

where A_{j,x_0} are the same transformations as in (4.29).

Moreover, as explained in §3.1, for any given $\delta > 0$, if we choose δ_1 small enough depending on δ, φ, K (but not on x_0) then we have the derivative bounds

$$
\sup |\partial^\alpha \widetilde{\psi}_{x_0}| \le \delta \quad \text{for all} \quad x_0 \in K, \quad 2 \le |\alpha| \le N+1. \tag{4.35}
$$

In fact, one can take $\delta_1 := \delta/C$ where C is some constant depending on φ, K.

4.5. **Proof of Theorem 4.** We now give the proof of the Stable/Unstable Manifold Theorem for hyperbolic maps. We fix $\tilde{\lambda}$ such that $\lambda < \tilde{\lambda} < 1$. Let $\delta > 0$ be the constant in Theorem 3 where we use $\tilde{\lambda}, \tilde{\lambda}^{-1}$ in place of λ, μ and C_0 is the constant from (4.30). Choose the rescaling parameter $\delta_1 > 0$ in §4.4 small enough so that (4.35) holds.

We use the rescaled adapted charts $\widetilde{\varkappa}_{x_0}$, $x_0 \in K$, defined in (4.31), and the maps $\widetilde{\psi}_{x_0}$ giving the action of φ in these charts, defined in (4.32). For every $x \in K$ and $m \in \mathbb{Z}$ denote

$$
\psi_{x,m} := \widetilde{\psi}_{\varphi^m(x)} = \widetilde{\varkappa}_{\varphi^{m+1}(x)} \circ \varphi \circ \widetilde{\varkappa}^{-1}_{\varphi^m(x)}. \tag{4.36}
$$

The dynamics of the iterates of φ near the trajectory $(\varphi^m(x))$ is conjugated by the charts $\widetilde{\varkappa}_{\varphi^m(x)}$ to the dynamics of the compositions of the maps $\psi_{x,m}$. We will prove Theorem 4 by applying Theorem 3 to the maps $\psi_{x,m}$ and pulling back the resulting stable/unstable manifolds by $\widetilde{\varkappa}_{\varphi^m(x)}$ to get the stable/unstable manifolds for φ.

As shown in §4.4, for each $x \in K$ the sequence of maps $(\psi_{x,m})_{m\in\mathbb{Z}}$ satisfies the assumptions in §3.5, where we use $\tilde{\lambda}, \tilde{\lambda}^{-1}$ in place of λ, μ. We apply Theorem 3 to get the stable/unstable manifolds for this sequence, which we denote

$$
W^u_{x,m}, W^s_{x,m} \subset \overline{B}_\infty(0,1) \subset \mathbb{R}^d.
$$

We define the unstable/stable manifolds for φ at x as follows:

$$W_u(x) := \widetilde{\varkappa}_x^{-1}(W^u_{x,0}), \quad W_s(x) := \widetilde{\varkappa}_x^{-1}(W^s_{x,0}). \tag{4.37}$$

The unstable/stable manifolds at the iterates $\varphi^n(x)$ are given by

$$W_u(\varphi^n(x)) = \widetilde{\varkappa}^{-1}_{\varphi^n(x)}(W^u_{x,n}), \quad W_s(\varphi^n(x)) = \widetilde{\varkappa}^{-1}_{\varphi^n(x)}(W^s_{x,n}), \quad n \in \mathbb{Z}. \tag{4.38}$$

The statement (4.38) is not a tautology since the left-hand sides were obtained by applying Theorem 3 to the sequence of maps $(\psi_{\varphi^n(x),m})_{m\in\mathbb{Z}}$ while the right-hand sides were obtained using the maps $(\psi_{x,m})_{m\in\mathbb{Z}}$. To prove (4.38) we note that (4.36) implies

$$\psi_{\varphi^n(x),m} = \psi_{x,m+n}.$$

Therefore, the sequence $(\psi_{\varphi^n(x),m})_{m\in\mathbb{Z}}$ is just a shift of the sequence $(\psi_{x,m})_{m\in\mathbb{Z}}$. From the construction of the manifolds $W^u_{x,m}$, $W^s_{x,m}$ in §3.4 we see that

$$W^u_{\varphi^n(x),m} = W^u_{x,m+n}, \quad W^s_{\varphi^n(x),m} = W^s_{x,m+n}.$$

Putting $m = 0$ and recalling (4.37), we get (4.38).

We now show that the manifolds $W_u(x)$, $W_s(x)$ defined in (4.37) satisfy the statements (1)–(8) in Theorem 4. This is straightforward since the above construction effectively reduced Theorem 4 to Theorem 3.

(1): This follows from the definitions (3.15) of $W^u_{x,m}, W^s_{x,m}$. The uniform boundedness of the embeddings in C^N follows from the fact that the functions $F^u_{x,m}$, $F^s_{x,m}$ used to define $W^u_{x,m}$, $W^s_{x,m}$ are bounded by 1 in C^N norm, see §3.4 and (2.26).

(2): We have $W^u_{x,m} \cap W^s_{x,m} = \{0\}$ by statement (2) in Theorem 3. By (3.14) we have also $T_0 W^u_{x,m} = E_u(0)$, $T_0 W^s_{x,m} = E_s(0)$ where $E_u(0), E_s(0) \subset \mathbb{R}^d$ are defined in (4.26). It remains to use (4.37) and the fact that $d\widetilde{\varkappa}_x(x)$ maps $E_u(x), E_s(x)$ to $E_u(0), E_s(0)$ by Definition 4.5 and (4.31).

(3): Every $w = (w_1, w_2) \in \partial W^u_{x,m}$ satisfies $|w_1| = 1$, thus $|w| \geq 1$. The latter is also true for all $w \in \partial W^s_{x,m}$. It then suffices to choose ε_0 small enough so that

$$x \in K,\ d(x,y) \leq \varepsilon_0 \quad \Longrightarrow \quad |\widetilde{\varkappa}_x(y)| < 1 \tag{4.39}$$

which is possible since $\widetilde{\varkappa}_x(x) = 0$ and $\widetilde{\varkappa}_x$ are bounded in C^N uniformly in x.

(4): By statement (1) in Theorem 3 we have

$$\psi^{-1}_{x,-1}(W^u_{x,0}) \subset W^u_{x,-1}, \quad \psi_{x,0}(W^s_{x,0}) \subset W^s_{x,1}. \tag{4.40}$$

From (4.36) we have

$$\psi^{-1}_{x,-1} = \widetilde{\varkappa}_{\varphi^{-1}(x)} \circ \varphi^{-1} \circ \widetilde{\varkappa}_x^{-1}, \quad \psi_{x,0} = \widetilde{\varkappa}_{\varphi(x)} \circ \varphi \circ \widetilde{\varkappa}_x^{-1}.$$

Applying $\widetilde{\varkappa}^{-1}_{\varphi^{-1}(x)}$ to the first statement in (4.40) and $\widetilde{\varkappa}^{-1}_{\varphi(x)}$ to the second one and using (4.38) we get $\varphi^{-1}(W_u(x)) \subset W_u(\varphi^{-1}(x))$ and $\varphi(W_s(x)) \subset W_s(\varphi(x))$ as needed.

(5)–(6): Define the closed neighborhoods of x

$$\overline{B}_x := \widetilde{\varkappa}_x^{-1}(\overline{B}_\infty(0,1)), \quad x \in K.$$

Note that $W_u(x) \cup W_s(x) \subset \overline{B}_x$. By (4.36), if $n \geq 1$ and $\varphi^{-\ell}(y) \in \overline{B}_{\varphi^{-\ell}(x)}$ for all $\ell = 0, \dots, n-1$, then

$$\widetilde{\varkappa}_{\varphi^{-n}(x)}\big(\varphi^{-n}(y)\big) = \psi_{x,-n}^{-1} \cdots \psi_{x,-1}^{-1}(\widetilde{\varkappa}_x(y)). \tag{4.41}$$

Similarly if $n \geq 1$ and $\varphi^\ell(y) \in \overline{B}_{\varphi^\ell(x)}$ for all $\ell = 0, \dots, n-1$, then

$$\widetilde{\varkappa}_{\varphi^n(x)}\big(\varphi^n(y)\big) = \psi_{x,n-1} \cdots \psi_{x,0}(\widetilde{\varkappa}_x(y)). \tag{4.42}$$

If $y \in W_u(x)$, then for all $\ell \geq 0$ we have $\varphi^{-\ell}(y) \in W_u(\varphi^{-\ell}(x)) \subset \overline{B}_{\varphi^{-\ell}(x)}$. Moreover, $\widetilde{\varkappa}_x(y) \in W_{x,0}^u$ by (4.37). Applying the statement (3) in Theorem 3 with $m := 0$, we get $\psi_{x,-n}^{-1} \cdots \psi_{x,-1}^{-1}(\widetilde{\varkappa}_x(y)) \to 0$ as $n \to \infty$, thus $d(\varphi^{-n}(y), \varphi^{-n}(x)) \to 0$ by (4.41). The case $y \in W_s(x)$ is handled similarly.

(7)–(8): We show (7), with (8) proved similarly. Assume that $d(\varphi^{-n}(y), \varphi^{-n}(x)) \leq \varepsilon_0$ for all $n \geq 0$. By (4.39) this implies that $\varphi^{-n}(y) \in \overline{B}_{\varphi^{-n}(x)}$ for all $n \geq 0$. Then by (4.41) we have

$$\psi_{x,-n}^{-1} \cdots \psi_{x,-1}^{-1}(\widetilde{\varkappa}_x(y)) \in \overline{B}_\infty(0,1) \quad \text{for all } n \geq 0.$$

By statement (5) in Theorem 3 with $m := 0$ we have $\widetilde{\varkappa}_x(y) \in W_{x,0}^u$ and thus $y \in W_u(x)$ by (4.37).

The quantitative statements (4.6)–(4.8) follow from (3.19)–(3.21) similarly to the proofs of statements (5)–(8) above.

Finally, statement (9) in Theorem 4 essentially follows from the continuous dependence of $E_u(x), E_s(x)$ on x (Lemma 4.2) and the fact that $E_u(x)$, $E_s(x)$ are transversal to each other. We give a more detailed (straightforward but slightly tedious) explanation below.

Recall from (4.37) that $W_u(x) = \widetilde{\varkappa}_x^{-1}(W_{x,0}^u)$ and $W_s(x) = \widetilde{\varkappa}_x^{-1}(W_{x,0}^s)$ where $\widetilde{\varkappa}_x$ is the rescaled adapted chart defined in (4.31) and $W_{x,0}^u, W_{x,0}^s \subset \overline{B}_\infty(0,1)$ are the unstable/stable graphs constructed in Theorem 3. Writing elements of $\mathbb{R}^d$ as (w_1, w_2) where $w_1 \in \mathbb{R}^{d_u}$, $w_2 \in \mathbb{R}^{d_s}$ we have

$$\widetilde{\varkappa}_x(W_u(x)) = W_{x,0}^u = \{(w_1, w_2) \colon |w_1| \leq 1,\ w_2 = F_{x,u}(w_1)\}, \tag{4.43}$$

$$\widetilde{\varkappa}_x(W_s(x)) = W_{x,0}^s = \{(w_1, w_2) \colon |w_2| \leq 1,\ w_1 = F_{x,s}(w_2)\} \tag{4.44}$$

for some functions $F_{x,u} : \overline{B}_u(0,1) \to \overline{B}_s(0,1)$, $F_{x,s} : \overline{B}_s(0,1) \to \overline{B}_u(0,1)$, where $\overline{B}_u(0,1), \overline{B}_s(0,1)$ are defined in (3.6), such that (recalling (3.14), (2.29), and (4.35))

$$F_{x,u}(0) = 0, \quad F_{x,s}(0) = 0, \quad \max\big(\|F_{x,u}\|_{C^1}, \|F_{x,s}\|_{C^1}\big) \leq C\delta_1; \tag{4.45}$$

here $\delta_1 > 0$ is the rescaling parameter used in the definition (4.31) of the chart $\widetilde{\varkappa}_x$.

Now, we assume that $x, y \in K$ and $d(x,y) \le \varepsilon_0$ where $\varepsilon_0 > 0$ is small, in particular $\varepsilon_0 \ll \delta_1$. Then $W_u(y)$ is contained in the domain of the chart $\widetilde{\varkappa}_x$. The points in $W_s(x) \cap W_u(y)$ have the form $\widetilde{\varkappa}_x^{-1}(F_{x,s}(w_2), w_2) = \widetilde{\varkappa}_y^{-1}(w_1, F_{y,u}(w_1))$ where $w = (w_1, w_2) \in \overline{B}_\infty(0,1)$ solves the equation

$$G_{x,y}(w) = 0, \quad G_{x,y}(w_1, w_2) := \widetilde{\varkappa}_{x,y}(w_1, F_{y,u}(w_1)) - (F_{x,s}(w_2), w_2) \tag{4.46}$$

where we put

$$\widetilde{\varkappa}_{x,y} := \widetilde{\varkappa}_x \circ \widetilde{\varkappa}_y^{-1} = T^{-1} \circ \varkappa_{x,y} \circ T, \quad \varkappa_{x,y} := \varkappa_x \circ \varkappa_y^{-1}, \quad T : w \mapsto \delta_1 w,$$

and $\varkappa_x, \varkappa_y$ are the unrescaled charts defined in (4.27). Since $\varkappa_x, \varkappa_y$ are adapted charts (see Definition 4.5), $d(x,y) \le \varepsilon_0$, and $E_u(x), E_s(x)$ depend continuously on x, for small enough ε_0 we have

$$d\varkappa_{x,y}(0) = A_{x,y} + \mathcal{O}(\delta_1), \quad A_{x,y} := \begin{pmatrix} A_{1,x,y} & 0 \\ 0 & A_{2,x,y} \end{pmatrix}$$

where $A_{1,x,y} : \mathbb{R}^{d_u} \to \mathbb{R}^{d_u}$, $A_{2,x,y} : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$ are isometries. The rescaled change of coordinates map $\widetilde{\varkappa}_{x,y}$ then satisfies (assuming $\varepsilon_0 \le \delta_1^2$)

$$|\widetilde{\varkappa}_{x,y}(0)| \le C\delta_1, \quad \sup_{w \in \overline{B}_\infty(0,1)} \|d\widetilde{\varkappa}_{x,y}(w) - A_{x,y}\| \le C\delta_1.$$

Combining this with (4.45) we get

$$|G_{x,y}(0)| \le C\delta_1, \quad \sup_{w \in \overline{B}_\infty(0,1)} \left\| dG_{x,y}(w) - \begin{pmatrix} A_{1,x,y} & 0 \\ 0 & -I \end{pmatrix} \right\| \le C\delta_1.$$

If δ_1 is small enough, then by the Contraction Mapping Principle the equation (4.46) has a unique solution in $\overline{B}_\infty(0,1)$. Therefore, $W_s(x) \cap W_u(y)$ consists of a single point as required.

4.6. **Hyperbolic flows.** We finally consider the setting of hyperbolic flows. Let M be a d-dimensional manifold without boundary and $d = d_u + d_s + 1$ where $d_u, d_s \ge 0$. Let

$$\varphi^t = \exp(tX) : M \to M, \quad t \in \mathbb{R}$$

be the flow generated by a C^{N+1} vector field X on M. For simplicity we assume that φ^t is globally well-defined for all t, though in practice it is enough to require this in the neighborhood of the set K below. We assume that $K \subset M$ is a φ^t-invariant hyperbolic set in the following sense:

Definition 4.6. *Let $K \subset M$ be a compact set such that $\varphi^t(K) = K$ for all $t \in \mathbb{R}$. We say that the flow φ^t is* ***hyperbolic*** *on K if the generating vector field X does not vanish on K and there exists a splitting*

$$T_xM = E_0(x) \oplus E_u(x) \oplus E_s(x), \quad x \in K, \quad E_0(x) := \mathbb{R}X(x), \tag{4.47}$$

where $E_u(x), E_s(x) \subset T_xM$ are subspaces of dimensions d_u, d_s and:

- *E_u, E_s are invariant under the flow, namely for all $x \in K$ and $t \in \mathbb{R}$*

$$d\varphi^t(x)E_u(x) = E_u(\varphi^t(x)), \quad d\varphi^t(x)E_s(x) = E_s(\varphi^t(x)); \tag{4.48}$$

- *$d\varphi^t$ is expanding on E_u and contracting on E_s, namely there exist constants $C > 0$, $\nu > 0$ such that for some Riemannian metric $|\bullet|$ on M and all $x \in K$*

$$|d\varphi^t(x)v| \leq Ce^{-\nu|t|} \cdot |v|, \quad \begin{cases} v \in E_u(x), & t \leq 0; \\ v \in E_s(x), & t \geq 0. \end{cases} \tag{4.49}$$

Remarks. 1. The time-t map φ^t of a hyperbolic flow is **not** a hyperbolic map in the sense of Definition 4.1 because of the flow direction E_0.

2. The property (4.49) does not depend on the choice of the metric on M, though the constant C (but not ν) will depend on the metric.

3. The basic example of a hyperbolic set is a closed trajectory

$$K = \{\varphi^t(x_0) : 0 \leq t \leq T\} \quad \text{for some} \quad x_0 \in M,\ T > 0 \quad \text{such that} \quad \varphi^T(x_0) = x_0$$

which is hyperbolic, namely $d\varphi^T(x_0)$ has a simple eigenvalue 1 and no other eigenvalues on the unit circle. The opposite situation is when $K = M$; in this case φ^t is called an *Anosov flow*. An important class of Anosov flows are geodesic flows on negatively curved manifolds, discussed in §5.1 below.

As in §4.1, fix a distance function $d(\bullet, \bullet)$ on M and define the balls $\overline{B}_d(x, r)$ by (4.5). We now state the Stable/Unstable Manifold Theorem for hyperbolic flows, which is similar to the case of maps (Theorem 4):

Theorem 5. *Assume that the flow φ^t is hyperbolic on $K \subset M$. Then for each $x \in K$ there exist* ***local unstable/stable manifolds***

$$W_u(x), W_s(x) \subset M$$

which have the following properties for some $\varepsilon_0 > 0$ depending only on φ^t, K:

1. *$W_u(x), W_s(x)$ are C^N embedded disks of dimensions d_u, d_s, and the C^N norms of the embeddings are bounded uniformly in x;*
2. *$W_u(x) \cap W_s(x) = \{x\}$ and $T_x W_u(x) = E_u(x)$, $T_x W_s(x) = E_s(x)$;*
3. *the boundaries of $W_u(x), W_s(x)$ do not intersect $\overline{B}_d(x, \varepsilon_0)$;*
4. *$\varphi^{-1}(W_u(x)) \subset W_u(\varphi^{-1}(x))$ and $\varphi^1(W_s(x)) \subset W_s(\varphi^1(x))$;*
5. *for each $y \in W_u(x)$, we have $d(\varphi^t(y), \varphi^t(x)) \to 0$ as $t \to -\infty$;*
6. *for each $y \in W_s(x)$, we have $d(\varphi^t(y), \varphi^t(x)) \to 0$ as $t \to \infty$;*
7. *if $y \in M$, $d(\varphi^t(y), \varphi^t(x)) \leq \varepsilon_0$ for all $t \leq 0$, and $d(\varphi^t(y), \varphi^t(x)) \to 0$ as $t \to -\infty$, then $y \in W_u(x)$;*
8. *if $y \in M$, $d(\varphi^t(y), \varphi^t(x)) \leq \varepsilon_0$ for all $t \geq 0$, and $d(\varphi^t(y), \varphi^t(x)) \to 0$ as $t \to \infty$, then $y \in W_s(x)$.*

Remarks. 1. The statement (4) is somewhat artificial since it involves the time-one map φ^1 and its inverse φ^{-1}. By rescaling the flow we can construct local stable/unstable manifolds such that the statement (4) holds with $\varphi^{t_0}, \varphi^{-t_0}$ instead, where $t_0 > 0$ is any fixed number. A more natural statement would be that $\varphi^{-t}(W_u(x)) \subset W_u(\varphi^{-t}(x))$ and $\varphi^t(W_s(x)) \subset W_s(\varphi^t(x))$ for all $t \geq 0$, but this would be more difficult to arrange since our method of proof is tailored to discrete time evolution. However, below in §4.7.3 we explain that the *global* stable/unstable manifolds are invariant under φ^t for all t.

2. Compared to Theorem 4, properties (7) and (8) impose the additional condition that $d(\varphi^t(y), \varphi^t(x)) \to 0$. This is due to the presence of the flow direction: for instance, if $y = \varphi^s(x)$ where $s \neq 0$ is small, then $d(\varphi^t(y), \varphi^t(x)) \leq \varepsilon_0$ for all t but $y \notin W_u(x) \cup W_s(x)$. Without this additional condition we can only assert that y lies in the *weak* stable/unstable manifold of x, see §4.7.1.

3. The analog of statement (9) of Theorem 4 is given in (4.66). The analogs of the quantitative statements (4.6)–(4.8) are discussed in §4.7.2.

We now sketch the proof of Theorem 5. We follow the proof of Theorem 4 in §§4.2–4.5, indicating the changes needed along the way.

First of all, the proof of Lemma 4.2 applies without change, so the spaces $E_u(x), E_s(x)$ depend continuously on x. (The Hölder continuity Lemma 4.3 holds as well.) This implies the following version of uniform transversality (4.19): there exists a constant C such that for all $x \in K$

$$\max\big(|v_0|, |v_u|, |v_s|\big) \leq C|v_0+v_u+v_s| \quad \text{if} \quad v_0 \in E_0(x),\ v_u \in E_u(x),\ v_s \in E_s(x). \tag{4.50}$$

Next, existence of adapted metrics is given by the following analog of Lemma 4.4:

Lemma 4.7. *Assume that φ^t is hyperbolic on K. Fix $\tilde\nu$ such that $0 < \tilde\nu < \nu$ where ν is given in Definition 4.6. Then there exist C^N Riemannian metrics $|\bullet|_u$, $|\bullet|_s$ on M such that for all $x \in K$*

$$\begin{aligned} |d\varphi^t(x)v|_u &\leq e^{-\tilde\nu|t|} \cdot |v|_u \quad \textit{for all} \quad v \in E_u(x),\ t \leq 0;\\ |d\varphi^t(x)v|_s &\leq e^{-\tilde\nu|t|} \cdot |v|_s \quad \textit{for all} \quad v \in E_s(x),\ t \geq 0. \end{aligned} \tag{4.51}$$

Proof. This follows by a similar argument to the proof of Lemma 4.4, fixing a metric $|\bullet|$ on M and defining the adapted metrics as follows: for $v \in T_xM$,

$$|v|_u^2 := \int_0^T e^{2\tilde\nu s}|d\varphi^{-s}(x)v|^2\,ds, \quad |v|_s^2 := \int_0^T e^{2\tilde\nu s}|d\varphi^{s}(x)v|^2\,ds$$

where $T > 0$ is a sufficiently large constant. □

We next define adapted charts, similarly to §4.4. Fix $\tilde\nu \in (\nu, 1)$ and let $|\bullet|_u$, $|\bullet|_s$ be the adapted metrics constructed in Lemma 4.7. We write elements of $\mathbb{R}^d$ as (x_0, x_1, x_2)

where $x_0 \in \mathbb{R}$, $x_1 \in \mathbb{R}^{d_u}$, and $x_2 \in \mathbb{R}^{d_s}$. Consider the subspaces of $\mathbb{R}^d$

$$E_u(0) := \{(0, v_1, 0) \mid v_1 \in \mathbb{R}^{d_u}\}, \quad E_s(0) := \{(0, 0, v_2) \mid v_2 \in \mathbb{R}^{d_s}\}.$$

Definition 4.8. *Let $x \in K$. A C^{N+1} diffeomorphism*

$$\varkappa : U_\varkappa \to V_\varkappa, \quad x \in U_\varkappa \subset M, \quad 0 \in V_\varkappa \subset \mathbb{R}^d$$

is called an ***adapted chart*** *for φ^t centered at x, if:*

1. *$\varkappa(x) = 0$;*
2. *for each $y \in U_\varkappa$, $d\varkappa(y)$ sends the generator of the flow $X(y)$ to ∂_{x_0};*
3. *$d\varkappa(x)E_u(x) = E_u(0)$ and the restriction of $d\varkappa(x)$ to $E_u(x)$ is an isometry from the metric $|\bullet|_u$ to the Euclidean metric;*
4. *$d\varkappa(x)E_s(x) = E_s(0)$ and the restriction of $d\varkappa(x)$ to $E_s(x)$ is an isometry from the metric $|\bullet|_s$ to the Euclidean metric.*

Similarly to §4.4, it follows from the uniform transversality property (4.50) that we can select for each $x \in K$ an adapted chart for φ^t centered at x

$$\varkappa_x : U_x \to V_x, \quad x \in K$$

such that the set $\{\varkappa_x \mid x \in K\}$ is bounded in the class of C^{N+1} charts.

Similarly to (4.31) we next define rescaled charts

$$\widetilde{\varkappa}_x := T^{-1} \circ \varkappa_x, \quad x \in K; \quad T(w) = \delta_1 w \tag{4.52}$$

where $\delta_1 > 0$ is chosen small depending only on φ^t, K when we apply Theorem 3. The action of the time-one map φ^1 in the charts $\widetilde{\varkappa}_x$ is given by the maps

$$\widetilde{\psi}_x := \widetilde{\varkappa}_{\varphi^1(x)} \circ \varphi^1 \circ \widetilde{\varkappa}_x^{-1}. \tag{4.53}$$

Arguing as in §4.4, we see that $\widetilde{\psi}_x$ has the following properties:

- the domain and the range of $\widetilde{\psi}_x$ contain the closed ball in $\mathbb{R}^d$;
- we have
$$\widetilde{\psi}_x(0) = 0, \quad d\widetilde{\psi}_x(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_{1,x} & 0 \\ 0 & 0 & A_{2,x} \end{pmatrix} \tag{4.54}$$
where the linear maps $A_{1,x} : \mathbb{R}^{d_u} \to \mathbb{R}^{d_u}$, $A_{2,x} : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$ satisfy for all $x \in K$
$$\|A_{1,x}^{-1}\| \le e^{-\tilde{\nu}}, \quad \|A_{2,x}\| \le e^{-\tilde{\nu}}, \quad \max(\|A_{1,x}\|, \|A_{2,x}\|^{-1}) \le C_0$$
where C_0 is some constant depending only on φ^t, K;
- for any given $\delta > 0$, if we choose δ_1 small enough depending on δ, then
$$\sup |\partial^\alpha \widetilde{\psi}_x| \le \delta \quad \text{for all} \quad x \in K, \quad 2 \le |\alpha| \le N + 1. \tag{4.55}$$

The matrix $d\widetilde{\psi}_x(0)$ in (4.54) has eigenvalue 1 coming from the flow direction, which makes some parts of the proof problematic because Theorem 3 only partly applies when either the expansion or the contraction rate is equal to 1 – see §3.6. To deal with this problem we introduce the *reduced* maps ω_x, which act on subsets of $\mathbb{R}^{d-1}$ and correspond to the action of the flow on Poincaré sections. Define the projection map

$$\pi_{us} : (w_0, w_1, w_2) \mapsto (w_1, w_2)$$

where $w_0 \in \mathbb{R}$, $w_1 \in \mathbb{R}^{d_u}$, $w_2 \in \mathbb{R}^{d_s}$. From the definition of adapted charts and (4.53) we see that

$$d\widetilde{\psi}_x(w)\partial_{x_0} = \partial_{x_0} \quad \text{for all } w. \tag{4.56}$$

Therefore (shrinking the domain of $\widetilde{\psi}_x$ if necessary) there exists a diffeomorphism ω_x of open neighborhoods of $\{(x_1, x_2) \in \mathbb{R}^{d-1} \colon \max(|x_1|, |x_2|) \leq 1\}$ such that

$$\pi_{us}(\widetilde{\psi}_x(w)) = \omega_x(\pi_{us}(w)) \quad \text{for all } w. \tag{4.57}$$

The maps ω_x satisfy

$$\omega_x(0) = 0, \quad d\omega_x(0) = \begin{pmatrix} A_{1,x} & 0 \\ 0 & A_{2,x} \end{pmatrix}$$

where $A_{1,x}, A_{2,x}$ are the same matrices as in (4.54). They also satisfy the derivative bounds (4.55).

We can finally give the

Proof of Theorem 5. We argue similarly to §4.5. For each $x \in K$ and $m \in \mathbb{Z}$ define the maps

$$\psi_{x,m} := \widetilde{\psi}_{\varphi^m(x)} = \widetilde{\varkappa}_{\varphi^{m+1}(x)} \circ \varphi^1 \circ \widetilde{\varkappa}_{\varphi^m(x)}^{-1}.$$

The sequence of maps $(\psi_{x,m})_{m\in\mathbb{Z}}$ satisfies the assumptions in §3.5 where we absorb the ∂_{x_0} direction into the stable space and put

$$\lambda := 1, \quad \mu := e^{\tilde{\nu}} > 1.$$

As explained in §3.6, Theorem 3 still partially applies to the maps $(\psi_{x,m})_{m\in\mathbb{Z}}$ even though $\lambda = 1$, yielding the unstable manifolds $W^u_{x,m} \subset \mathbb{R}^d$. We then define the unstable manifold for φ^t at x by

$$W_u(x) := \widetilde{\varkappa}_x^{-1}(W^u_{x,0}). \tag{4.58}$$

If we instead absorb the ∂_{x_0} direction into the unstable space, then the sequence $(\psi_{x,m})_{m\in\mathbb{Z}}$ satisfies the assumptions in §3.5 where

$$\lambda := e^{-\tilde{\nu}} < 1, \quad \mu := 1.$$

As explained in §3.6, Theorem 3 gives the stable manifolds $W^s_{x,m} \subset \mathbb{R}^d$, and we define the stable manifold for φ^t at x by

$$W_s(x) := \widetilde{\varkappa}_x^{-1}(W^s_{x,0}). \tag{4.59}$$

Statements (1)–(4) of Theorem 5 are then proved in the same way as for Theorem 4. The proof of Theorem 4 also gives the convergence statements (5)–(6) for integer t, which imply these statements for all t.

To show statements (7)–(8) we use the reduced maps ω_x. The sequence

$$\omega_{x,m} := \omega_{\varphi^m(x)}$$

satisfies the assumptions in §3.5 with

$$\lambda := e^{-\tilde\nu} < 1 < \mu := e^{\tilde\nu}.$$

Therefore Theorem 3 applies to give unstable/stable manifolds for the sequence $(\omega_{x,m})_{m\in\mathbb{Z}}$. Recalling the construction of these manifolds in §3.4 it is straightforward to see that the unstable/stable manifolds for $(\omega_{x,m})_{m\in\mathbb{Z}}$ are equal to $\pi_{us}(W^u_{x,m})$, $\pi_{us}(W^s_{x,m})$ where $W^u_{x,m}$, $W^s_{x,m}$ are the unstable/stable manifolds for the sequence $(\psi_{x,m})_{m\in\mathbb{Z}}$.

We now show statement (7), with the statement (8) proved similarly. Assume that $y \in M$ and $d(\varphi^t(y),\varphi^t(x)) \leq \varepsilon_0$ for all $t \leq 0$. Arguing similarly to the proof of Theorem 4 we see that

$$\psi^{-1}_{x,-n}\cdots\psi^{-1}_{x,-1}(\widetilde{\varkappa}_x(y)) \in \{(w_0,w_1,w_2)\colon \max(|w_1|,|(w_0,w_2)|) \leq 1\} \quad \text{for all} \quad n \geq 0.$$

It follows from (4.57) that

$$\omega^{-1}_{x,-n}\cdots\omega^{-1}_{x,-1}(\pi_{us}(\widetilde{\varkappa}_x(y))) \in \{(w_1,w_2)\colon \max(|w_1|,|w_2|) \leq 1\} \quad \text{for all} \quad n \geq 0.$$

Applying statement (5) in Theorem 3 for the maps $(\omega_{x,m})_{m\in\mathbb{Z}}$, we see that $\pi_{us}(\widetilde{\varkappa}_x(y)) \in \pi_{us}(W^u_{x,0})$ and thus

$$\widetilde{\varkappa}_x(y) = w + (s,0,0) \quad \text{for some} \quad w \in W^u_{x,0}, \quad s \in [-2,2]. \tag{4.60}$$

Now, assume additionally that $d(\varphi^t(y),\varphi^t(x)) \to 0$ as $t \to -\infty$. Then

$$\psi^{-1}_{x,-n}\cdots\psi^{-1}_{x,-1}(\widetilde{\varkappa}_x(y)) \to 0 \quad \text{as} \quad n \to \infty. \tag{4.61}$$

Using (4.56) and (4.60), we see that

$$\psi^{-1}_{x,-n}\cdots\psi^{-1}_{x,-1}(\widetilde{\varkappa}_x(y)) = \psi^{-1}_{x,-n}\cdots\psi^{-1}_{x,-1}(w) + (s,0,0).$$

Since $w \in W^u_{x,0}$, by part (3) of Theorem 3 we have $\psi^{-1}_{x,-n}\cdots\psi^{-1}_{x,-1}(w) \to 0$ as $n \to \infty$. Together with (4.61) this shows that $s = 0$. Therefore $\widetilde{\varkappa}_x(y) = w \in W^u_{x,0}$ which implies that $y \in W_u(x)$ as needed. ∎

4.7. **Further properties of hyperbolic flows.** We now discuss some further properties of the stable/unstable manifolds constructed in Theorem 5. Throughout this section we assume that the conditions of Theorem 5 hold.

4.7.1. *Weak stable/unstable manifolds.* Let $\delta_1 > 0$ be the rescaling parameter used in the proof of Theorem 5, see (4.52). Recall that δ_1 is chosen small depending only on φ^t, K. For each $x \in K$ define the *weak unstable/stable manifolds*

$$W_{u0}(x) := \bigcup_{|s|\leq 2\delta_1} \varphi^s(W_u(x)), \quad W_{s0}(x) := \bigcup_{|s|\leq 2\delta_1} \varphi^s(W_s(x)).$$

In the adapted chart $\widetilde{\varkappa}_x$ from (4.52), we have (using (4.58)–(4.59) and the fact that $d\widetilde{\varkappa}_x$ maps the generator X of the flow to $\delta_1^{-1}\partial_{x_0}$)

$$\begin{aligned} W_{u0}(x) &= \widetilde{\varkappa}_x^{-1}(W^{u0}_{x,0}), \quad W^{u0}_{x,0} := \{w + (s, 0, 0) \colon w \in W^u_{x,0},\ |s| \leq 2\};\\ W_{s0}(x) &= \widetilde{\varkappa}_x^{-1}(W^{s0}_{x,0}), \quad W^{s0}_{x,0} := \{w + (s, 0, 0) \colon w \in W^s_{x,0},\ |s| \leq 2\} \end{aligned} \tag{4.62}$$

where $W^u_{x,0}$, $W^s_{x,0}$ are the unstable/stable manifolds for the maps $\widetilde{\psi}_x$, see the proof of Theorem 5. Since $W^u_{x,0}$, $W^s_{x,0}$ are graphs of functions of x_1, x_2 (see (3.15)), we see that $W^{u0}_{x,0}, W^{s0}_{x,0}$, and thus $W_{u0}(x), W_{s0}(x)$, are embedded C^N submanifolds of dimensions $d_u + 1, d_s + 1$. It is also clear that

$$T_xW_{u0}(x) = E_u(x) \oplus E_0(x), \quad T_xW_{s0}(x) = E_s(x) \oplus E_0(x). \tag{4.63}$$

It follows from statement (4) in Theorem 5 that the weak stable/unstable manifolds are invariant under the positive/negative integer time maps of the flow φ^t:

$$\varphi^{-1}(W_{u0}(x)) \subset W_{u0}(\varphi^{-1}(x)), \quad \varphi^1(W_{s0}(x)) \subset W_{s0}(\varphi^1(x)). \tag{4.64}$$

We next have the following versions of the statements (7)–(8) in Theorem 5: for all $y \in M$,

$$\begin{aligned} d(\varphi^t(y), \varphi^t(x)) \leq \varepsilon_0 \quad \text{for all } t \leq 0 \quad &\Longrightarrow \quad y \in W_{u0}(x);\\ d(\varphi^t(y), \varphi^t(x)) \leq \varepsilon_0 \quad \text{for all } t \geq 0 \quad &\Longrightarrow \quad y \in W_{s0}(x). \end{aligned} \tag{4.65}$$

The properties (4.65) follow immediately from (4.60) (and its analog for stable manifolds) and (4.62).

Similarly to the proof of statement (9) of Theorem 4 one can show the following transversality properties: if $x, y \in K$ and $d(x, y) \leq \varepsilon_0$ then

$$W_{s0}(x) \cap W_u(y), W_s(x) \cap W_{u0}(y) \quad \text{have exactly one point each.} \tag{4.66}$$

4.7.2. *Quantitative statements.* We now discuss quantitative versions of the statements of Theorem 5, which are the analogs of (4.6)–(4.8). We fix $\tilde{\nu}$ such that

$$0 < \tilde{\nu} < \nu$$

and allow the manifolds W_u, W_s and the constant ε_0 to depend on $\tilde{\nu}$.

The analog of (4.6) is given by the following: there exists a constant C such that for all $x \in K$ and $t \geq 0$

$$\begin{aligned} y, \tilde{y} \in W_u(x) \quad &\Longrightarrow \quad d(\varphi^{-t}(y), \varphi^{-t}(\tilde{y})) \leq Ce^{-\tilde{\nu}t}d(y, \tilde{y});\\ y, \tilde{y} \in W_s(x) \quad &\Longrightarrow \quad d(\varphi^{t}(y), \varphi^{t}(\tilde{y})) \leq Ce^{-\tilde{\nu}t}d(y, \tilde{y}). \end{aligned} \tag{4.67}$$

To show (4.67), it suffices to consider the case of integer t. The latter case is proved similarly to (4.6) (see §4.5), since (3.19) still applies to the maps $\psi_{x,m}$.

The analog of (4.7) is given by the following: for all $x \in K$, $y \in M$, $t \geq 0$, and $0 \leq \sigma \leq \varepsilon_0$

$$\begin{aligned} d(\varphi^{-s}(y), \varphi^{-s}(x)) \leq \sigma \text{ for all } s \in [0,t] \quad &\Longrightarrow \quad d(y, W_{u0}(x)) \leq Ce^{-\tilde{\nu}t}\sigma, \\ d(\varphi^{s}(y), \varphi^{s}(x)) \leq \sigma \text{ for all } s \in [0,t] \quad &\Longrightarrow \quad d(y, W_{s0}(x)) \leq Ce^{-\tilde{\nu}t}\sigma. \end{aligned} \tag{4.68}$$

This is proved by applying (3.20) for the reduced maps $\omega_{x,m}$ defined in (4.57), see the proof of (4.60).

Finally the analog of (4.8) is given by the following: for all $x \in K$, $y \in M$, $t \geq 0$, and $0 \leq \sigma \leq \varepsilon_0$,

$$\begin{aligned} & d(\varphi^{s}(y), \varphi^{s}(x)) \leq \sigma \quad \text{for all} \quad s \in [-t,t] \\ \Longrightarrow \quad & d(y, \varphi^{r}(x)) \leq Ce^{-\tilde{\nu}t}\sigma \quad \text{for some } r \in [-2\delta_1, 2\delta_1]. \end{aligned} \tag{4.69}$$

This is proved by applying (3.21) to the reduced maps $\omega_{x,m}$, arguing similarly to the proof of (4.60).

4.7.3. *Local invariance and global stable/unstable manifolds.* In this section we discuss local invariance of the stable/unstable manifolds. We first discuss invariance under the flow φ^t. Theorem 5 does not imply that $\varphi^{-t}(W_u(x)) \subset W_u(x)$, $\varphi^t(W_s(x)) \subset W_s(x)$ for non-integer $t \geq 0$, however local versions of this statement are established in (4.71)–(4.74) below.

Similarly to (4.9), (4.13) for each $x \in K$ and integer $k \geq 0$ define the iterated unstable/stable manifolds

$$W_u^{(k)}(x) := \varphi^k(W_u(\varphi^{-k}(x))), \quad W_s^{(k)}(x) := \varphi^{-k}(W_s(\varphi^k(x))). \tag{4.70}$$

It follows from statement (4) in Theorem 5 that for all $k \geq 0$

$$W_u^{(k)}(x) \subset W_u^{(k+1)}(x), \quad W_s^{(k)}(x) \subset W_s^{(k+1)}(x).$$

Local invariance of the unstable/stable manifolds under the flow is given by the following statements: there exist $k_0 \geq 0$ and $\varepsilon_1 > 0$ depending only on φ^t, K such that for all $x \in K$ and $s \in [-1,1]$

$$\varphi^s(W_u(x)) \subset W_u^{(k_0)}(\varphi^s(x)), \tag{4.71}$$

$$\varphi^s(W_s(x)) \subset W_s^{(k_0)}(\varphi^s(x)), \tag{4.72}$$

$$\varphi^s(W_u(x)) \cap \overline{B}_d(\varphi^s(x), \varepsilon_1) \subset W_u(\varphi^s(x)), \tag{4.73}$$

$$\varphi^s(W_s(x)) \cap \overline{B}_d(\varphi^s(x), \varepsilon_1) \subset W_s(\varphi^s(x)). \tag{4.74}$$

Note that (4.71), (4.73) can be extended to all $s \leq 0$ and (4.72), (4.74) to all $s \geq 0$ using statement (4) in Theorem 5.

We show (4.72), (4.74), with (4.71), (4.73) proved similarly. We start with (4.72). Assume that $y \in W_s(x)$. Then by (4.67) we have for all $t \geq 0$

$$d(\varphi^{t+s}(y), \varphi^{t+s}(x)) \leq C d(\varphi^t(y), \varphi^t(x)) \leq C e^{-\tilde{\nu} t}. \tag{4.75}$$

Therefore, $d(\varphi^{t+s}(y), \varphi^{t+s}(x)) \to 0$ as $t \to \infty$ and there exists $k_0 \geq 0$ such that

$$d(\varphi^{t+s+k_0}(y), \varphi^{t+s+k_0}(x)) \leq \varepsilon_0 \quad \text{for all} \quad t \geq 0. \tag{4.76}$$

It follows from statement (8) in Theorem 5 that $\varphi^{s+k_0}(y) \in W_s(\varphi^{s+k_0}(x))$ and thus $\varphi^s(y) \in W_s^{(k_0)}(\varphi^s(x))$ as needed.

To show (4.74), assume that $y \in W_s(x)$ and $d(\varphi^s(y), \varphi^s(x)) \leq \varepsilon_1$. Then (4.76) holds. If ε_1 is small enough, then (4.76) holds also for all $t \in [-k_0, 0]$, and thus for all $t \geq -k_0$. It then follows from statement (8) in Theorem 5 that $\varphi^s(y) \in W_s(\varphi^s(x))$ as needed.

Next, we note that the statements (4.10), (4.12), (4.14), and (4.15) relating the stable/unstable manifolds at different points are still valid in the case of flows, with very similar proofs.

Finally, similarly to (4.16) we can define the *global unstable/stable manifolds*: for $x \in K$,

$$W_u^{(\infty)}(x) := \bigcup_{k \geq 0} W_u^{(k)}(x), \quad W_s^{(\infty)}(x) := \bigcup_{k \geq 0} W_s^{(k)}(x). \tag{4.77}$$

By statements (5)–(8) in Theorem 5, these can be characterized as follows:

$$\begin{aligned} y \in W_u^{(\infty)}(x) \quad &\iff \quad d(\varphi^t(y), \varphi^t(x)) \to 0 \quad \text{as } t \to -\infty; \\ y \in W_s^{(\infty)}(x) \quad &\iff \quad d(\varphi^t(y), \varphi^t(x)) \to 0 \quad \text{as } t \to \infty. \end{aligned} \tag{4.78}$$

The manifolds $W_u^{(\infty)}(x)$, $W_s^{(\infty)}(x)$ are d_u and d_s-dimensional immersed submanifolds without boundary in M. They are invariant under the flow φ^t and disjoint from each other.

5. Examples

In this section we provide two examples of hyperbolic systems: geodesic flows on negatively curved surfaces (§5.1) and billiard ball maps on Euclidean domains with concave boundary (§5.2).

5.1. **Surfaces of negative curvature.**

Let (M, g) be a closed (compact without boundary) oriented surface. To simplify the computations below, we will often use (positively oriented) *isothermal coordinates* (x, y) in which the metric is conformally flat:

$$g = e^{2G(x,y)}(dx^2 + dy^2). \tag{5.1}$$

Such coordinates exist locally near each point of M. In isothermal coordinates, the Gauss curvature is given by

$$K(x,y) = -e^{-2G(x,y)}(\partial_x^2 + \partial_y^2)G(x,y). \tag{5.2}$$

We show below in Theorem 6 that if $K < 0$ then the geodesic flow on (M, g) is hyperbolic. We define the geodesic flow as the Hamiltonian flow

$$\begin{aligned} \varphi^t &:= \exp(tX) : S^*M \to S^*M, \qquad X := H_p, \\ S^*M &:= \{(x,\xi) \in T^*M : p(x,\xi) = 1\}, \qquad p(x,\xi) := |\xi|_g \end{aligned} \tag{5.3}$$

where T^*M is the cotangent bundle of M and S^*M is the unit cotangent bundle (with respect to the metric g). In local coordinates (x, y), if (ξ, η) are the corresponding momentum variables (i.e. coordinates on the fibers of T^*M) then

$$X = H_p = (\partial_\xi p)\partial_x + (\partial_\eta p)\partial_y - (\partial_x p)\partial_\xi - (\partial_y p)\partial_\eta. \tag{5.4}$$

In isothermal coordinates (5.1) we have

$$p(x,y,\xi,\eta) = e^{-G(x,y)}\sqrt{\xi^2 + \eta^2}. \tag{5.5}$$

Theorem 6. *Let (M, g) be a closed oriented surface with geodesic flow $\varphi^t : S^*M \to S^*M$ defined in (5.3). Assume that the Gauss curvature K is negative everywhere. Then φ^t is an Anosov flow, that is φ^t is hyperbolic on the entire S^*M in the sense of Definition 4.6.*

Remark. Theorem 6 extends to higher dimensional manifolds of *negative sectional curvature* – see for instance [KaHa97, Theorem 17.6.2]. The orientability hypothesis is made only for convenience of the proof, one can remove it for instance by passing to a double cover of M.

In the remainder of this section we prove Theorem 6. We first define a convenient frame on S^*M. Let V be the vector field on S^*M generating rotations on the circle fibers (counterclockwise with respect to the fixed orientation). If (x, y) are isothermal coordinates (5.1), we use local coordinates (x, y, θ) on S^*M where θ is defined by

$$\xi = e^{G(x,y)}\cos\theta, \quad \eta = e^{G(x,y)}\sin\theta,$$

and we have (here X is the generator of the geodesic flow)

$$V = \partial_\theta, \quad X = e^{-G(x,y)}\big(\cos\theta\,\partial_x + \sin\theta\,\partial_y + (\partial_y G(x,y)\cos\theta - \partial_x G(x,y)\sin\theta)\partial_\theta\big).$$

Define the vector field

$$X_\perp := [X, V],$$

in isothermal coordinates

$$X_\perp = e^{-G(x,y)}\big(\sin\theta\,\partial_x - \cos\theta\,\partial_y + (\partial_x G(x,y)\cos\theta + \partial_y G(x,y)\sin\theta)\partial_\theta\big).$$

The vector fields $X, V, X_\perp$ form a global frame on S^*M and we have (using (5.2))

$$[X,V]=X_\perp,\quad [X_\perp,V]=-X,\quad [X,X_\perp]=-KV. \tag{5.6}$$

For any vector field W on S^*M, we have (by the standard properties of Lie bracket)

$$\partial_t(\varphi_*^{-t}W)=\varphi_*^{-t}[X,W]\quad\text{where}\quad \varphi_*^{-t}W(\rho):=d\varphi^{-t}(\rho)W(\varphi^t(\rho)),\ \rho\in S^*M. \tag{5.7}$$

It then follows from (5.6) that the following two-dimensional subbundle of $T(S^*M)$ is invariant under the flow φ^t:

$$E_{us}:=\operatorname{span}(V,X_\perp).$$

The space E_{us} will be the direct sum of the stable and the unstable subspaces for the flow φ^t. For a vector $v\in E_{us}(\rho)$, $\rho\in S^*M$, we define its coordinates (a,b) with respect to the frame $V, X_\perp$:

$$v=aV(\rho)+bX_\perp(\rho). \tag{5.8}$$

We have the following differential equations for the action of $d\varphi^t$ on E_{us} (which are a special case of Jacobi's equations):

Lemma 5.1. *Let $\rho\in S^*M$, $v\in E_{us}(\rho)$, and denote*

$$\rho(t)=(x(t),\xi(t)):=\varphi^t(\rho)\in S^*M,\quad v(t):=d\varphi^t(\rho)v\in E_{us}(\rho(t)).$$

Let $a(t),b(t)$ be the coordinates of $v(t)$ defined in (5.8). Put $K(t):=K(x(t))$. Then, denoting by dots derivatives with respect to t, the functions $a(t),b(t)$ satisfy the ordinary differential equation

$$\dot a=K(t)b,\quad \dot b=-a. \tag{5.9}$$

Proof. Define the vector field $W_t:=a(t)V+b(t)X_\perp$. Then $W_t(\varphi^t(\rho))=v(t)=d\varphi^t(\rho)v$ and thus $v=\varphi_*^{-t}W_t(\rho)$. By (5.7) we have

$$0=\partial_t(\varphi_*^{-t}W_t)(\rho)=\varphi_*^{-t}\big(\partial_tW_t+[X,W_t]\big)(\rho).$$

Using (5.6), we get

$$0=(\partial_tW_t+[X,W_t])(\rho(t))=\big(\dot a(t)-K(x(t))b(t)\big)V(\rho(t))+\big(\dot b(t)+a(t)\big)X_\perp(\rho(t)).$$

This implies (5.9). □

Lemma 5.1 immediately implies Theorem 6 in the special case of *constant curvature* $K\equiv -1$, with expansion rate $\nu=1$ in (4.49) and E_u, E_s given by

$$E_u=\operatorname{span}(V-X_\perp),\quad E_s=\operatorname{span}(V+X_\perp).$$

To handle the case of variable curvature we first construct cones in E_{us} which are invariant under the flow φ^t for positive and negative times (see Figure 10):

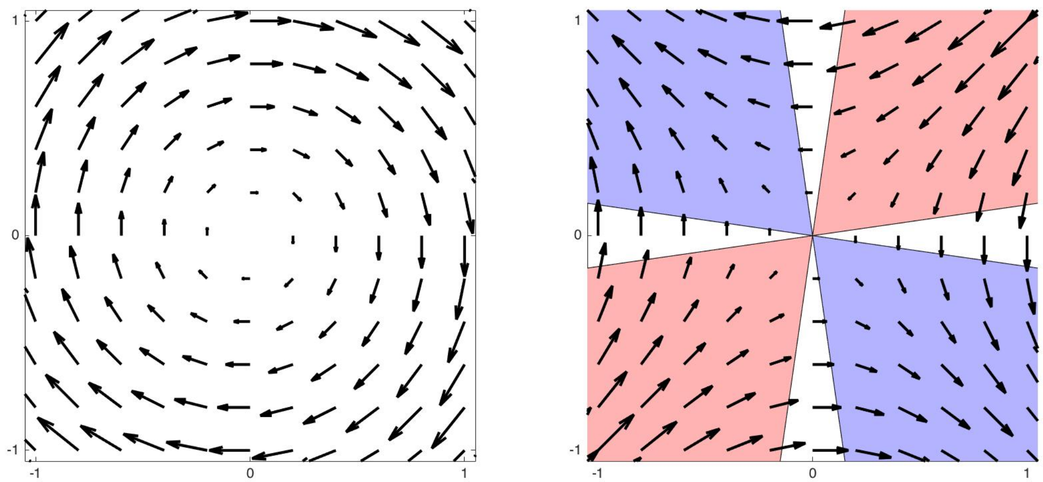

FIGURE 10. Direction fields for the equation (5.9) for $K = 1$ (left) and $K = -1$ (right) in the (a, b) plane. The cones on the right are $\mathcal{C}^u_\gamma$ (blue) and $\mathcal{C}^s_\gamma$ (red).

Lemma 5.2. *For each $\rho \in S^*M$, define the closed cones $\mathcal{C}^u_0(\rho), \mathcal{C}^s_0(\rho) \subset E_{us}(\rho)$:*

$$\mathcal{C}^u_0(\rho) := \{aV(\rho) + bX_\perp(\rho) \mid ab \leq 0\}, \quad \mathcal{C}^s_0(\rho) := \{aV(\rho) + bX_\perp(\rho) \mid ab \geq 0\}.$$

Assume that $K \leq 0$ everywhere on M. Then for all $t \geq 0$

$$d\varphi^t(\mathcal{C}^u_0(\rho)) \subset \mathcal{C}^u_0(\varphi^t(\rho)), \quad d\varphi^{-t}(\mathcal{C}^s_0(\rho)) \subset \mathcal{C}^s_0(\varphi^{-t}(\rho)). \tag{5.10}$$

Proof. In the notation of Lemma 5.1 we have

$$\partial_t(a(t)b(t)) = -a(t)^2 + K(x(t))b(t)^2 \leq 0$$

and (5.10) follows immediately. □

We now prove an upgraded version of Lemma 5.2, constructing invariant cones on which the differentials $d\varphi^t$, $d\varphi^{-t}$ are expanding. Fix small constants $\zeta > 0$, $\gamma > 0$ to be chosen later in Lemma 5.3. Define the norm $|\bullet|$ on the fibers of E_{us} as follows:

$$|aV(\rho) + bX_\perp(\rho)| := \sqrt{\zeta a^2 + b^2}.$$

Define also the following dilation invariant function Θ on the fibers of $E_{us} \setminus 0$:

$$\Theta(aV(\rho) + bX_\perp(\rho)) := \frac{ab}{\zeta a^2 + b^2}.$$

The upgraded invariant cones are constructed in

Lemma 5.3. *Assume that $K < 0$ everywhere on M. Then there exist $\zeta > 0$, $\gamma > 0$, $\nu > 0$ such that the closed cones $\mathcal{C}^u_\gamma(\rho), \mathcal{C}^s_\gamma(\rho) \subset E_{us}(\rho)$ defined by (see Figure 10)*

$$\mathcal{C}^u_\gamma(\rho) := \{v \in E_{us}(\rho) \colon \Theta(v) \leq -\gamma\} \cup \{0\}, \quad \mathcal{C}^s_\gamma(\rho) := \{v \in E_{us}(\rho) \colon \Theta(v) \geq \gamma\} \cup \{0\}$$

have the following properties for all $\rho \in S^*M$ *and* $t \geq 0$

$$d\varphi^t(\rho)\mathcal{C}^u_\gamma(\rho) \subset \mathcal{C}^u_\gamma(\varphi^t(\rho)); \tag{5.11}$$

$$d\varphi^{-t}(\rho)\mathcal{C}^s_\gamma(\rho) \subset \mathcal{C}^s_\gamma(\varphi^{-t}(\rho)); \tag{5.12}$$

$$|d\varphi^t(\rho)v| \geq e^{\nu t}|v| \quad \text{for all} \quad v \in \mathcal{C}^u_\gamma(\rho); \tag{5.13}$$

$$|d\varphi^{-t}(\rho)v| \geq e^{\nu t}|v| \quad \text{for all} \quad v \in \mathcal{C}^s_\gamma(\rho). \tag{5.14}$$

Proof. We fix constants $K_0, K_1 > 0$ such that

$$0 < K_0 \leq -K(x) \leq K_1 \quad \text{for all} \quad x \in M$$

and put

$$\zeta := \frac{1}{K_1}, \quad \gamma := \frac{\sqrt{K_0}}{3}, \quad \nu := \gamma(1 + \zeta K_0). \tag{5.15}$$

Let $\rho \in S^*M$, $v \in E_{us}(\rho)$, and put $\rho(t) := \varphi^t(\rho)$, $v(t) = d\varphi^t(\rho)v$. We write $v(t) = a(t)V(\rho(t)) + b(t)X_\perp(\rho(t))$ and recall that $a(t), b(t)$ satisfy the differential equations (5.9). Denote $K(t) := K(x(t))$ where $\rho(t) = (x(t), \xi(t))$ and

$$R(t) := |v(t)|^2 = \zeta a(t)^2 + b(t)^2, \quad \Theta(t) := \Theta(v(t)) = \frac{a(t)b(t)}{\zeta a(t)^2 + b(t)^2}.$$

Then it follows from (5.9) that (denoting by dots derivatives with respect to t)

$$\dot{R} = -2(1 - \zeta K(t))\Theta R, \quad \dot{\Theta} = -\frac{a^2 - Kb^2}{R} + 2(1 - \zeta K(t))\Theta^2. \tag{5.16}$$

Therefore

$$\dot{\Theta} \leq -\frac{a^2 + K_0 b^2}{R} + 2(1 + \zeta K_1)\Theta^2 \leq -K_0 + 4\Theta^2.$$

Thus if $|\Theta(t)| = \gamma$ for some t, then $\dot{\Theta}(t) < 0$. In particular, if $\Theta(0) \leq -\gamma$, then $\Theta(t) \leq -\gamma$ for all $t \geq 0$, which implies (5.11). Similarly if $\Theta(0) \geq \gamma$, then $\Theta(t) \geq \gamma$ for all $t \leq 0$, which implies (5.12).

We next prove (5.13). Assume that $v \in \mathcal{C}^u_\gamma(\rho) \setminus 0$, then $\Theta(t) \leq -\gamma$ for all $t \geq 0$. From (5.16) we have

$$\dot{R}(t) \geq 2\gamma(1 + \zeta K_0)R(t) = 2\nu R(t) \quad \text{for all} \quad t \geq 0.$$

Therefore $R(t) \geq e^{2\nu t}R(0)$ for all $t \geq 0$, which gives (5.13). The bound (5.14) is proved similarly. □

We finally use the cones from Lemma 5.3 to construct the stable/unstable spaces, finishing the proof of Theorem 6:

Lemma 5.4. *Let* γ *be chosen in Lemma 5.3. For each* $\rho \in S^*M$ *and* $t \geq 0$ *define the subsets of* $E_{us}(\rho)$

$$\mathcal{C}^u_{\gamma,t}(\rho) := d\varphi^t(\varphi^{-t}(\rho))\mathcal{C}^u_\gamma(\varphi^{-t}(\rho)), \quad \mathcal{C}^s_{\gamma,t}(\rho) := d\varphi^{-t}(\varphi^t(\rho))\mathcal{C}^s_\gamma(\varphi^t(\rho)),$$

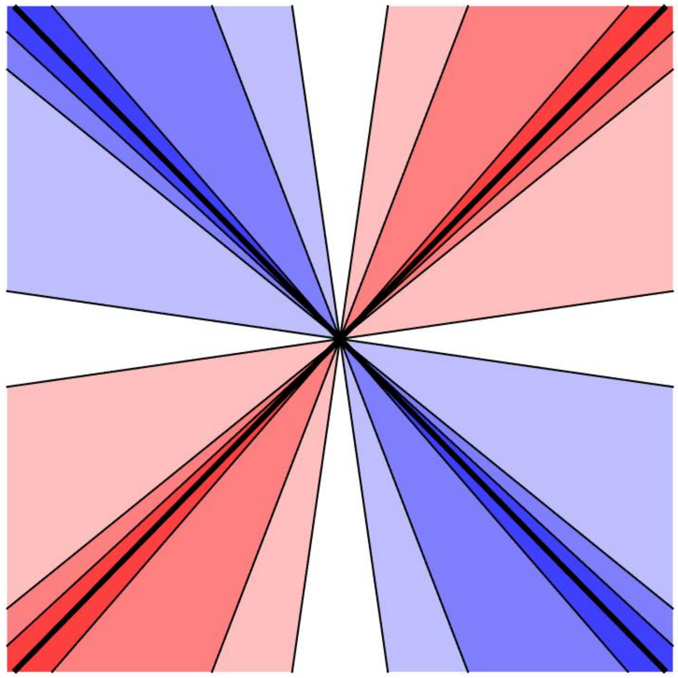

FIGURE 11. The cones $\mathcal{C}^u_{\gamma,t}(\rho)$ (blue) and $\mathcal{C}^s_{\gamma,t}(\rho)$ (red) for several values of t, with the darker colors representing larger values of t. The solid lines are the spaces $E_u(\rho), E_s(\rho)$.

and consider their intersections (see Figure 11)

$$E_u(\rho) := \bigcap_{t\geq 0} \mathcal{C}^u_{\gamma,t}(\rho), \quad E_s(\rho) := \bigcap_{t\geq 0} \mathcal{C}^s_{\gamma,t}(\rho). \tag{5.17}$$

Then $E_u(\rho), E_s(\rho)$ are one-dimensional subspaces of $E_{us}(\rho)$ which satisfy the conditions of Definition 4.6.

Proof. We show the properties of $E_u(\rho)$. The properties of $E_s(\rho)$ are proved similarly and the transversality of $E_u(\rho)$ and $E_s(\rho)$ follows from the fact that $E_u(\rho) \subset \mathcal{C}^u_\gamma(\rho)$, $E_s(\rho) \subset \mathcal{C}^s_\gamma(\rho)$, and $\mathcal{C}^u_\gamma(\rho) \cap \mathcal{C}^s_\gamma(\rho) = \{0\}$.

It follows from (5.11) that

$$\mathcal{C}^u_{\gamma,s}(\rho) \subset \mathcal{C}^u_{\gamma,t}(\rho) \quad \text{when} \quad s \geq t \geq 0.$$

We first claim that $E_u(\rho)$ contains a one-dimensional subspace of $E_{us}(\rho)$. Indeed, let $\mathscr{G}$ be the Grassmanian of all one-dimensional subspaces of $E_{us}(\rho)$ and $\mathscr{V}_t \subset \mathscr{G}$ consist of the subspaces which are contained in $\mathcal{C}^u_{\gamma,t}(\rho)$. Then $\mathscr{V}_s \subset \mathscr{V}_t$ for $s \geq t$ and all the sets $\mathscr{V}_t$ are compact. Moreover, each $\mathscr{V}_t$ is nonempty since (recalling (5.15))

$$V^0_u(\rho) := \{aV(\rho) + bX_\perp(\rho) \mid b = -a\sqrt{\zeta}\} \subset \mathcal{C}^u_\gamma(\rho). \tag{5.18}$$

Therefore the intersection $\bigcap_{t\geq 0}\mathscr{V}_t\subset\mathscr{G}$ is nonempty. Take an element $V_u(\rho)$ of this intersection, then $V_u(\rho)$ is a one-dimensional subspace of $E_{us}(\rho)$ and $V_u(\rho)\subset E_u(\rho)$.

We now claim that

$$E_u(\rho)=V_u(\rho). \tag{5.19}$$

For that, it suffices to show that every $v\in E_u(\rho)$ lies in $V_u(\rho)$. Define the one-dimensional space $V_s^0(\rho)\subset\mathcal{C}_\gamma^s(\rho)$ similarly to (5.18) but putting $b=a\sqrt{\zeta}$. Since $V_u(\rho)\subset\mathcal{C}_\gamma^u(\rho)$, the spaces $V_u(\rho)$ and $V_s^0(\rho)$ are transverse to each other. Thus we can write

$$v=v_1+v_2\quad\text{for some}\quad v_1\in V_u(\rho),\quad v_2\in V_s^0(\rho).$$

Denote

$$v(t):=d\varphi^{-t}(\rho)v,\quad v_1(t):=d\varphi^{-t}(\rho)v_1,\quad v_2(t):=d\varphi^{-t}(\rho)v_2.$$

Since $v,v_1\in E_u(\rho)$, we have $v(t),v_1(t)\in\mathcal{C}_\gamma^u(\varphi^{-t}(\rho))$ for all $t\geq 0$. It follows from (5.13) applied to $v(t),v_1(t)$ that

$$|v_2(t)|\leq|v(t)|+|v_1(t)|\leq e^{-\nu t}(|v|+|v_1|)\quad\text{for all}\quad t\geq 0. \tag{5.20}$$

On the other hand, since $v_2\in\mathcal{C}_\gamma^s(\rho)$ we have by (5.14)

$$|v_2|\leq e^{-\nu t}|v_2(t)|\quad\text{for all}\quad t\geq 0. \tag{5.21}$$

Combining (5.20) and (5.21) and letting $t\to\infty$ we get $v_2=0$, thus $v=v_1\in V_u(\rho)$ which gives (5.19).

Now, (5.19) implies immediately that $E_u(\rho)$ is a one-dimensional subspace of $E_{us}(\rho)$. We have $d\varphi^{-t}(\rho)E_u(\rho)\subset E_u(\varphi^{-t}(\rho))$ for all $t\geq 0$, which gives the invariance property (4.48). The expansion property (4.49) follows from (5.13) and the inclusion $E_u(\rho)\subset\mathcal{C}_\gamma^u(\rho)$. □

Remark. The proof of Lemma 5.4 can be used to show the stability of Anosov maps/flows under perturbations, namely a small C^N perturbation of an Anosov map/flow is still Anosov. This uses the fact that (a slightly modified version of) the properties (5.11)–(5.14) is stable under perturbations; note it is enough to require these properties for $0\leq t\leq 1$. See [KaHa97, Corollary 6.4.7 and Proposition 17.4.4] for details.

5.2. **A simple case of the billiard ball map.** We finally briefly discuss two-dimensional billiard ball maps. Let $\Omega\subset\mathbb{R}^2$ be a domain with smooth boundary, referring the reader to [ChMa06] for a comprehensive treatment and history of the subject. We do not require Ω to be bounded but we require that the boundary $\partial\Omega$ be compact; this implies that either Ω is compact (*interior case*) or $\mathbb{R}^2\setminus\Omega^\circ$ is compact (*exterior case*).

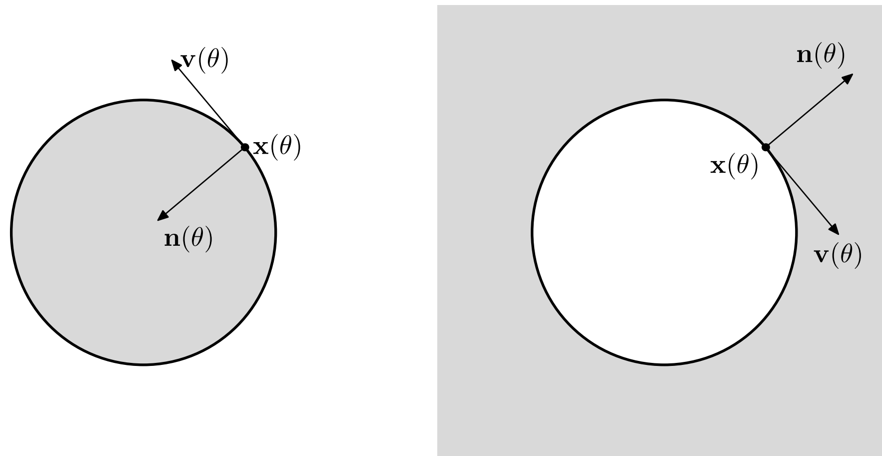

FIGURE 12. The interior (left) and exterior (right) case when the boundary is a circle of radius r. The domain Ω is shaded. The curvature is equal to $1/r$ in the interior case and to $-1/r$ in the exterior case.

The boundary $\partial\Omega$ is diffeomorphic to the union of finitely many circles. We parametrize it locally by a real number variable θ, denoting the parametrization

$$\mathbf{x} : \partial\Omega \to \mathbb{R}^2.$$

We assume that $\mathbf{x}$ is a unit speed parametrization:

$$|\mathbf{v}| \equiv 1 \quad \text{where} \quad \mathbf{v}(\theta) := \partial_\theta \mathbf{x}(\theta).$$

We choose the inward pointing unit normal vector field

$$\mathbf{n} : \partial\Omega \to \mathbb{R}^2.$$

We assume that $(\mathbf{v}, \mathbf{n})$ is a positively oriented frame on $\mathbb{R}^2$ at every point of $\partial\Omega$. If $\partial\Omega$ consists of a single circle, then the direction of increasing θ is counterclockwise in the interior case and clockwise in the exterior case. For each $\theta \in \partial\Omega$ we define the *curvature* $K(\theta)$ of the boundary at θ by the following identity:

$$\partial_\theta \mathbf{v}(\theta) = K(\theta)\mathbf{n}(\theta).$$

We say that the boundary is (strictly) *convex* at some point θ if $K(\theta) > 0$ and (strictly) *concave* if $K(\theta) < 0$. See Figure 12.

The billiard ball map acts on the phase space M which consists of inward pointing unit vectors at the boundary:

$$M = \{(\theta, \mathbf{w}) \mid \theta \in \partial\Omega,\ \mathbf{w} \in \mathbb{R}^2,\ |\mathbf{w}| = 1,\ \langle \mathbf{w}, \mathbf{n}(\theta)\rangle \geq 0\}.$$

The boundary of the phase space ∂M consists of *glancing* directions, where $\mathbf{w}$ is tangent to $\partial\Omega$. These directions are what makes billiards much more difficult to handle than closed manifolds; in these notes we ignore entirely the complications resulting from

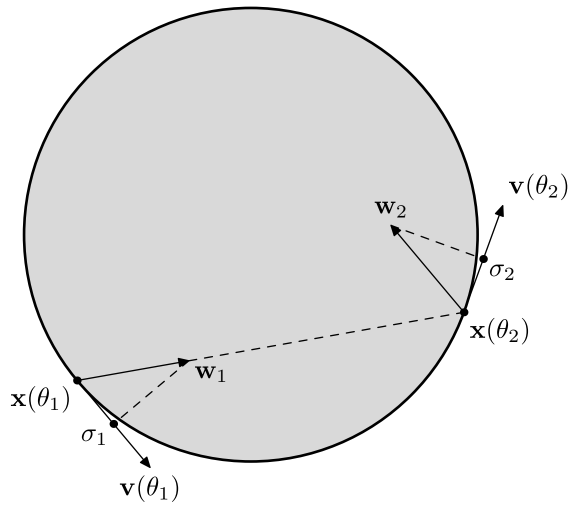

FIGURE 13. The billiard ball map for the disk (interior case), given in case of the unit disk by the formulas $\theta_2 = \theta_1 + 2\arccos\sigma_1$, $\sigma_2 = \sigma_1$.

glancing, restricting to the interior of M. The interior M° can be parametrized by two numbers $\theta \in \partial\Omega$ and $\sigma \in (-1,1)$ defined by

$$\sigma := \langle \mathbf{w}, \mathbf{v}(\theta)\rangle.$$

In particular $\sigma = 0$ corresponds to vectors which are orthogonal to the boundary.

We now define the billiard ball map

$$\varphi : M^\circ \to M^\circ, \quad \varphi(\theta_1, \sigma_1) = (\theta_2, \sigma_2)$$

where $\mathbf{x}(\theta_2)$ is the first intersection of $\partial\Omega$ with the ray $\{\mathbf{x}(\theta_1) + t\mathbf{w}_1 \mid t > 0\}$ and $\mathbf{w}_2$ is obtained by the law of reflection – see Figure 13. The map φ is in fact only defined on an open subset of M° since the ray might intersect $\partial\Omega$ in a glancing direction or (in the exterior case) may escape to infinity without intersecting $\partial\Omega$, we ignore here the issues arising from this fact.

Define the function on $\partial\Omega \times \partial\Omega$

$$\Phi(\theta_1, \theta_2) := |\mathbf{x}(\theta_1) - \mathbf{x}(\theta_2)|.$$

Then Φ is the generating function of φ, namely

$$\varphi(\theta_1, \sigma_1) = (\theta_2, \sigma_2) \quad \Longleftrightarrow \quad \sigma_1 = -\partial_{\theta_1}\Phi(\theta_1, \theta_2),\ \sigma_2 = \partial_{\theta_2}\Phi(\theta_1, \theta_2). \tag{5.22}$$

(Strictly speaking, we should restrict the right-hand side above to $\theta_1 \neq \theta_2$ such that the interior of the line segment between $\mathbf{x}(\theta_1)$ and $\mathbf{x}(\theta_2)$ does not intersect $\partial\Omega$.)

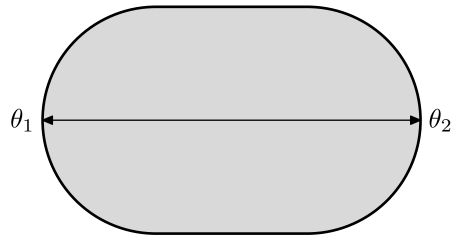

FIGURE 14. A hyperbolic trajectory on a Bunimovich stadium

We have

$$\partial^2_{\theta_1}\Phi=\frac{1-\sigma_1^2}{\Phi}-K(\theta_1)\sqrt{1-\sigma_1^2},\quad \partial^2_{\theta_2}\Phi=\frac{1-\sigma_2^2}{\Phi}-K(\theta_2)\sqrt{1-\sigma_2^2},$$
$$\partial_{\theta_1\theta_2}\Phi=\frac{\sqrt{(1-\sigma_1^2)(1-\sigma_2^2)}}{\Phi}.$$

Therefore, if $\varphi(\theta_1,\sigma_1)=(\theta_2,\sigma_2)$ and we denote $\ell:=\Phi(\theta_1,\theta_2)$ (the distance traveled between the bounces) and $K_1:=K(\theta_1)$, $K_2:=K(\theta_2)$, then

$$d\varphi(\theta_1,\sigma_1)=\begin{pmatrix}\dfrac{\ell K_1-\sqrt{1-\sigma_1^2}}{\sqrt{1-\sigma_2^2}} & -\dfrac{\ell}{\sqrt{(1-\sigma_1^2)(1-\sigma_2^2)}}\\[2ex] K_1\sqrt{1-\sigma_2^2}+K_2\sqrt{1-\sigma_1^2}-\ell K_1K_2 & \dfrac{\ell K_2-\sqrt{1-\sigma_2^2}}{\sqrt{1-\sigma_1^2}}\end{pmatrix}. \quad (5.23)$$

Note that $\det d\varphi\equiv 1$.

We now discuss under which conditions φ is hyperbolic. We start with the simplest case of a closed trajectory of period 2 (which is necessarily orthogonal to the boundary):

Proposition 5.5. *Assume that $(\theta_1,0)$ is a periodic point for φ with period 2, namely $\varphi(\theta_1,0)=(\theta_2,0)$ and $\varphi(\theta_2,0)=(\theta_1,0)$ for some $\theta_2\in\partial\Omega$. Let $\ell:=|\mathbf{x}(\theta_1)-\mathbf{x}(\theta_2)|$, $K_1:=K(\theta_1)$, $K_2:=K(\theta_2)$. Then φ is hyperbolic on the closed trajectory $\{(\theta_j,0)\}$, in the sense of Definition 4.1, if and only if the following condition holds:*

$$(1-\ell K_1)(1-\ell K_2)\notin[0,1]. \quad (5.24)$$

Remark. Condition (5.24) always holds in the concave case (when $K_1,K_2<0$). However it sometimes also holds in the convex case, see Figure 14.

Proof. Put

$$A:=d\varphi^2(\theta_1,0)=d\varphi(\theta_2,0)d\varphi(\theta_1,0).$$

Then φ is hyperbolic on $\{(\theta_j,0)\}$ if and only if A has no eigenvalues on the unit circle. Since $\det A=1$, this is equivalent to $|\operatorname{tr}A|>2$. Computing

$$\operatorname{tr}A=2+4\ell(\ell K_1K_2-K_1-K_2)$$

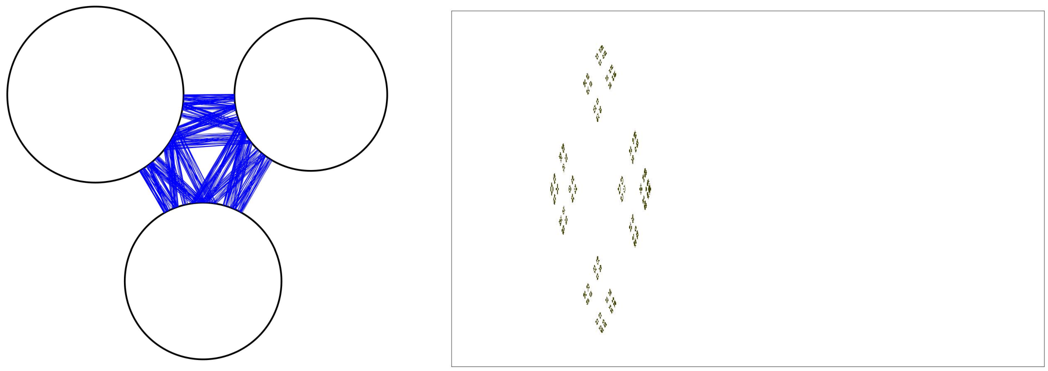

FIGURE 15. Left: the set of trapped trajectories in the exterior of 3 disks. Right: the corresponding reduction to the boundary $\mathcal{K}$, that is the trapped set of the billiard ball map, restricted to the bottom circle; the horizontal direction is θ and the vertical direction is σ. Both sets exhibit fractal structure.

we arrive to the condition (5.24). □

For general sets we restrict to the concave case (the corresponding billiards are sometimes called *dispersing*):

Proposition 5.6. *Assume that $\mathcal{K} \subset M^\circ$ is a φ-invariant compact set and the curvature K satisfies $K(\theta) < 0$ for all $(\theta, \sigma) \in \mathcal{K}$. Then the billiard ball map φ is hyperbolic on $\mathcal{K}$ in the sense of Definition 4.1.*

Remark. One example of a compact φ-invariant set is a closed (non-glancing) trajectory. Another example is when Ω is the complement of several strictly convex obstacles (that is, the exterior concave case), we impose the *no-eclipse condition* that no obstacle intersects the convex hull of the union of any two other obstacles, and $\mathcal{K}$ is the reduction to boundary of the *trapped set* which consists of all billiard ball trajectories which stay in a bounded set for all times. The no-eclipse condition ensures that trapped trajectories cannot be glancing. See Figure 15.

Sketch of the proof. We argue similarly to §5.1. Consider the cones in $\mathbb{R}^2$

$$\mathcal{C}_0^u := \{(v_\theta, v_\sigma) \mid v_\theta \cdot v_\sigma \geq 0\}, \quad \mathcal{C}_0^s := \{(v_\theta, v_\sigma) \mid v_\theta \cdot v_\sigma \leq 0\}.$$

By the concavity condition, for all $\rho := (\theta_1, \sigma_1) \in \mathcal{K}$ we have $K_1, K_2 < 0$ in (5.23). Thus all entries of the matrix $d\varphi(\rho)$ are negative. It follows that

$$d\varphi(\rho)\mathcal{C}_0^u \subset \mathcal{C}_0^u, \quad d\varphi(\rho)^{-1}\mathcal{C}_0^s \subset \mathcal{C}_0^s. \tag{5.25}$$

Next, at each $\rho = (\theta, \sigma) \in \mathcal{K}$ define the norm $|\bullet|_\rho$ by

$$|(v_\theta, v_\sigma)|_\rho^2 := (1 - \sigma^2)v_\theta^2 + \frac{v_\sigma^2}{1 - \sigma^2}.$$

Then there exists $\lambda > 1$ such that for all $\rho \in \mathcal{K}$ and $v \in \mathbb{R}^2$

$$v \in \mathcal{C}_0^u \quad \Longrightarrow \quad |d\varphi(\rho)v|_{\varphi(\rho)} \geq \lambda |v|_\rho, \tag{5.26}$$

$$v \in \mathcal{C}_0^s \quad \Longrightarrow \quad |d\varphi^{-1}(\rho)v|_{\varphi(\rho)} \geq \lambda |v|_\rho. \tag{5.27}$$

The hyperbolicity of φ on $\mathcal{K}$ now follows by adapting the proof of Lemma 5.4, using (5.25)–(5.27) in place of (5.11)–(5.14). □

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