Fractal uncertainty principle and applications

Semyon Dyatlov (UC Berkeley/MIT/Clay Mathematics Institute)

October 9, 2017

- This talk presents several recent results in quantum chaos
- Central ingredient: fractal uncertainty principle (FUP)

No function can be localized
in both position and frequency
near a fractal set

- Using tools from
  - Microlocal analysis (classical/quantum correspondence)
  - Hyperbolic dynamics (classical chaos)
  - Fractal geometry
  - Harmonic analysis
- Despite recent progress, many open problems remain

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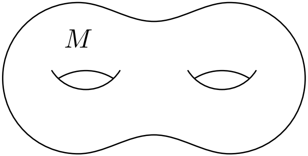
Application: control of eigenfunctions

- (M, g) compact hyperbolic surface
- Geodesic flow $\varphi_t : T^*M \to T^*M$ is a standard model of classical chaos
- Eigenfunctions of the Laplacian $-\Delta_g$ studied by quantum chaos



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Theorem 1 [Bourgain–D '16, D–Jin '17]

Let $\Omega \subset M$ be an arbitrary nonempty open set. Then

$$\|u\|_{L^2(\Omega)} \geq c > 0$$

where c depends on M, Ω but not on λ

For bounded λ this follows from unique continuation principle

The new result is in the high frequency limit $\lambda \to \infty$

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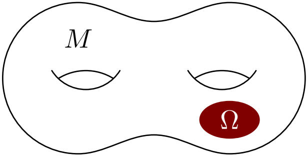

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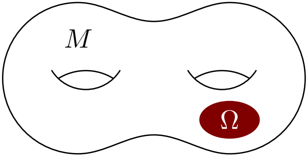

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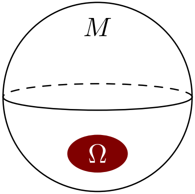

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The chaotic nature of geodesic flow is important

For example, Theorem 1 is false if M is the round sphere

Microlocal analysis

Localization in position and frequency using semiclassical quantization

$$a(x,\xi)\in C^\infty(T^*M)\ \mapsto\ \operatorname{Op}_h(a)=a\Big(x,\frac{h}{i}\partial_x\Big):C^\infty(M)\to C^\infty(M)$$

Examples (on $\mathbb{R}^n$): $\operatorname{Op}_h(x_j)u=x_ju$, $\operatorname{Op}_h(\xi_j)u=\frac{h}{i}\partial_{x_j}u$

Properties of quantization in the semiclassical limit $h\to 0$

- $\operatorname{Op}_h(a)\operatorname{Op}_h(b)=\operatorname{Op}_h(ab)+\mathcal{O}(h)$
- $\operatorname{Op}_h(a)^*=\operatorname{Op}_h(\bar a)+\mathcal{O}(h)$
- $[\operatorname{Op}_h(a),\operatorname{Op}_h(b)]=-ih\operatorname{Op}_h(\{a,b\})+\mathcal{O}(h^2)$
- $\sup|a|<\infty\quad\Longrightarrow\quad\|\operatorname{Op}_h(a)\|_{L^2\to L^2}=\mathcal{O}(1)$

Rescale $(-\Delta_g-\lambda^2)u=0$, $\lambda\to\infty$

to obtain $(-h^2\Delta_g-1)u=0$, $h=\lambda^{-1}\to 0$

where $-h^2\Delta_g-1=\operatorname{Op}_h(p^2-1)$, $p(x,\xi)=|\xi|_g$

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Microlocal version of Theorem 1

General elliptic estimate

If $a, b \in C^\infty(T^*M)$ and $\operatorname{supp} b \subset \{a \neq 0\}$ then for all $u \in L^2(M)$

$$\|\operatorname{Op}_h(b)u\| \leq C\|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^\infty)\|u\|$$

Localization of eigenfunctions to $S^*M := \{(x,\xi) \in T^*M : |\xi|_g = 1\}$

$$\text{Assume} \quad (-h^2\Delta_g - 1)u = 0, \quad \|u\|_{L^2(M)} = 1. \tag{1}$$

Then $\operatorname{supp} b \cap S^*M = \emptyset \quad \Longrightarrow \quad \|\operatorname{Op}_h(b)u\|_{L^2} = \mathcal{O}(h^\infty)$

Theorem 1′ [Bourgain–D '16, D–Jin '17]

Let $a \in C_c^\infty(T^*M)$ satisfy $a|_{S^*M} \not\equiv 0$, u satisfy (1). Then for $h \ll 1$

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Semiclassical measures

Take a high frequency sequence of Laplacian eigenfunctions

$$(-h_j^2\Delta_g - 1)u_j = 0, \quad \|u_j\|_{L^2(M)} = 1, \quad h_j \to 0$$

We say u_j converges weakly to a measure μ on T^*M if

$$\forall a \in C_c^\infty(T^*M): \quad \langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle_{L^2} \to \int_{T^*M} a\, d\mu \quad \text{as } j \to \infty$$

Call such limits μ semiclassical measures

Basic properties

- μ is a probability measure, $\operatorname{supp}\mu \subset S^*M$
- μ is invariant under the geodesic flow $\varphi_t : S^*M \to S^*M$
- Natural candidate: Liouville measure $\mu_L \sim d\,\mathrm{vol}$ (equidistribution)
- Natural enemy: delta measure δ_γ on a closed geodesic (scarring)

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Theorem 1″ [Bourgain–D '16, D–Jin '17]

Let μ be a semiclassical measure on M. Then $\operatorname{supp}\mu = S^*M$

Brief overview of history

- Quantum Ergodicity [Shnirelman '74, Zelditch '87, Colin de Verdière '85]: $\mu = \mu_L$ for density 1 sequence of eigenfunctions
- Quantum Unique Ergodicity conjecture [Rudnick–Sarnak '94]: $\mu = \mu_L$ for all eigenfunctions, that is μ_L is the only semiclassical measure. Proved in the arithmetic case [Lindenstrauss '06]

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Brief overview of history, continued

- Entropy bound [Anantharaman '08, A–Nonnenmacher '07]: $H_{\mathrm{KS}}(\mu) \geq \frac{1}{2}$, in particular $\mu \neq \delta_\gamma$. Here H_{KS} denotes Kolmogorov–Sinai entropy. Note $H_{\mathrm{KS}}(\mu_L) = 1$ and $H_{\mathrm{KS}}(\delta_\gamma) = 0$
- Theorem 1″: between QE and QUE and 'orthogonal' to entropy bound. There exist φ_t-invariant μ with $\operatorname{supp}\mu \neq S^*M$, $H_{\mathrm{KS}}(\mu) > \frac{1}{2}$

Proof of Theorem 1′

$$(-h^2\Delta_g - 1)u = 0, \quad \|u\|_{L^2} = 1, \quad a \in C_c^\infty(T^*M), \quad a|_{S^*M} \not\equiv 0$$

We say u is controlled on an open set $V \subset T^*M$ if

$$\|\operatorname{Op}_h(b)u\|_{L^2} \leq C\|\operatorname{Op}_h(a)u\|_{L^2} + o(1)_{h\to 0} \quad \text{when } \operatorname{supp} b \subset V$$

Goal: show u is controlled on T^*M (then can take $b \equiv 1$, $\operatorname{Op}_h(b)u = u$)

- u is controlled away from S^*M (by ellipticity)
- u is controlled on $\{a \neq 0\}$ (also by ellipticity)
- Use the half-wave propagator $U(t) = \exp(-it\sqrt{-\Delta_g})$

$$U(t)u = e^{-it/h}u \quad \Longrightarrow \quad \|U(-t)\operatorname{Op}_h(a)U(t)u\|_{L^2} = \|\operatorname{Op}_h(a)u\|_{L^2}$$

  Egorov's Theorem: $U(-t)\operatorname{Op}_h(a)U(t) = \operatorname{Op}_h(a\circ\varphi_t) + \mathcal{O}(h)$ where $\varphi_t = \exp(tH_p) : T^*M \to T^*M$ is the homogeneous geodesic flow
- Thus u is controlled on $\varphi_t(\{a \neq 0\})$ for all t, $|t| \leq \rho\log(1/h)$, $\rho < 1$

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- u is controlled on $\varphi_t(\{a \neq 0\})$ for $|t| \leq T(h) := \rho \log(1/h)$
- Thus $u = \mathrm{Op}_h(b_\pm)u + \text{(controlled)}$ for some $b_\pm$, $\operatorname{supp} b_\pm \subset \Gamma_\pm(h)$,

$$\Gamma_\pm(h) = \big\{(x,\xi) \in T^*M \colon\ \varphi_{\mp t}(x,\xi) \notin \{a \neq 0\} \quad \forall\, t \in [0, T(h)]\big\}$$

- Hyperbolicity of φ_t + unique ergodicity of horocycle flows $\implies$

  $\Gamma_+(h)$ smooth in the unstable direction, porous in the stable direction

  $\Gamma_-(h)$ smooth in the stable direction, porous in the unstable direction

  (using Arnold cat map model for the figures)

$\Gamma_-(h)$, $T = 0$ $\qquad$ $\{a = 0\}$ $\qquad$ $\Gamma_+(h)$, $T = 0$

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- Hyperbolicity of φ_t + unique ergodicity of horocycle flows $\implies$

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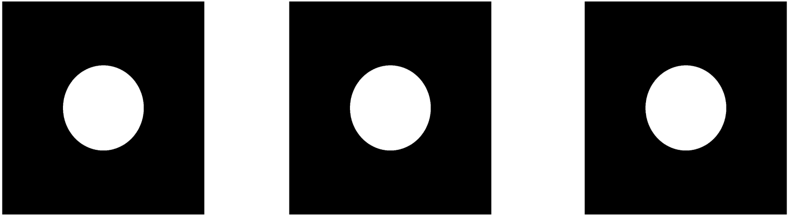

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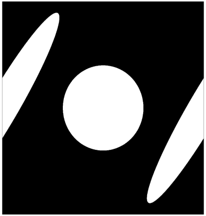
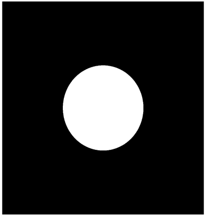
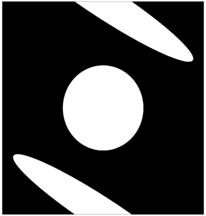


$\Gamma_-(h)$, $T = 1$



$\{a = 0\}$



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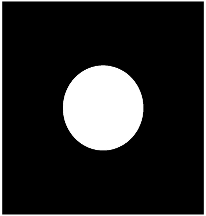
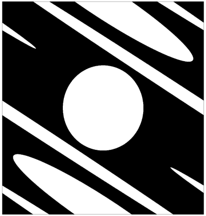


$\{a = 0\}$



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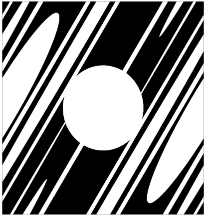
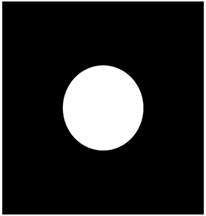


$\Gamma_-(h)$, $T = 3$



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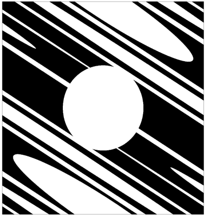

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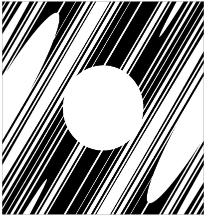
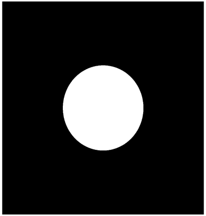


$\Gamma_-(h)$, $T = 4$



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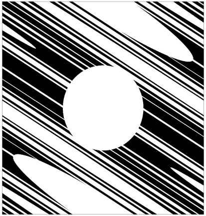

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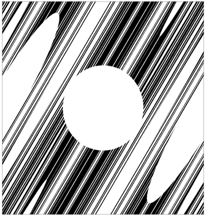

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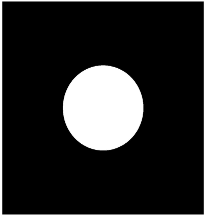

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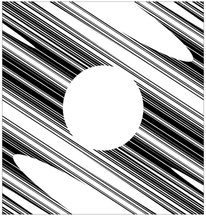

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$$u = \mathrm{Op}_h(b_-)\,\mathrm{Op}_h(b_+)u + \text{(controlled)}$$

$$\operatorname{supp} b_\pm \subset \Gamma_\pm(h)$$

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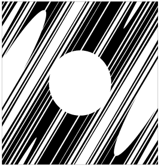
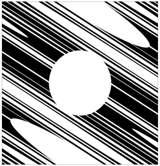
$$\nu = \nu(M, \{a \neq 0\}) > 0$$



$\Gamma_-(h)$



$\Gamma_+(h)$

Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is ν-porous up to scale h if for each interval $I \subset R$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$

Fractal uncertainty principle + porosity of $\operatorname{supp} b_\pm$ gives

$$\|\mathrm{Op}_h(b_-)\,\mathrm{Op}_h(b_+)\|_{L^2(M)\to L^2(M)} = \mathcal{O}(h^\beta) \quad \text{for some } \beta = \beta(\nu) > 0$$

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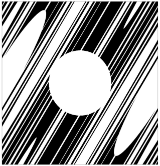

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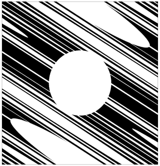

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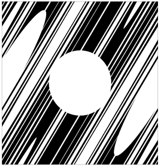

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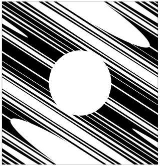

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Fractal uncertainty principle

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Simplified setting on $\mathbb{R}$ using unitary semiclassical Fourier transform

$$\mathcal{F}_h f(\xi) = (2\pi h)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi/h} u(x)\, dx, \quad u \in L^2(\mathbb{R})$$

Localization in stable direction	$\to$	Localization in position
$\mathrm{Op}_h(b_+)$	$\to$	$\mathbb{1}_X$, $X \subset \mathbb{R}$
Localization in unstable direction	$\to$	Localization in frequency
$\mathrm{Op}_h(b_-)$	$\to$	$\mathcal{F}_h^* \mathbb{1}_Y \mathcal{F}_h$, $Y \subset \mathbb{R}$
$\\|\mathrm{Op}_h(b_-)\mathrm{Op}_h(b_+)\\|_{L^2(M)\to L^2(M)}$	$\to$	$\\|\mathbb{1}_Y \mathcal{F}_h \mathbb{1}_X\\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}$

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Theorem 2 [Bourgain–D '16]

Assume that $X, Y \subset [0,1]$ are ν-porous up to scale h. Then

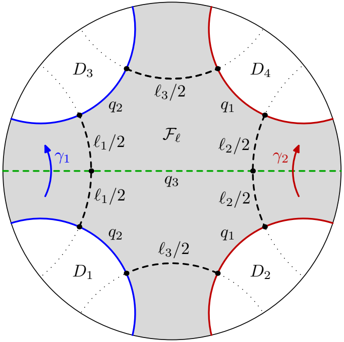
$$\| \mathbb{1}_Y \, \mathcal{F}_h \, \mathbb{1}_X \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as } h \to 0$$

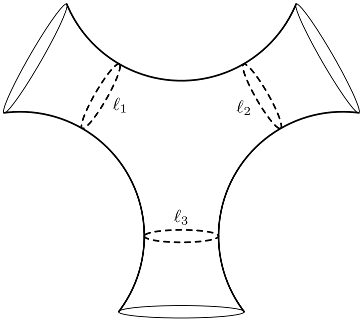
where $\beta = \beta(\nu) > 0$

The proof uses tools from harmonic analysis, in particular the Beurling–Malliavin theorem, and iteration on scale

Another application: spectral gaps

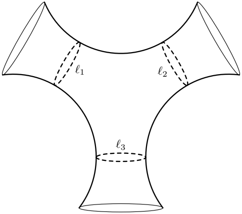
$(M, g) = \Gamma\backslash\mathbb{H}^2$ convex co-compact hyperbolic surface





An example: three-funnel surface with neck lengths ℓ_1, ℓ_2, ℓ_3

Resonances of hyperbolic surfaces



(M, g) convex co-compact hyperbolic surface

Δ_g Laplace–Beltrami operator on $L^2(M)$

The L^2 spectrum of $-\Delta_g$ consists of

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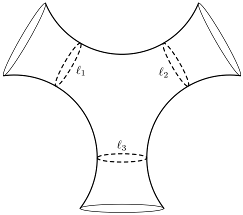
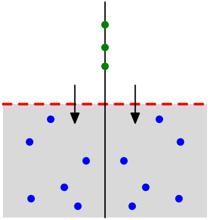


Resonances are poles of the meromorphic continuation

$$R(\lambda) = \Big(-\Delta_g - \lambda^2 - \frac{1}{4}\Big)^{-1} : \begin{cases} L^2 \to H^2, & \operatorname{Im}\lambda > 0 \\ L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}, & \operatorname{Im}\lambda \leq 0 \end{cases}$$



Essential spectral gaps

Definition

M has an essential spectral gap of size $\beta \geq 0$ if the half-plane $\{\operatorname{Im} \lambda \geq -\beta\}$ only has finitely many resonances

Applications of spectral gaps

- Resonance expansions of linear waves with $\mathcal{O}(e^{-\beta t})$ remainder
- Strichartz estimates [Burq–Guillarmou–Hassell '10]
- Diophantine problems [Bourgain–Gamburd–Sarnak '11, Magee–Oh–Winter '14]

Previous results ($\delta \in (0,1)$ dimension of the limit set)

- Patterson '76, Sullivan '79: $\beta = \frac{1}{2} - \delta$. Related to pressure gap
- Naud '05: $\beta > \frac{1}{2} - \delta$

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Application of FUP to spectral gaps

Theorem 3 [D–Zahl '16, Bourgain–D '16]

Every convex co-compact surface M has an essential spectral gap of some size $\beta = \beta(M) > 0$

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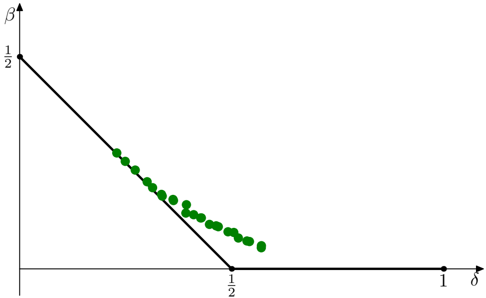

Numerics for 3- and 4-funneled surfaces by Borthwick–Weich '14
+ standard gap $\beta = \max(0, \frac{1}{2} - \delta)$

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Theorem 3 [D–Zahl '16, Bourgain–D '16]

Every convex co-compact surface M has an essential spectral gap of some size $\beta = \beta(M) > 0$

The proof uses fractal uncertainty principle

$$\| \operatorname{Op}_h(b_-) \operatorname{Op}_h(b_+) \|_{L^2(M) \to L^2(M)} = \mathcal{O}(h^\beta) \quad \text{as } h \to 0,$$
$$\operatorname{supp} b_\pm \subset \Gamma_\pm(h)$$

but this time $\Gamma_\pm(h)$ are the sets of forward/backward trapped geodesics:

$$\Gamma_\pm(h) = B \cap \varphi_{\pm T}(B), \quad T = \log(1/h)$$

where $B \subset T^*M$ is large but bounded set and φ_t is the geodesic flow

Open problems

- Can Theorem 1 (control of eigenfunctions) and Theorem 3 (spectral gap) be extended to surfaces of variable negative curvature and more general systems with hyperbolic classical dynamics?
- Can Theorems 1 and 3 be extended to higher dimensional manifolds?
- Is the exponent in FUP bigger for generic systems?

Thank you for your attention!