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POLLICOTT–RUELLE RESONANCES FOR OPEN SYSTEMS

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Abstract. We define Pollicott–Ruelle resonances for geodesic flows on noncompact asymptotically hyperbolic negatively curved manifolds, as well as for more general open hyperbolic systems related to Axiom A flows. These resonances are the poles of the meromorphic continuation of the resolvent of the generator of the flow and they describe decay of classical correlations. As an application, we show that the Ruelle zeta function extends meromorphically to the entire complex plane.

For an Anosov flow on a compact manifold, *Pollicott–Ruelle resonances* are complex numbers which describe fine features of decay of correlations [Po85, Ru86]. They also are the singularities of the meromorphic extension of the Ruelle zeta function, whose existence (conjectured by Smale [Sm]) has recently been proved on compact manifolds by Giulietti–Liverani–Pollicott [GLP], see also Dyatlov–Zworski [DyZw13].

The purpose of this paper is to define Pollicott–Ruelle resonances for *open hyperbolic systems*. An example is the geodesic flow $\varphi^t = e^{tX} : SM \to SM$ on an asymptotically hyperbolic negatively curved noncompact Riemannian manifold M (see §6.3). Building on the microlocal approach of Faure–Sjöstrand [FaSj] and [DyZw13], we show that:

- the resolvent $(X+\lambda)^{-1} : L^2(SM) \to L^2(SM)$, $\operatorname{Re}\lambda > 0$, continues meromorphically to $\mathbf{R}(\lambda) : C_0^\infty(SM) \to \mathcal{D}'(SM)$, $\lambda \in \mathbb{C}$ (Theorem 1);
- the singular part of $\mathbf{R}(\lambda)$ at its poles (called Pollicott–Ruelle resonances) is described in terms of support and wavefront set (Theorem 2);
- the Ruelle zeta function $\zeta(\lambda) = \prod_{\gamma^\sharp}(1 - e^{-\lambda T_{\gamma^\sharp}})$, where $T_{\gamma^\sharp} > 0$ are the lengths of primitive closed geodesics, extends meromorphically to $\lambda \in \mathbb{C}$ (Theorem 3).

These results are motivated by decay of correlations, counting closed trajectories, linear response, and boundary rigidity in geometric inverse problems – see the discussion below.

Rather than consider the flow on the entire SM, it suffices to work with its restriction to $\mathcal{U} = SU$, where $U \subset M$ is a large convex compact set containing all trapped trajectories. Our results hold under the following general assumptions (see §6.1 for a basic example and §6.2 for applications to boundary problems):

(A1) $\overline{\mathcal{U}}$ is an n-dimensional compact manifold with interior $\mathcal{U}$ and boundary $\partial\mathcal{U}$, X is a smooth (C^∞) nonvanishing vector field on $\overline{\mathcal{U}}$, and $\varphi^t = e^{tX}$ is the corresponding flow;

(A2) $\rho \in C^\infty(\overline{\mathcal{U}})$ is a boundary defining function, that is $\rho > 0$ on $\mathcal{U}$, $\rho = 0$ on $\partial\mathcal{U}$, and $d\rho \neq 0$ on $\partial\mathcal{U}$;

(A3) the boundary $\partial\mathcal{U}$ is *strictly convex* in the sense that

$$x \in \partial\mathcal{U},\ X\rho(x) = 0 \quad \Longrightarrow \quad X^2\rho(x) < 0. \tag{1.1}$$

The condition (1.1) does not depend on the choice of ρ. We embed $\overline{\mathcal{U}}$ into some *compact* manifold $\mathcal{M}$ without boundary (which is unrelated to the noncompact manifold SM used in the example above) and extend the vector field X there, so that the flow φ^t is defined for all times, see §2.1. We choose the extension of X to $\mathcal{M}$ (also denoted X) so that $\mathcal{U}$ is convex in the sense that (see Lemma 2.10)

$$x, \varphi^T(x) \in \mathcal{U},\ T \geq 0 \quad \Longrightarrow \quad \varphi^t(x) \in \mathcal{U} \quad \text{for all } t \in [0, T]. \tag{1.2}$$

Define the *incoming/outgoing tails* $\Gamma_\pm \subset \overline{\mathcal{U}}$ and the *trapped set* K by

$$\Gamma_\pm := \bigcap_{\pm t \geq 0} \varphi^t(\overline{\mathcal{U}}), \quad K := \Gamma_+ \cap \Gamma_-. \tag{1.3}$$

We have $K \subset \mathcal{U}$, see §2.1. We make the assumption that the flow is *hyperbolic* on K:

(A4) for each $x \in K$, there is a splitting

$$T_x\mathcal{M} = E_0(x) \oplus E_s(x) \oplus E_u(x), \quad E_0(x) := \mathbb{R}X(x) \tag{1.4}$$

continuous in x, invariant under φ^t, and such that for some constants $C, \gamma > 0$,

$$|d\varphi^t(x) \cdot v| \leq Ce^{-\gamma|t|}|v|, \quad \begin{cases} t \geq 0,\ v \in E_s(x); \\ t \leq 0,\ v \in E_u(x). \end{cases} \tag{1.5}$$

In the terminology of [KaHa, Definitions 6.4.18 and 17.4.1], K is a *locally maximal hyperbolic set* for the flow φ^t. We expect that, by constructing strictly convex neighborhoods, our results hold for any locally maximal hyperbolic set, and thus for any basic set of an Axiom A flow [Po86, §7]. (We keep the strict convexity assumption because it simplifies the proofs and is satisfied in many cases.) We remark that our proofs never use that the periodic orbits are dense in K (which is part of Axiom A).

We finally assume that

(A5) we fix a smooth complex vector bundle $\mathcal{E}$ over $\overline{\mathcal{U}}$ and a first order differential operator $\mathbf{X} : C^\infty(\overline{\mathcal{U}}; \mathcal{E}) \to C^\infty(\overline{\mathcal{U}}; \mathcal{E})$ such that

$$\mathbf{X}(f\mathbf{u}) = (Xf)\mathbf{u} + f(\mathbf{X}\mathbf{u}), \quad f \in C^\infty(\overline{\mathcal{U}}),\ \mathbf{u} \in C^\infty(\overline{\mathcal{U}}; \mathcal{E}). \tag{1.6}$$

In the scalar case, where $\mathcal{E} = \mathbb{R}$ is the trivial bundle, (1.6) means that $\mathbf{X} = X - V$, where $V \in C^\infty(\overline{\mathcal{U}}; \mathbb{C})$ is a potential. We extend $\mathbf{X}$ arbitrarily to $\mathcal{M}$ so that (1.6) holds.

One important special case is that of *Anosov flows*, when $\mathcal{U} = \mathcal{M}$ is a compact manifold without boundary, and thus $K = \mathcal{U}$. In this situation, Pollicott–Ruelle resonances have been studied extensively, see the overview of previous work below.

Meromorphic continuation of the resolvent. Fix a smooth measure μ on $\mathcal{M}$ and a smooth metric on the fibers of $\mathcal{E}$ (neither needs to be invariant under the flow); this fixes a norm on $L^2(\mathcal{M};\mathcal{E})$. We consider the *transfer operator* $e^{-t\mathbf{X}} : L^2(\mathcal{M};\mathcal{E}) \to L^2(\mathcal{M};\mathcal{E})$. For the scalar case $\mathcal{E} = \mathbb{R}$, $\mathbf{X} = X - V$, it has the form

$$e^{-t\mathbf{X}} f(x) = \exp\Big(\int_0^t V(\varphi^{-s}(x))\,ds\Big) f(\varphi^{-t}(x)). \tag{1.7}$$

Note that (1.6) implies the following property characterizing the support of $e^{-t\mathbf{X}}$:

$$e^{-t\mathbf{X}}(f\mathbf{u}) = (f\circ\varphi^{-t})e^{-t\mathbf{X}}\mathbf{u}, \quad f\in C^\infty(\mathcal{M}),\ \mathbf{u}\in C^\infty(\mathcal{M};\mathcal{E}). \tag{1.8}$$

For a large constant C_0, we have

$$\|e^{-t\mathbf{X}}\|_{L^2(\mathcal{M};\mathcal{E})\to L^2(\mathcal{M};\mathcal{E})} \le e^{C_0 t}, \quad t\ge 0. \tag{1.9}$$

For $\operatorname{Re}\lambda > C_0$, the resolvent $(\mathbf{X}+\lambda)^{-1}$ on $L^2(\mathcal{M};\mathcal{E})$ is given by the formula

$$(\mathbf{X}+\lambda)^{-1}\mathbf{f} = \int_0^\infty e^{-t(\mathbf{X}+\lambda)}\mathbf{f}\,dt. \tag{1.10}$$

For the purposes of meromorphic continuation, we consider the *restricted resolvent*

$$\mathbf{R}(\lambda) = \mathbb{1}_{\mathcal{U}}(\mathbf{X}+\lambda)^{-1}\,\mathbb{1}_{\mathcal{U}} : C_0^\infty(\mathcal{U};\mathcal{E}) \to \mathcal{D}'(\mathcal{U};\mathcal{E}), \quad \operatorname{Re}\lambda > C_0. \tag{1.11}$$

Here $\mathcal{D}'(\mathcal{U};\mathcal{E})$ is the space of distributions on $\mathcal{U}$ with values in $\mathcal{E}$. By (1.2), (1.8), and (1.10), $\mathbf{R}(\lambda)$ does not depend on the values of $\mathbf{X}$ outside of $\mathcal{U}$. Our main result is

Theorem 1. *Under the assumptions (A1)–(A5), the family $\mathbf{R}(\lambda)$ defined in (1.11) continues meromorphically to $\lambda\in\mathbb{C}$, with poles of finite rank. These poles are called Pollicott–Ruelle resonances of $\mathbf{X}$.*

In fact, we can define $\mathbf{R}(\lambda)$ as the restricted resolvent of a Fredholm problem on certain anisotropic Sobolev spaces – see §4.2. For the case of Anosov flows, our definition of resonances coincides with that of [FaSj], the only difference being the convention for the spectral parameter – if $\{\lambda_j\}$ are the resonances in Theorem 1, then the resonances of [FaSj] are $\{i\lambda_j\}$.

Characterization of resonant states. We next study the singular parts of $\mathbf{R}(\lambda)$. For each $\lambda\in\mathbb{C}$, $j\ge 1$ define the space of *generalized resonant states*

$$\operatorname{Res}^{(j)}_{\mathbf{X}}(\lambda) = \{\mathbf{u}\in\mathcal{D}'(\mathcal{U};\mathcal{E}) \mid \operatorname{supp}\mathbf{u}\subset\Gamma_+,\ \operatorname{WF}(\mathbf{u})\subset E_+^*,\ (\mathbf{X}+\lambda)^j\mathbf{u}=0\}. \tag{1.12}$$

Here $E_+^*\supset E_u^*$ is the extended unstable bundle over Γ_+, constructed in Lemma 2.10. We will also use the extended stable bundle $E_-^*\supset E_s^*$ over Γ_-. The symbol WF denotes the wavefront set, see §3.1.

Theorem 2. *For each $\lambda_0 \in \mathbb{C}$, we have the expansion*

$$\mathbf{R}(\lambda) = \mathbf{R}_H(\lambda) + \sum_{j=1}^{J(\lambda_0)} \frac{(-1)^{j-1}(\mathbf{X}+\lambda_0)^{j-1}\Pi_{\lambda_0}}{(\lambda-\lambda_0)^j} \tag{1.13}$$

for some family $\mathbf{R}_H(\lambda) : C_0^\infty(\mathcal{U};\mathcal{E}) \to \mathcal{D}'(\mathcal{U};\mathcal{E})$ holomorphic near λ_0 and some finite rank operator $\Pi_{\lambda_0} : C_0^\infty(\mathcal{U};\mathcal{E}) \to \mathcal{D}'(\mathcal{U};\mathcal{E})$. Moreover, if $K_{\Pi_{\lambda_0}}$ is the Schwartz kernel of Π_{λ_0} and $\mathrm{WF}'(\Pi_{\lambda_0})$ is its wavefront set (see (3.1), (3.2)), then

$$\operatorname{supp} K_{\Pi_{\lambda_0}} \subset \Gamma_+ \times \Gamma_-, \quad \mathrm{WF}'(\Pi_{\lambda_0}) \subset E_+^* \times E_-^*; \tag{1.14}$$

$$\Pi_{\lambda_0}^2 = \Pi_{\lambda_0}, \quad \mathbf{X}\Pi_{\lambda_0} = \Pi_{\lambda_0}\mathbf{X}, \quad \operatorname{Ran}(\Pi_{\lambda_0}) = \operatorname{Res}_{\mathbf{X}}^{(J(\lambda_0))}(\lambda_0). \tag{1.15}$$

The operator products in (1.15) are understood in the sense of distributions. The operator $\Pi_{\lambda_0}^2 : C_0^\infty(\mathcal{U};\mathcal{E}) \to \mathcal{D}'(\mathcal{U};\mathcal{E})$ is well-defined due to (1.14), since $\Gamma_+ \cap \Gamma_- = K$ is a compact subset of $\mathcal{U}$ and E_+^*, E_-^* only intersect at the zero section – see [HöI, Theorem 8.2.14].

Note that Theorem 2 implies that λ_0 is a resonance if and only if the space $\operatorname{Res}_{\mathbf{X}}^{(1)}(\lambda_0)$ of resonant states is nontrivial.

We can apply Theorem 2 to the operator $\mathbf{X}^*$, which satisfies (1.6) with X replaced by $-X$ and $\mathcal{E}$ replaced by $\mathcal{E}^* \otimes |\Omega|^1$, with $|\Omega|^1$ the bundle of densities on $\mathcal{U}$. The direction of the flow is reversed, which means that Γ_+, E_+^* switch places with Γ_-, E_-^*. Therefore,

$$\operatorname{Ran}((\Pi_{\lambda_0})^*) = \operatorname{Res}_{\mathbf{X}^*}^{J(\lambda_0)}(\lambda_0),$$

where $\operatorname{Res}_{\mathbf{X}^*}^{(j)}(\lambda)$ is the space of generalized coresonant states:

$$\operatorname{Res}_{\mathbf{X}^*}^{(j)}(\lambda) = \{\mathbf{v} \in \mathcal{D}'(\mathcal{U};\mathcal{E}^* \otimes |\Omega|^1) \mid \operatorname{supp}\mathbf{v} \subset \Gamma_-,\ \mathrm{WF}(\mathbf{v}) \subset E_-^*,\ (\mathbf{X}^* + \bar\lambda)^j \mathbf{v} = 0\}.$$

Note that for each $\mathbf{u} \in \operatorname{Res}_{\mathbf{X}}^{(1)}(\lambda), \mathbf{v} \in \operatorname{Res}_{\mathbf{X}^*}^{(1)}(\lambda)$, the pointwise product $\mathbf{u}\cdot\overline{\mathbf{v}} \in \mathcal{D}'(\mathcal{U};|\Omega|^1)$ is well-defined and supported on K, thanks to [HöI, Theorem 8.2.10]. Moreover, $\mathcal{L}_X(\mathbf{u}\cdot\overline{\mathbf{v}}) = 0$.

Ruelle zeta function. Let $V \in C^\infty(\mathcal{U};\mathbb{C})$. For a primitive closed trajectory $\gamma^\sharp : [0, T_{\gamma^\sharp}] \to K$ of φ^t of period $T_{\gamma^\sharp}$, let

$$V_{\gamma^\sharp} = \frac{1}{T_{\gamma^\sharp}} \int_0^{T_{\gamma^\sharp}} V(\gamma^\sharp(t))\, dt \tag{1.16}$$

be the average of V over $\gamma^\sharp$. Define the *Ruelle zeta function* as the following product over all primitive closed trajectories of φ^t on K:

$$\zeta_V(\lambda) := \prod_{\gamma^\sharp} \big(1 - \exp(-T_{\gamma^\sharp}(\lambda + V_{\gamma^\sharp}))\big), \quad \operatorname{Re}\lambda \gg 1.$$

The product converges for $\operatorname{Re}\lambda$ large enough since the number of closed trajectories of period no more than T grows at most exponentially in T – see Lemma 2.17.

Theorem 3. *Assume that the stable/unstable foliations E_u, E_s are orientable. Then the function $\zeta_V(\lambda)$ admits a meromorphic continuation to $\lambda \in \mathbb{C}$.*

Theorem 3 was established in [GLP] in the special case of Anosov flows when additionally $V = 0$. Another argument based on microlocal methods was presented in [DyZw13] and served as the starting point of our proof. The singularities (zeroes and poles) or ζ_V are Pollicott–Ruelle resonances for certain operators on the bundle of differential forms, see §5.2. Our methods actually prove meromorphic continuation of more general dynamical traces – see Theorem 4 in §5.1. The orientability condition can be relaxed, see (5.11) and the remarks following it.

Applications to boundary value problems. A useful corollary of our work is the well-posedness (up to a finite dimensional space corresponding to resonant states) of the two boundary value problems for the transport equation

$$(X - V)u = f \text{ in } \mathcal{U}, \quad u|_{\partial_\pm \mathcal{U}} = 0$$

for u, f in certain anisotropic Sobolev spaces, where $\partial_\pm \mathcal{U} := \{x \in \partial\mathcal{U} \mid \mp X\rho > 0\}$ and V is a potential; see for instance Proposition 6.1 and particularly [G, §4]. The microlocal description of solutions is crucial in the proof of lens rigidity of surfaces with hyperbolic trapped sets and no conjugate points in [G].

Motivation and discussion. We call a resonance λ_0 the *first resonance* of $\mathbf{X}$ if λ_0 is simple (that is, $\operatorname{rank} \Pi_{\lambda_0} = 1$), $\lambda_0 \in \mathbb{R}$, and there are no other resonances with $\operatorname{Re}\lambda \geq \operatorname{Re}\lambda_0$. We say that there is a *spectral gap* of size $\nu > 0$ if there are no resonances with $\operatorname{Re}\lambda \geq \operatorname{Re}\lambda_0 - \nu$, and an *essential spectral gap* if the number of resonances with $\operatorname{Re}\lambda \geq \operatorname{Re}\lambda_0 - \nu$ is finite. The size of an essential spectral gap on compact manifolds is bounded from above, see Jin–Zworski [JiZw].

The first resonances and the corresponding resonant states capture key dynamical features of the flow. For Anosov flows in the scalar case $\mathcal{E} = \mathbb{C}, \mathbf{X} = X$, zero is always a resonance since the function 1 is a resonant state. Moreover, resonances with $\operatorname{Re}\lambda = 0$ have equal algebraic and geometric multiplicity ($\operatorname{Res}^{(j)}_{\mathbf{X}}(\lambda) = \operatorname{Res}^{(1)}_{\mathbf{X}}(\lambda)$) and the associated projectors Π_λ are bounded on the space $C^0(\mathcal{U})$ of continuous functions; this follows from Theorem 2 and the fact that $\|\mathbf{R}(\lambda)\|_{C^0 \to C^0} \leq (\operatorname{Re}\lambda)^{-1}$ for $\operatorname{Re}\lambda > 0$. The space of coresonant states $\operatorname{Res}^{(1)}_{\mathbf{X}^*}(0)$ consists of Sinai–Ruelle–Bowen (SRB) measures. One consequence of this relation is the fact that SRB measures depend smoothly on a parameter, if the vector field X depends smoothly on that parameter – this is known as *linear response*. Moreover, ergodicity of the flow with respect to the SRB measure is equivalent to zero being a simple resonance, and mixing is equivalent to zero being the first resonance. See [BuLi] for details.

For Axiom A flows, the first pole of the Ruelle zeta function $\zeta_V(\lambda)$ for $V \equiv 0$ is the topological entropy h_{top} of the flow φ^t. This implies the asymptotic formula

$N^\sharp(T) \sim e^{h_{\mathrm{top}}T}/(h_{top}T)$ for the number $N^\sharp(T)$ of primitive closed trajectories of period less than T – see [PaPo83]. The associated (co)resonant states are related to Margulis measures on the stable/unstable foliations and their product is the measure of maximal entropy – see [Ma, BoMa, GLP].

If one has an essential spectral gap of size ν with a polynomial (in λ) resolvent bound, then there is a *resonance expansion* with remainder $\mathcal{O}(e^{-(\nu-\operatorname{Re}\lambda_0)t})$ for correlations $\langle e^{-t\mathbf{X}}\mathbf{u},\mathbf{v}\rangle$, $\mathbf{u}\in C_0^\infty(\mathcal{U};\mathcal{E})$, $\mathbf{v}\in C_0^\infty(\mathcal{U};\mathcal{E}^*\otimes|\Omega|^1)$ – see [NoZw13, Corollary 5]. For the Anosov case and $\mathcal{E}=\mathbb{C}$, $\mathbf{X}=X$, existence of a spectral gap was proved by Dolgopyat [Do] and Liverani [Li04] (with subexponential decay of correlations earlier established by Chernov [Ch]); the precise size of the essential gap was given by Tsujii [Ts]. This followed earlier work of Ratner [Ra] and Moore [Mo] for locally symmetric spaces, including geodesic flows on compact hyperbolic manifolds. (See [DFG] for a detailed description of resonances in the latter case.)

Regarding the noncompact case, Naud [Na] established a spectral gap for $\mathcal{E}=\mathbb{C}$, $\mathbf{X}=X$ on convex co-compact hyperbolic surfaces. The first resonance in that case is given by $\delta-1$, where δ is the exponent of convergence of the Poincaré series of the fundamental group. This result was later extended to higher dimensions by Petkov–Stoyakov [PeSt]. Similarly, a spectral gap for the Ruelle zeta function implies an asymptotic formula for $N^\sharp(T)$ with an exponentially small remainder, see [GLP]. We also mention the work of Stoyanov [St11, St13a] on the spectral gap for the Ruelle zeta function of Axiom A flows under additional assumptions, as well as decay of correlations for Gibbs measures, as well as the work [St13b] addressing these questions for contact Anosov flows. The recent preprint of Petkov–Stoyanov [PeSt14] provides a spectral gap for transfer operators depending on two complex parameters. We note that results mentioned in this paragraph give a holomorphic continuation of the zeta function to a small strip past the domain of convergence under additional assumptions, such as contact structure of the flow or the local non-integrability condition; our result gives a meromorphic continuation to the entire complex plane without such additional assumptions, but at the cost of not establishing a spectral gap.

Previous work and methods of the proofs. We finally give a brief overview of the history of the subject and explain the methods used in the present paper.

Smale [Sm] defined Axiom A flows and formulated a conjecture [Sm, pp. 802–803] whether a certain zeta function, related trivially to the Ruelle zeta function, extends meromorphically to $\mathbb{C}$, admitting that 'a positive answer would be a little shocking'. Ruelle [Ru76] gave a positive answer to Smale's question for real analytic Anosov flows with analytic stable/unstable foliations; the analyticity assumption on the foliations, but not on the flow itself, was removed in the works of Rugh [Ru] in the case of Axiom A maps and Fried [Fr] for Axiom A flows. Pollicott [Po86, Po85] and Ruelle [Ru86, Ru87] extended the zeta function to a small strip past the first pole for general Axiom A flows

and related its poles to decay of correlations. These papers, as well as the previously mentioned work [Do, Na, St11, St13b] use *Ruelle transfer operators*, which conjugate the flow to a shift on a space constructed by symbolic dynamics. We refer the reader to the book of Parry–Pollicott [PaPo90] for a detailed description of this approach.

Later, Pollicott–Ruelle resonances for the special case of Anosov flows were interpreted as the eigenvalues of the generator of the flow (or of the transfer operator $e^{-t\mathbf{X}}$) on suitably designed *anisotropic spaces* which consist of functions which are regular in the stable directions and irregular in the unstable directions. These spaces fall into two categories:

- anisotropic Hölder spaces, studied by Liverani [Li04], Butterley–Liverani [BuLi], and for the related case of Anosov maps, by Blank–Keller–Liverani [BKL], Liverani [Li05], and Gouëzel–Liverani [GoLi]; and
- anisotropic Sobolev spaces, studied for Anosov maps by Baladi–Tsujii [BaTs] and used in the microlocal works discussed below.

Some similar ideas appeared already in the works of Rugh [Ru] in the analytic category and Kitaev [Ki]. Using anisotropic spaces, Pollicott–Ruelle resonances were defined in the entire complex plane and a meromorphic continuation of the Ruelle zeta function to $\mathbb{C}$ for Anosov flows was proved by Giulietti–Liverani–Pollicott [GLP].

The work of Faure–Sjöstrand [FaSj] for Anosov flows (following the earlier work [FRS] for Anosov maps and the work [Fa] on the prequantum cat map) interpreted the equation $(\mathbf{X}+\lambda)\mathbf{u} = \mathbf{f}$ on anisotropic Sobolev spaces as a *scattering problem*. This used the methods of *microlocal analysis* to consider the operator $e^{-t\mathbf{X}}$ as quantizing a Hamiltonian flow e^{tH_p} (see (2.9)) on the phase space $T^*\mathcal{U}$. In contrast with standard scattering problems (for the operator $-\Delta - \lambda^2$ on a noncompact Riemannian manifold), waves escape not to the spatial infinity $\{|x| = \infty\}$, but to the fiber infinity $\{|\xi| = \infty\}$, and the anisotropic spaces provide the correct regularity at the fiber infinity to make $(\mathbf{X} + \lambda)\mathbf{u} = \mathbf{f}$ into a Fredholm problem. The microlocal approach made it possible to apply the methods of scattering theory to Anosov flows, resulting in:

- sharp upper bound for the number of resonances in strips [DDZ] (improving the bound of [FaSj]);
- an essential spectral gap of optimal size [Ts, NoZw13];
- band structure for resonances of contact Anosov flows, including a Weyl law under the pinching condition and meromorphic continuation of the Gutzwiller–Voros zeta function [FaTs12, FaTs13a, FaTs13b];
- a microlocal proof of meromorphic continuation of Ruelle zeta function [DyZw13] (recovering the result of [GLP]);
- definition of resonances as limits of the eigenvalues of $\mathbf{X} - \varepsilon\Delta$ as $\varepsilon \to 0+$, and stochastic stability of resonances [DyZw14].

Our present work uses the microlocal method to obtain results for general open hyperbolic systems. In particular, we use anisotropic Sobolev spaces to control the singularities at fiber infinity. However, more work is needed to capture singularities which escape from $\mathcal{U}$. To do that, we surround $\mathcal{U}$ by a slightly larger strictly convex set and multiply X by a boundary defining function of this set to make it vanish on the boundary. We then use complex absorbing potentials on the boundary and complex absorbing pseudodifferential operators beyond the boundary (and near the glancing points) to obtain a global Fredholm problem for the extension of $\mathbf{X}$ to a compact manifold without boundary $\mathcal{M}$. The restriction of the resulting resolvent to $\mathcal{U}$ gives the meromorphic continuation of $\mathbf{R}(\lambda)$. To show meromorphic continuation of the zeta function, we use a wavefront set condition on $\mathbf{R}(\lambda)$ to ensure that a certain flat trace can be defined; this trace is the continuation of ζ_V'/ζ_V.

We finally remark that the work of Arnoldi–Faure–Weich [AFW] defined resonances for certain open hyperbolic maps, while [FaTs13b] defined resonances for the Grassmanian bundle of an Anosov flow, which can be viewed as special cases of the open hyperbolic systems studied in the present paper.

Structure of the paper. Sections 2 and 3 contain the necessary preliminary constructions; Section 2 concerns hyperbolic dynamical systems and Section 3, microlocal and semiclassical analysis. The proofs of Theorems 1 and 2 are contained in Section 4. Theorem 3 and the closely related Theorem 4 are proved in Section 5. Finally, Section 6 gives several examples of open hyperbolic systems, including geodesic flows on certain complete negatively curved Riemannian manifolds.

2. Dynamical preliminaries

In this section, we discuss several dynamical corollaries of assumptions (A1)–(A5) in the introduction. In particular, in §2.2, we show how to extend the stable/unstable bundles to $\Gamma_\pm$ (Lemma 2.10) and construct the components of the weight function for the anisotropic Sobolev space (Lemma 2.12).

2.1. Basic properties.

We start by showing that the vector field X can be extended from $\overline{\mathcal{U}}$ to a compact manifold without boundary so that $\mathcal{U}$ is convex:

Lemma 2.1. *Let $\mathcal{U}, X, \rho$ satisfy the assumptions (A1)–(A3) in the introduction. Then there exists a compact manifold without boundary $\mathcal{M} \supset \overline{\mathcal{U}}$ and a smooth extension of X to a vector field on $\mathcal{M}$ such that (1.2) holds. Moreover, $\overline{\mathcal{U}}$ satisfies the convexity condition (1.2) as well.*

Proof. We first embed $\overline{\mathcal{U}}$ into some compact manifold without boundary $\mathcal{M}$ (for example, by letting $\mathcal{M}$ be the doubling of $\overline{\mathcal{U}}$ across the boundary) and extend the function

ρ to $\mathcal{M}$ so that $\rho < 0$ on $\mathcal{M} \setminus \overline{\mathcal{U}}$. We next extend X in an abitrary way to $\mathcal{M}$ and call the resulting vector field X_1. It follows from (1.1) that for some constant $C > 0$,

$$X^2\rho < C(X\rho)^2 \quad \text{on } \partial\mathcal{U}.$$

By continuity, there exists $\varepsilon > 0$ such that

$$X_1^2\rho < C(X_1\rho)^2 \quad \text{on } \{|\rho| \leq 2\varepsilon\}. \tag{2.1}$$

We now take $\psi \in C^\infty(\mathbb{R})$ such that

$$\psi(s) = 1 \quad \text{for } s \geq 0, \quad \operatorname{sgn}\psi(s) = \operatorname{sgn}(s+\varepsilon), \quad \psi'(-\varepsilon) > 0.$$

The extension of X to $\mathcal{M}$ is then defined by

$$X := \psi(\rho)X_1; \quad X\rho = \psi(\rho)X_1\rho, \quad X^2\rho = \psi(\rho)^2X_1^2\rho + \psi(\rho)\psi'(\rho)(X_1\rho)^2.$$

It follows from (2.1) that

$$|\rho(x)| \leq 2\varepsilon,\ \rho(x) \neq -\varepsilon,\ X\rho(x) = 0 \quad \Longrightarrow \quad X^2\rho(x) < 0. \tag{2.2}$$

We now show that (1.2) holds. Assume that $x, \varphi^T(x) \in \mathcal{U}$ for some $T \geq 0$, but $\varphi^t(x) \notin \mathcal{U}$ for some $t \in [0,T]$. Denote $f(t) = \rho(\varphi^t(x))$. Let t_0 be the point that minimizes the value of f on $[0,T]$. By our assumptions, $f(t_0) \leq 0$ and thus $t_0 \in (0,T)$; it follows that $f'(t_0) = 0$ and $f''(t_0) \geq 0$. On the other hand, since X vanishes on $\{\rho = -\varepsilon\}$, we have $f(t_0) > -\varepsilon$. By (2.2), we have $f''(t_0) < 0$, giving a contradiction. The condition (1.2) is verified for $\overline{\mathcal{U}} = \{\rho \geq 0\}$ by the same argument. □

We henceforth assume that X is extended to $\mathcal{M}$ in the manner described in Lemma 2.1, and put $\varphi^t := e^{tX}$. We next establish the topological properties of $\Gamma_\pm$ and K:

Lemma 2.2. *Let K be defined in (1.3). Then $K \subset \mathcal{U}$.*

Proof. From (1.3), we see that $K \subset \overline{\mathcal{U}}$; therefore it suffices to show that $K \cap \partial\mathcal{U} = \emptyset$. Assume that $x \in K \cap \partial\mathcal{U}$. Then $\varphi^t(x) \in \overline{\mathcal{U}}$ for all $t \in \mathbb{R}$. Therefore, the function $f(t) := \rho(\varphi^t(x))$ has a local minimum at $t = 0$, which contradicts (1.1). □

Lemma 2.3. *Assume that $x \in \Gamma_\pm$. Then we have uniformly in x,*

$$\varphi^t(x) \to K \quad \text{as } t \to \mp\infty,$$

where convergence is understood as follows: for each neighborhood of K, $\varphi^t(\Gamma_\pm)$ lies inside that neighborhood for $\mp t$ large enough.

Proof. We consider the case of Γ_-; the case of Γ_+ is handled similarly. Since $\mathcal{M}$ is compact, it suffices to show that for each sequences $t_j \to +\infty$, $x_j \in \Gamma_-$, if $\varphi^{t_j}(x_j) \to x_\infty \in \mathcal{M}$, then $x_\infty \in K$; that is, $\varphi^t(x_\infty) \in \overline{\mathcal{U}}$ for all $t \in \mathbb{R}$. This is true since $\varphi^t(x_\infty)$ is the limit of $\varphi^{t+t_j}(x_j)$; it remains to use that $\varphi^{t+t_j}(x_j) \in \overline{\mathcal{U}}$ whenever $t + t_j \geq 0$, which happens for j large enough. □

Lemma 2.4. *Let $V \subset \mathcal{U}$ be a neighborhood of K. Then there exists $T > 0$ such that for each $x \in \overline{\mathcal{U}}$ such that $\varphi^T(x), \varphi^{-T}(x) \in \overline{\mathcal{U}}$, we have $x \in V$.*

Proof. It suffices to show that for each sequences $T_j \to +\infty$, $x_j \in \overline{\mathcal{U}}$, if $x_j \to x_\infty$ and $\varphi^{T_j}(x_j), \varphi^{-T_j}(x_j) \in \overline{\mathcal{U}}$, then $x_\infty \in K$. By Lemma 2.1, we have $\varphi^t(x_j) \in \overline{\mathcal{U}}$ for $|t| \le T_j$. Therefore, $\varphi^t(x_\infty) \in \overline{\mathcal{U}}$ for all t, implying that $x_\infty \in K$. □

Lemma 2.5. *Assume that $\chi \in C_0^\infty(\mathcal{U})$. Then there exists $\chi' \in C_0^\infty(\mathcal{U})$ such that*

$$x, \varphi^T(x) \in \operatorname{supp}(\chi),\ T \ge 0 \implies \varphi^t(x) \notin \operatorname{supp}(1-\chi') \quad \text{for all } t \in [0,T]. \tag{2.3}$$

Proof. Take $\delta \in (0, 2\varepsilon)$ small enough such that $\operatorname{supp}\chi \subset \{\rho \ge \delta\}$ and choose $\chi' \in C_0^\infty(\mathcal{U})$ such that $\chi' = 1$ near $\{\rho \ge \delta\}$. Similarly to the proof of Lemma 2.1, we derive from (2.1) that $\{\rho \ge \delta\}$ is convex; therefore, (2.3) holds. □

We next derive several properties of the vector fields X and X_1 near $\partial\mathcal{U}$:

Lemma 2.6. *Let ε, X_1, ρ be chosen in the proof of Lemma 2.1 and take $\alpha, \beta \in [-2\varepsilon, 2\varepsilon]$ such that $\alpha \le \beta$. Let $x \in \{\alpha \le \rho \le \beta\}$.*

1. There exists $T \ge 0$ such that $e^{TX_1}(x) \in \{\rho = \alpha\} \cup \{\rho = \beta\}$.

2. If additionally $X_1\rho(x) \le 0$, then there exists $T \ge 0$ such that $e^{TX_1}(x) \in \{\rho = \alpha\}$ and $X_1\rho(e^{tX_1}(x)) < 0$, $\rho(e^{tX_1}(x)) \in [\alpha, \beta)$ for all $t \in (0, T]$.

Same is true when X_1 is replaced by $-X_1$.

Proof. Denote $f(t) := \rho(e^{tX_1}(x))$. Then by (2.1), there exists $\delta > 0$ such that

$$f''(t) + \delta \le C(f'(t))^2 \quad \text{if } |f(t)| \le 2\varepsilon.$$

Then for some $\delta_1 > 0$,

$$g''(t) > \delta_1 \quad \text{if } |f(t)| \le 2\varepsilon, \quad g(t) := e^{-Cf(t)}.$$

It follows immediately that we cannot have $f(t) \in [\alpha, \beta]$ for all $t \ge 0$; this implies part 1. To see part 2, we note that $X_1\rho(x) \le 0$ implies that $g'(0) \ge 0$; then there exists $T \ge 0$ such that $g(T) = e^{-C\alpha}$ and $g'(t) > 0$, $g(t) \in (e^{-C\beta}, e^{-C\alpha}]$ for all $t \in (0, T]$. □

Lemma 2.7. *Let ε, X_1 be chosen in the proof of Lemma 2.1 and consider the sets*

$$\Sigma_\pm := \bigcup_{\pm t \ge 0} \varphi^t(\mathcal{U}), \quad \Sigma := \Sigma_+ \cup \Sigma_-. \tag{2.4}$$

Then $\overline{\Sigma} \cap \{\rho = -\varepsilon\} \cap \{X_1\rho = 0\} = \emptyset$.

Proof. Take $x \in \{\rho = -\varepsilon\} \cap \{X_1\rho = 0\}$. Then by (2.1), the function $t \mapsto \rho(e^{tX_1}(x))$ has a nondegenerate local maximum at $t = 0$. Therefore, there exists $\delta > 0$ such that

$$e^{\pm\delta X_1}(x) \in \{\rho < -\varepsilon\}, \quad e^{tX_1}(x) \notin \overline{\mathcal{U}} \quad \text{for all } t \in [-\delta, \delta]. \tag{2.5}$$

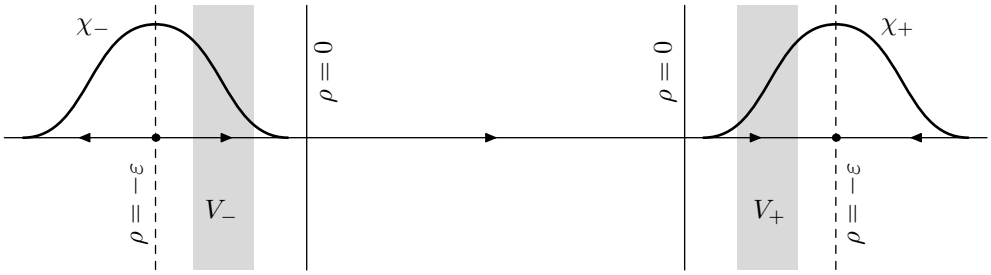


FIGURE 1. A nontrapped trajectory of the vector field X_1 with the sets $V_\pm$ and the functions $\chi_\pm$. The arrows indicate the direction of the field $X = \psi(\rho)X_1$.

Fix δ and take x' in a small neighborhood of x. Then (2.5) holds also for x'. Since X is a multiple of X_1 which vanishes on $\{\rho = -\varepsilon\}$, it follows that the trajectory $\varphi^t(x')$ never passes through $\overline{\mathcal{U}}$; that is, $x' \notin \Sigma$. It follows that $x \notin \overline{\Sigma}$, finishing the proof. □

Lemma 2.8. *Let $V_\pm \subset \Sigma_\pm \setminus \mathcal{U}$ be a compact set. Then there exists a function*

$$\chi_\pm \in C_0^\infty(\{-2\varepsilon < \rho < \varepsilon\} \cap \{\pm X_1\rho < 0\}; [0,1])$$

such that $\pm X\chi_\pm \geq 0$ everywhere and $\pm X\chi_\pm > 0$ on $V_\pm$. (See Figure 1.)

Proof. We construct χ_+; the function χ_- is constructed similarly, reversing the direction of the flow. By compactness of V_+, it suffices to prove the lemma for the case when $V_+ = \{x_0\}$, where $x_0 \in \Sigma_+ \setminus \mathcal{U}$. Note that $x_0 \in \{\rho > -\varepsilon\}$ since $x_0 \in \Sigma_+$ and X vanishes on $\{\rho = -\varepsilon\}$.

We first claim that $X_1\rho(x_0) < 0$. Indeed, assume that $X_1\rho(x_0) \geq 0$. Then by part 2 of Lemma 2.6 (with $[\alpha, \beta] = [-\varepsilon, 0]$), there exists $T \geq 0$ such that $e^{-TX_1}(x_0) \in \{\rho = -\varepsilon\}$ and $e^{-tX_1}(x_0) \notin \mathcal{U}$ for all $t \in [0, T]$. Since $X = \psi(\rho)X_1$, we see that $\varphi^{-t}(x_0) \notin \mathcal{U}$ for all $t \geq 0$, contradicting the fact that $x_0 \in \Sigma_+$.

By part 2 of Lemma 2.6 (with $[\alpha, \beta] = [-\varepsilon, 0]$), there exists $T \geq 0$ such that $x_1 := e^{TX_1}(x_0) \in \{\rho = -\varepsilon\}$ and $e^{tX_1}(x_0) \in \{-\varepsilon \leq \rho \leq 0\}$, $X_1\rho(e^{tX_1}(x_0)) < 0$ for all $t \in [0, T]$. Let U_1 be a small neighborhood of x_1 in the surface $\{\rho = -\varepsilon\}$. Then for $\delta > 0$ small enough, the map

$$(x', t) \in U_1 \times (-T - \delta, \delta) \;\mapsto\; e^{tX_1}(x') \in \mathcal{M} \tag{2.6}$$

is a diffeomorphism onto some open subset of $\{-2\varepsilon < \rho < \varepsilon\} \cap \{X_1\rho < 0\}$. Note that in the (x', t) coordinates, $X = \psi(\rho)\partial_t$ and $\operatorname{sgn}\psi(\rho) = -\operatorname{sgn} t$. It remains to put in the (x', t) coordinates,

$$\chi_+(x', t) = \chi_0(x')\chi_1(t),$$

where $\chi_0 \in C_0^\infty(U_1;[0,1])$ satisfies $\chi_0(x_1)=1$ and $\chi_1 \in C_0^\infty((-T-\delta,\delta);[0,1])$ satisfies $t\chi_1'(t)\le 0$ everywhere and $\chi_1'(-T)>0$. We finally extend χ_+ by zero to the entire $\mathcal{M}$. □

We finally give the following property of the resolvent $\mathbf{R}(\lambda)$ defined in (1.11).

Lemma 2.9. *Assume that $\psi_1,\psi_2 \in C_0^\infty(\mathcal{U})$ satisfy $\operatorname{supp}\psi_1 \cap \Gamma_- = \operatorname{supp}\psi_2 \cap \Gamma_+ = \emptyset$. Then the operators*

$$\mathbf{R}(\lambda)\psi_1,\ \psi_2\mathbf{R}(\lambda) : C_0^\infty(\mathcal{U}) \to \mathcal{D}'(\mathcal{U}), \quad \operatorname{Re}\lambda > C_0,$$

extend holomorphically to $\lambda\in\mathbb{C}$.

Proof. We establish holomorphic extension of $\mathbf{R}(\lambda)\psi_1$; the extension of $\psi_2\mathbf{R}(\lambda)$ is handled similarly. There exists $T>0$ such that $\varphi^t(\operatorname{supp}\psi_1)\cap\mathcal{U}=\emptyset$ for all $t\ge T$. Indeed, it is enough to show this for some $T=T(x_0)$ when the compact set $\operatorname{supp}\psi_1$ is replaced by a small neighborhood U_0 of some fixed $x_0\in\operatorname{supp}\psi_1$. Since $x_0\notin\Gamma_-$, there exists $T>0$ such that $\varphi^T(x_0)\notin\overline{\mathcal{U}}$. It follows that $\varphi^T(x)\notin\overline{\mathcal{U}}$ when x lies in a small neighborhood of U_0 of x_0. By convexity of $\mathcal{U}$, it follows that $\varphi^t(x)\notin\mathcal{U}$ for $t\ge T$ and $x\in U_0$.

The holomorphic extension of $\mathbf{R}(\lambda)\psi_1$ is now given by the formula

$$\mathbf{R}(\lambda)\psi_1\mathbf{f} = \int_0^T \big(e^{-t(\mathbf{X}+\lambda)}\psi_1\mathbf{f}\big)\big|_{\mathcal{U}}\,dt, \tag{2.7}$$

where we used the fact that $\big(e^{-t(\mathbf{X}+\lambda)}\psi_1\mathbf{f}\big)\big|_{\mathcal{U}}=0$ for $t\ge T$, following from (1.8). □

2.2. **Hyperbolic sets.** We next express the assumption (A4) from the introduction in terms of the action of the differential on the dual space. Define the function on the cotangent bundle $T^*\mathcal{M}$

$$p(x,\xi) = \langle X(x),\xi\rangle, \tag{2.8}$$

then its Hamiltonian flow is the action of $d\varphi^t$ on covectors:

$$e^{tH_p}(x,\xi) = (\varphi^t(x),(d\varphi^t(x))^{-T}\cdot\xi), \quad x\in\mathcal{M},\ \xi\in T_x^*\mathcal{M}, \tag{2.9}$$

where $(d\varphi^t(x))^{-T} : T_x^*\mathcal{M}\to T^*_{\varphi^t(x)}\mathcal{M}$ is the inverse transpose of $d\varphi^t(x) : T_x\mathcal{M}\to T_{\varphi^t(x)}\mathcal{M}$. For each $x\in K$, define the dual stable/unstable decomposition

$$T_x^*\mathcal{M} = E_0^*(x)\oplus E_s^*(x)\oplus E_u^*(x), \tag{2.10}$$

where E_0^* is the annihilator of $E_s\oplus E_u$, E_s^* is the annihilator of $E_0\oplus E_s$, and E_u^* is the annihilator of $E_0\oplus E_u$. Note the reversal of roles of E_s,E_u. By (1.5), we have

$$|(d\varphi^t(x))^{-T}\cdot\xi| \le Ce^{-\gamma|t|}|\xi|, \quad \begin{cases} t\ge 0,\ \xi\in E_s^*(x);\\ t\le 0,\ \xi\in E_u^*(x).\end{cases} \tag{2.11}$$

We now extend the bundles E_s^*, E_u^* to Γ_-, Γ_+ respectively, and study the global dynamics of the flow e^{tH_p}:

Lemma 2.10. *There exist vector subbundles $E_\pm^* \subset T^*_{\Gamma_\pm}\mathcal{M}$ over $\Gamma_\pm$ such that:*

1. *$E_+^*|_K = E_u^*$, $E_-^*|_K = E_s^*$, and $E_\pm^*(x)$ depend continuously on $x \in \Gamma_\pm$.*
2. *$E_\pm^*$ are invariant under the flow φ^t and $\langle X, \eta\rangle = 0$ for $\eta \in E_\pm^*$.*
3. *If $x \in \Gamma_\pm$ and $\xi \in E_\pm^*(x)$, then as $t \to \mp\infty$*

$$|(d\varphi^t(x))^{-T}\xi| \leq \widetilde{C}e^{-\tilde{\gamma}|t|}|\xi| \tag{2.12}$$

for some constants $\widetilde{C}, \tilde{\gamma} > 0$ independent of x, ξ.

4. *If $x \in \Gamma_\pm$ and $\xi \in T_x^*\mathcal{M}$ satisfies $p(x,\xi) = 0$ and $\xi \notin E_\pm^*(x)$, then as $t \to \mp\infty$*

$$|(d\varphi^t(x))^{-T}\xi| \to \infty, \quad \frac{(d\varphi^t(x))^{-T}\xi}{|(d\varphi^t(x))^{-T}\xi|} \to E_\mp^*|_K. \tag{2.13}$$

Proof. We construct E_-^*; the bundle E_+^* is contructed similarly. The lemma is a natural consequence of the lamination of Γ_- by the weak stable manifolds $(W_s(x))_{x\in K}$ of the flow, where we put E_-^* to be the annihilator of the tangent space of $W_s(x)$, see for example [NoZw09, §3.3]; the construction of $W_s(x)$ ultimately relies on the Hadamard–Perron Theorem [KaHa, Theorem 6.2.8]. However, to make the paper more self-contained and since we only need a small portion of the proof of the Hadamard–Perron theorem, we sketch a direct proof of the lemma below.

We fix some smooth Riemannian metric $\tilde{g}$ on $\mathcal{M}$ and measure the norms of cotangent vectors with respect to this metric. Denote by $d_{\tilde{g}}(\cdot,\cdot)$ the distance function induced by $\tilde{g}$. Take $\varepsilon > 0$ small enough to be fixed later; we in particular let ε be smaller than the injectivity radius of $(\mathcal{M}, \tilde{g})$. (This constant is unrelated to the one in Lemma 2.1.) For $x, y \in \mathcal{M}$ such that $d_{\tilde{g}}(x,y) < \varepsilon$, let

$$\tau_{x\to y} : T_x^*\mathcal{M} \to T_y^*\mathcal{M}$$

be the parallel transport along the shortest geodesic from x to y.

Using (2.11), fix $t_0 > 0$ such that for each $t \geq t_0$, $y \in K$ and $\eta \in T_y^*\mathcal{M}$,

$$\begin{aligned} |(d\varphi^t(y))^{-T}\eta| &\leq \frac{1}{10}|\eta|, \quad \eta \in E_s^*(y); \\ |(d\varphi^t(y))^{-T}\eta| &\geq 10|\eta|, \quad \eta \in E_u^*(y). \end{aligned}$$

For each $y \in K$, let

$$\pi_s(y) : T_y^*\mathcal{M} \to E_s^*(y), \quad \pi_u(y) : T_y^*\mathcal{M} \to E_u^*(y)$$

be the projection maps corresponding to the decomposition (2.11).

For $x \in \mathcal{M}$, $y \in K$, and $d_{\tilde g}(x,y) < \varepsilon$, define the *dual stable/unstable cones* inside the annihilator of X in $T^*_x\mathcal{M}$:

$$\begin{aligned}\mathcal{C}_s^{(y)}(x) &= \{\xi \in T^*_x\mathcal{M} \mid p(x,\xi) = 0,\ |\pi_s(y)\tau_{x\to y}\xi| \geq |\pi_u(y)\tau_{x\to y}\xi|\},\\ \mathcal{C}_u^{(y)}(x) &= \{\xi \in T^*_x\mathcal{M} \mid p(x,\xi) = 0,\ |\pi_u(y)\tau_{x\to y}\xi| \geq |\pi_s(y)\tau_{x\to y}\xi|\}.\end{aligned} \tag{2.14}$$

Then for ε small enough and each $t \in [t_0, 2t_0]$, $y, y' \in K$, and $x \in \mathcal{M}$ such that $d_{\tilde g}(x,y) < \varepsilon$, $d_{\tilde g}(\varphi^t(x), y') < \varepsilon$, we have similarly to [KaHa, Lemma 6.2.10]

$$\begin{aligned}(d\varphi^t(x))^{-T}\mathcal{C}_u^{(y)}(x) &\Subset \mathcal{C}_u^{(y')}(\varphi^t(x)),\\ (d\varphi^t(x))^{-T}\mathcal{C}_s^{(s)}(x) &\Supset \mathcal{C}_s^{(y')}(\varphi^t(x)).\end{aligned} \tag{2.15}$$

Indeed, (2.15) is verified directly for the case $x = y, \varphi^t(x) = y'$, and it follows for small ε by continuity. Moreover, similarly to [KaHa, Lemma 6.2.11] we find for $t \in [t_0, 2t_0]$,

$$\begin{aligned}|(d\varphi^t(x))^{-T}\xi| &\geq 4|\xi|, \quad \xi \in \mathcal{C}_u^{(y)}(x);\\ |(d\varphi^{-t}(x))^{-T}\xi| &\geq 4|\xi|, \quad \xi \in \mathcal{C}_s^{(y)}(x).\end{aligned} \tag{2.16}$$

For $x \in \Gamma_-$, we define $E^*_-(x)$ as follows: $\xi \in T^*_x\mathcal{M}$ lies in $E^*_-(x)$ if and only if $p(x,\xi) = 0$ and there exists $t_1 \geq 0$ such that for all $t \geq t_1$ and each $y \in K$ such that $d_{\tilde g}(\varphi^t(x), y) < \varepsilon$, we have $(d\varphi^t(x))^{-T}\xi \in \mathcal{C}_s^{(y)}(\varphi^t(x))$. (Recall that $d_{\tilde g}(\varphi^t(x), K) \to 0$ as $t \to +\infty$ by Lemma 2.3.)

By a straightforward adaptation of the proof of [KaHa, Proposition 6.2.12], we see that $E^*_-(x)$ is a linear subbundle of $T^*_{\Gamma_-}\mathcal{M}$ invariant under φ^t. In fact, for each $t_j \to +\infty$ and $y_j \in K$ with $d_{\tilde g}(\varphi^{t_j}(x), y_j) < \varepsilon$, we have

$$E^*_-(x) = \lim_{j\to\infty} (d\varphi^{t_j}(x))^T \tau_{y_j \to \varphi^{t_j}(x)} E^*_s(y_j)$$

where the limit is taken in the Grassmanian of $T^*_x\mathcal{M}$. The fact that $E^*_-(x) = E^*_s(x)$ for $x \in K$ follows from here immediately, as we can take $y_j := \varphi^{t_j}(x)$. The bound (2.12) follows directly from (2.16).

To show (2.13), take $x \in \Gamma_-$ and $\xi \in T^*_x\mathcal{M}$ such that $p(x,\xi) = 0$ and $\xi \notin E^*_-(x)$. By Lemma 2.3, there exists $t_1 \geq 0$ and $y_1 \in K$ such that $d_{\tilde g}(\varphi^{t_1}(x), y_1) < \varepsilon$ and

$$d_{\tilde g}(\varphi^t(x), K) < \varepsilon \quad \text{for } t \geq t_1; \quad (d\varphi^{t_1}(x))^{-T}\xi \notin \mathcal{C}_s^{(y_1)}(\varphi^{t_1}(x)). \tag{2.17}$$

Iterating (2.15), we see that

$$(d\varphi^t(x))^{-T}\xi \in \mathcal{C}_u^{(y)}(\varphi^t(x)), \quad t \geq t_0 + t_1, \tag{2.18}$$

for each $y \in K$ such that $d_{\tilde g}(\varphi^t(x), y) < \varepsilon$. Iterating (2.16), we get $|(d\varphi^t(x))^{-T}\xi| \to \infty$ as $t \to +\infty$. To see the second part of (2.13), it suffices to take an arbitrary sequence $t_j \to +\infty$ such that

$$\varphi^{t_j}(x) \to x_\infty \in K, \quad \frac{(d\varphi^{t_j}(x))^{-T}\xi}{|(d\varphi^{t_j}(x))^{-T}\xi|} \to \xi_\infty \in T^*_{x_\infty}\mathcal{M}$$

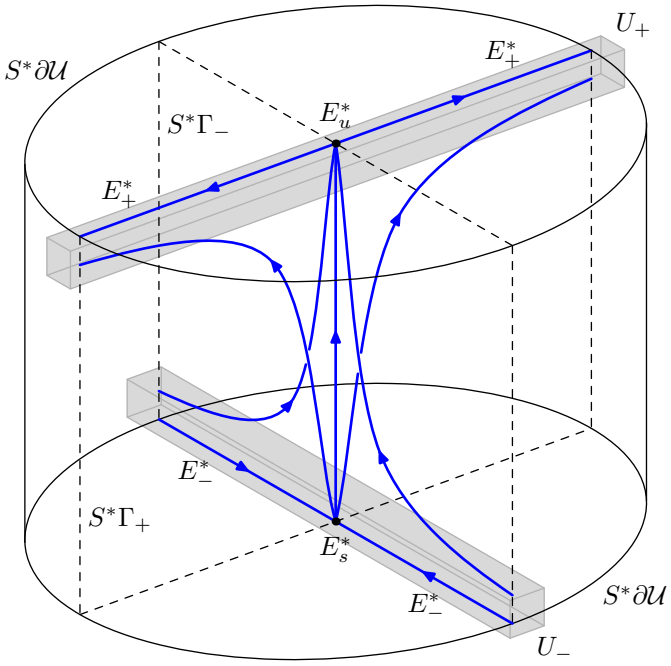


FIGURE 2. A schematic representation of the flow lines e^{tH_p} (thick blue lines) on $S^*\mathcal{U}$, which is depicted by the cylinder. The vertical direction in the picture corresponds to the fibers of $S^*\mathcal{U}$. The two vertical planes are $S^*\Gamma_\pm$, containing the subbundles $E_\pm^*$ (formally speaking, their images under κ); the vertical line in the middle is S^*K, containing the subbundles E_u^*, E_s^*. The shaded regions are the neighborhoods $U_\pm \supset E_\pm^*$.

and prove that $\xi_\infty \in E_u^*(x_\infty)$. Clearly $p(x_\infty, \xi_\infty) = 0$. Next, for each $t \geq 0$, we have

$$(d\varphi^{-t}(x_\infty))^{-T}\xi_\infty = \lim_{j\to\infty} \frac{(d\varphi^{t_j - t}(x))^{-T}\xi}{|(d\varphi^{t_j}(x))^{-T}\xi|} \in \mathcal{C}_u^{(\varphi^{-t}(x_\infty))}(\varphi^{-t}(x_\infty)),$$

which implies that $\xi_\infty \in E_u^*(x_\infty)$ as needed.

Finally, to show that $E_-^*(x)$ depends continuously on x, note that the condition (2.17) is stable under perturbations of x, ξ (recall that the convergence of Lemma 2.3 is uniform in x); on the other hand, similarly to (2.18) the condition (2.17) implies that $\xi \notin E_-^*(x)$. □

The subbundle E_+^* is a *generalized radial sink* and E_-^* is a *generalized radial source* in the following sense (this definition is a modification of [DyZw13, (2.12)]).

Lemma 2.11. *Let $\kappa : T^*\mathcal{M} \setminus 0 \to S^*\mathcal{M}$ be the canonical projection, where $S^*\mathcal{M}$ is the cosphere bundle over $\mathcal{M}$. Fix open neighborhoods $U_\pm \subset S^*\mathcal{M}$ of $\kappa(E_\pm^*)$ such that $\overline{U_\pm} \cap \kappa(E_\mp^*) = \emptyset$ (see Figure 2). Then for all $(x, \xi) \in T^*\mathcal{M} \setminus 0$ such that $p(x, \xi) = 0$,*

$\kappa(x,\xi)\in U_\pm$, and $x,\varphi^t(x)\in\overline{\mathcal{U}}$, we have

$$\begin{aligned} d\big(\kappa(e^{tH_p}(x,\xi)),\kappa(E_\pm^*)\big)&\to 0\quad \text{as } t\to\pm\infty;\\ |(d\varphi^t(x))^{-T}\xi|&\geq C^{-1}e^{\tilde\gamma|t|}|\xi|\quad \text{for } \pm t\geq 0,\end{aligned}\tag{2.19}$$

uniformly in (x,ξ). Here d denotes any distance function on $S^*\mathcal{M}$ and $C,\tilde\gamma>0$ are constants independent of x,ξ.

Proof. We study the trajectories starting in U_+ for $t\geq 0$; the behavior in U_- for $t\leq 0$ is proved similarly. It suffices to show that for each sequences $(x_j,\xi_j),(y_j,\eta_j)\in T^*\mathcal{M}\setminus 0$ and $t_j\to+\infty$ such that $e^{t_jH_p}(x_j,\xi_j)=(y_j,\eta_j)$, $p(x_j,\xi_j)=0$, $\kappa(x_j,\xi_j)\in U_+$, and $x_j,y_j\in\overline{\mathcal{U}}$, we have

$$d\big(\kappa(y_j,\eta_j),\kappa(E_\pm^*)\big)\to 0,\quad |\eta_j|\geq C^{-1}e^{\tilde\gamma t_j}|\xi_j|.\tag{2.20}$$

By passing to a subsequence, we may assume that $x_j\to x_\infty\in\overline{\mathcal{U}}$, $y_j\to y_\infty\in\overline{\mathcal{U}}$. For each $t\geq 0$, $\varphi^t(x_\infty)=\lim_{j\to\infty}\varphi^t(x_j)\in\overline{\mathcal{U}}$, therefore $x_\infty\in\Gamma_-$. Similarly $y_\infty\in\Gamma_+$. We also pass to a subsequence to make $\xi_j/|\xi_j|\to\xi_\infty\in T^*_{x_\infty}\mathcal{M}$, $\eta_j/|\eta_j|\to\eta_\infty\in T^*_{y_\infty}\mathcal{M}$, with $|\xi_\infty|=|\eta_\infty|=1$. Since $\kappa(x_j,\xi_j)\in U_+$ and $\overline{U_+}$ does not intersect $\kappa(E_-^*)$, we have $\xi_\infty\notin E_-^*(x_\infty)$.

For the first part of (2.20), we need to prove that $\eta_\infty\in E_+^*(y_\infty)$. Assume the contrary. We will use the proof of Lemma 2.10. Similarly to (2.18), for $t_2>0$ large enough and each $x',y'\in K$ such that $d_{\tilde g}(\varphi^{t_2}(x_\infty),x'),d_{\tilde g}(\varphi^{-t_2}(y_\infty),y')<\varepsilon$, we have

$$(d\varphi^{t_2}(x_\infty))^{-T}\xi_\infty\notin\mathcal{C}_s^{(x')}(\varphi^{t_2}(x_\infty)),\quad (d\varphi^{-t_2}(y_\infty))^{-T}\eta_\infty\notin\mathcal{C}_u^{(y')}(\varphi^{-t_2}(y_\infty)).$$

It follows that for t_2 large and fixed, and j large enough depending on t_2, we have

$$(d\varphi^{t_2}(x_j))^{-T}\xi_j\notin\mathcal{C}_s^{(x')}(\varphi^{t_2}(x_j)),\quad (d\varphi^{-t_2}(y_j))^{-T}\eta_j\notin\mathcal{C}_u^{(y')}(\varphi^{-t_2}(y_j)),\tag{2.21}$$

for each $x',y'\in K$ such that $d_{\tilde g}(\varphi^{t_2}(x_j),x'),d_{\tilde g}(\varphi^{-t_2}(y_j),y')<\varepsilon$. By Lemma 2.4, we can furthermore fix t_2 large enough so that for $t_j\geq 2t_2$, $d_{\tilde g}(\varphi^t(x_j),K)<\varepsilon$ for $t\in[t_2,t_j-t_2]$. Now, by the first statement of (2.21) and iterating (2.15), for j large enough and $x'\in K$, $d_{\tilde g}(x',\varphi^{t_j/2}(x_j))<\varepsilon$, we have $(d\varphi^{t_j/2}(x_j))^{-T}\xi_j\notin\mathcal{C}_s^{(x')}(\varphi^{t_j/2}(x_j))$. Similarly from the second statement of (2.21) we get $(d\varphi^{-t_j/2}(y_j))^{-T}\eta_j\notin\mathcal{C}_u^{(y')}(\varphi^{-t_j/2}(y_j))$. However, these two vectors are the same, giving a contradiction and implying the first part of (2.20).

The proof of the second part of (2.20) works in a similar fashion, using (2.16). □

To construct the weight function for anisotropic Sobolev spaces, we need the following adaptation of [FaSj, Lemma 2.1] (see also [DyZw13, Lemma C.1]). We consider e^{tH_p} as a flow on the sphere bundle $S^*\mathcal{M}$, by pulling it back by the projection $\kappa:T^*\mathcal{M}\setminus 0\to S^*\mathcal{M}$. Consider also the projection $\pi:S^*\mathcal{M}\to\mathcal{M}$.

Lemma 2.12. *Let $U_\pm \subset S^*\mathcal{M}$ be the neighborhoods of $\kappa(E_\pm^*)$ introduced in Lemma 2.11. Then there exist functions $m_\pm \in C^\infty(S^*\mathcal{M})$ such that:*

(1) *$m_\pm = 1$ near $\kappa(E_\pm^*)$ and $0 \leq m_\pm \leq 1$ everywhere;*
(2) *$\operatorname{supp} m_\pm \cap \{p = 0\} \cap \pi^{-1}(\overline{\mathcal{U}}) \subset U_\pm$;*
(3) *$\operatorname{supp} m_\pm \subset \pi^{-1}(\Sigma_\pm)$, where $\Sigma_\pm$ is defined in (2.4);*
(4) *$\pm H_p m_\pm \geq 0$ on $V \cap \pi^{-1}(\mathcal{U})$, where V is a neighborhood of $\{p = 0\}$.*

Proof. We construct m_+; the function m_- is constructed similarly, reversing the direction of propagation. Let $W \Subset \mathcal{U}$ be an open neighborhood of K. Fix $m_0 \in C_0^\infty(U_+ \cap \pi^{-1}(\mathcal{U}))$ such that $m_0 = 1$ in a neighborhood of $\kappa(E_+^*) \cap \pi^{-1}(\overline{W})$ and $0 \leq m_0 \leq 1$ everywhere.

We show that for $T > 0$ large enough and fixed, the function

$$m_+(x,\xi) = \frac{1}{T}\int_T^{2T} m_0(e^{-tH_p}(x,\xi))\,dt$$

has the required properties:

(1) Clearly $0 \leq m_+ \leq 1$ everywhere. Now, take $(x,\xi) \in \kappa(E_+^*)$. Then $x \in \Gamma_+$. By Lemma 2.3, $\varphi^{-t}(x) \in W$ for all $t \in [T, 2T]$ and T large enough. Since E_+^* is invariant under the flow, we have $e^{-tH_p}(x,\xi) \in \kappa(E_+^*)$ and thus $m_0(e^{-tH_p}(x,\xi)) = 1$ for $t \in [T, 2T]$, implying that $m(x,\xi) = 1$. Same argument works when (x,ξ) lies in a small neighborhood of $\kappa(E_+^*)$.
(2) Assume that $(x,\xi) \in \operatorname{supp} m_+ \cap \{p = 0\} \cap \pi^{-1}(\overline{\mathcal{U}})$. Then there exists $t \in [T, 2T]$ such that $e^{-tH_p}(x,\xi) \in \operatorname{supp} m_0$. Note that $x, \varphi^{-t}(x) \in \overline{\mathcal{U}}$ and $e^{-tH_p}(x,\xi) \in U_+$. Then by Lemma 2.11, for T large enough and $t \in [T, 2T]$, we have $(x,\xi) \in U_+$ as required.
(3) This follows immediately from (2.4) and the fact that $\operatorname{supp} m_0 \subset \pi^{-1}(\mathcal{U})$.
(4) Assume that $(x,\xi) \in S^*\mathcal{M}$, $x \in \mathcal{U}$, and $p(x,\xi) = 0$. Then

$$H_p m_+(x,\xi) = \frac{1}{T}\big(m_0(e^{-TH_p}(x,\xi)) - m_0(e^{-2TH_p}(x,\xi))\big).$$

We then need to show that $m_0(e^{-TH_p}(x,\xi)) \geq m_0(e^{-2TH_p}(x,\xi))$. Since $0 \leq m_0 \leq 1$, we only need to handle the case when $m_0(e^{-TH_p}(x,\xi)) < 1$ and $m_0(e^{-2TH_p}(x,\xi)) > 0$. In particular, we have $\varphi^{-2T}(x) \in \mathcal{U}$, and by Lemma 2.4, for T large enough, we have $\varphi^{-T}(x) \in W$. Then $e^{-TH_p}(x,\xi)$ does not lie in some fixed neighborhood W_1 of $\kappa(E_+^*)$, depending only on m_0. On the other hand, $e^{-2TH_p}(x,\xi) \in U_+$ and $\varphi^{-2T}(x), \varphi^{-T}(x) \in \overline{\mathcal{U}}$. By Lemma 2.11, we reach a contradiction for T large enough. Same reasoning applies if we replace the condition $p(x,\xi) = 0$ by $(x,\xi) \in V$ for some neighborhood V of $\{p = 0\}$. □

2.3. **Estimates on recurrence.** We finally give an extension of the recurrence estimates [DyZw13, Appendix A] to our situation, used in §5.1. Throughout this subsection, we fix $t_e > 0$ and a compact subset $V \subset \mathcal{U}$. We also consider the distance function $d_{\tilde{g}}$ and the parallel transport operators $\tau_{x\to y}$ introduced in the proof of Lemma 2.10, defined for $d_{\tilde{g}}(x,y) < \varepsilon$, where $\varepsilon > 0$ is a small constant (unrelated to the constant in Lemma 2.1). We however ask that $\tau_{x\to y}$ act on the tangent spaces $T_x\mathcal{M} \to T_y\mathcal{M}$ instead of the cotangent spaces. We start with

Lemma 2.13. *For each $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that*

$$d_{\tilde{g}}(x,\varphi^t(x)) < \delta_1,\ t \geq t_e,\ x \in V \implies d_{\tilde{g}}(x,K) < \varepsilon_1.$$

Proof. It suffices to show that for each sequences $x_j \in V$, $t_j \geq t_0$ such that $x_j \to x_\infty \in \mathcal{U}$ and $d(x_j,\varphi^{t_j}(x_j)) \to 0$, we have $x_\infty \in K$. We have $\varphi^{t_j}(x_j) \to x_\infty$. By passing to a subsequence we may assume that $t_j \to t_\infty \in (0,\infty]$. If $t_\infty < \infty$, then $\varphi^{t_\infty}(x_\infty) = x_\infty$ and thus $x_\infty \in K$. Assume now that $t_\infty = \infty$. For each $t \geq 0$ and j large enough depending on t, we have $t_j \geq t$ and $x_j, \varphi^{t_j}(x_j) \in \mathcal{U}$; by (1.2), $\varphi^t(x_j) \in \mathcal{U}$ and $\varphi^{t_j-t}(x_j) \in \mathcal{U}$. Passing to the limit, we see that $\varphi^t(x_\infty)$ and $\varphi^{-t}(x_\infty)$ lie in $\overline{\mathcal{U}}$; since t was chosen arbitrarily, we get $x_\infty \in K$. □

Denote by $\pi^\perp : T\mathcal{U} \to T\mathcal{U}$ the orthogonal projection onto the orthogonal complement of $E_0 = \mathbb{R}X$ (with respect to some fixed Riemannian metric). This operator need not be invariant under $d\varphi^t$ and its image need not be equal to $E_u \oplus E_s$. However, there exists a constant $C > 0$ such that for each $x \in K$ and $v = v_0 + v_u + v_s \in T_x\mathcal{M}$, $v_0 \in E_0(x)$, $v_u \in E_u(x)$, $v_s \in E_s(x)$,

$$C^{-1}(|v_u| + |v_s|) \leq |\pi^\perp(v)| \leq C(|v_u| + |v_s|). \tag{2.22}$$

Moreover, $\pi^\perp(d\varphi^t(x)\cdot v)$ depends only on $\pi^\perp(v)$:

$$\pi^\perp(d\varphi^t(x)\cdot v) = \pi^\perp(d\varphi^t(x)\cdot \pi^\perp(v)) \tag{2.23}$$

The next lemma gives a convexity property for the absolute value of a vector propagated along the flow.

Lemma 2.14. *There exists $T_0 > 0$ such that for each $t \geq T_0$, $\varepsilon_t > 0$ small enough depending on t, and each $(x,v) \in T\mathcal{U}$ with $d_{\tilde{g}}(x,K) < \varepsilon_t$,*

$$|\pi^\perp(v)| \leq \frac{|\pi^\perp(d\varphi^t(x)\cdot v)| + |\pi^\perp(d\varphi^{-t}(x)\cdot v)|}{4}. \tag{2.24}$$

Proof. Assume first that $x \in K$. Then $v = v_0 + v_u + v_s$, where $v_0 \in E_0(x), v_u \in E_u(x), v_s \in E_s(x)$. By (1.5), there exists a constant C such that for all $t \geq T_0$,

$$|v_u| \leq Ce^{-\gamma T_0}|d\varphi^t(x)\cdot v_u|, \quad |v_s| \leq Ce^{-\gamma T_0}|d\varphi^{-t}(x)\cdot v_s|,$$

Adding these up and using (2.22), we get for some other constant C,

$$|\pi^\perp(v)| \leq Ce^{-\gamma T_0}(|\pi^\perp(d\varphi^t(x)\cdot v)| + |\pi^\perp(d\varphi^{-t}(x)\cdot v)|).$$

It remains to take T_0 large enough. The case of x with $d_{\tilde g}(x,K) < \varepsilon_t$ follows by continuity. □

The following is a generalization of [DyZw13, Lemma A.1]:

Lemma 2.15. *There exist $\delta > 0$ and C such that for each $x \in V, t \geq t_e, v \in T_x\mathcal{M}$ satisfying $d_{\tilde g}(x,\varphi^t(x)) < \delta$ and $v \perp X(x)$,*

$$|v| + |w| \leq C|\pi^\perp(w)|, \quad w := (d\varphi^t(x) - \tau_{x\to\varphi^t(x)})v.$$

Proof. It suffices to show that for each sequences

$$x_j \in V, \quad t_j \geq t_e, \quad v_j \perp X(x_j), \quad w_j := (d\varphi^{t_j}(x_j) - \tau_{x_j\to\varphi^{t_j}(x_j)})v_j$$

such that

$$d_{\tilde g}(x_j,\varphi^{t_j}(x_j)) \to 0, \quad \pi^\perp(w_j) \to 0,$$

we have $v_j \to 0$ and $w_j \to 0$. By passing to a subsequence and using Lemma 2.13, we may assume that

$$x_j \to x_\infty \in K, \quad \varphi^{t_j}(x_j) \to x_\infty, \quad t_j \to t_\infty \in (0,\infty].$$

Assume first that $t_\infty < \infty$. By passing to a subsequence, we may assume that $v_j/|v_j| \to v_\infty \perp X(x_\infty)$. We have $\varphi^{t_\infty}(x_\infty) = x_\infty$ and $w_j/|v_j| \to d\varphi^{t_\infty}(x_\infty)\cdot v_\infty - v_\infty$. By (1.5), $\pi^\perp(w_j)/|v_j|$ has a nonzero limit; since $\pi^\perp(w_j) \to 0$, we get $v_j \to 0$ and thus $w_j \to 0$.

We henceforth assume that $t_\infty = \infty$. We first show that $v_j \to 0$. Assume the contrary, then by passing to a subsequence and rescaling, we can make

$$v_j \to v_\infty \perp X(x_\infty), \quad |v_\infty| = 1.$$

Consider the following two cases:

Case 1: v_∞ has a nonzero E_u component in the decomposition (1.4). By (1.5), we have $|\pi^\perp(d\varphi^t(x_\infty)\cdot v_\infty)| \to \infty$ as $t \to +\infty$. Let T_0 be the constant from Lemma 2.14. Fix $T \geq T_0$ so that $|\pi^\perp(d\varphi^T(x_\infty)\cdot v_\infty)| > 2$. For j large enough, we have

$$|\pi^\perp(d\varphi^T(x_j)\cdot v_j)| \geq 2|v_j|. \tag{2.25}$$

Moreover, if ε_T is chosen in Lemma 2.14, then for j large enough,

$$d_{\tilde g}(e^{tH_p}(x_j),K) < \varepsilon_T \quad \text{for all } t \in [0,t_j]. \tag{2.26}$$

Indeed, for $t \in [T', t_j - T']$ and T' large enough depending on ε_T, this follows from Lemma 2.4; for other values of t, it follows from continuity and the fact that both x_j and $\varphi^{t_j}(x_j)$ converge to $x_\infty \in K$.

We have for each $\ell \in \mathbb{N}_0$ such that $\ell T \le t_j$,

$$|\pi^\perp(d\varphi^{(\ell+1)T}(x_j)\cdot v_j)| \ge 2|\pi^\perp(d\varphi^{\ell T}(x_j)\cdot v_j)|.$$

This is proved by induction on ℓ; the base $\ell = 0$ of the induction is given by (2.25) and the inductive step follows from (2.24) applied to $v = d\varphi^{\ell T}(x_j)\cdot v_j$, $t = T$. We can modify T a tiny bit depending on j so that t_j/T is an integer; then we obtain

$$|\pi^\perp(d\varphi^{t_j}(x_j)\cdot v_j)| \ge 2^{t_j/T}|v_j|.$$

This implies that $|\pi^\perp(w_j)| \to \infty$, a contradiction.

Case 2: v_∞ has a nonzero E_s component in the decomposition (1.4). Since $\pi^\perp(w_j) \to 0$, we have $\pi^\perp(d\varphi^{t_j}(x_j)\cdot v_j) \to v_\infty$. Arguing as in case (i), with $\varphi^{t_j}(x_j), d\varphi^{t_j}(x_j)\cdot v_j$ replacing x_j, v_j, and going backwards along the flow, we get

$$|\pi^\perp(d\varphi^{t_j-(\ell+1)T}(x_j)\cdot v_j)| \ge 2|\pi^\perp(d\varphi^{t_j-\ell T}(x_j)\cdot v_j)|$$

which implies

$$|\pi^\perp(d\varphi^{t_j}(x_j)\cdot v_j)| \le 2^{-t_j/T}|v_j|.$$

Then $\pi^\perp(w_j) \to -v_\infty$, a contradiction.

We now show that $w_j \to 0$. Let T_0 be the constant from Lemma 2.14 and fix $T > T_0$; we will modify it a little bit depending on j so that $L := t_j/T$ is an integer. For large j, (2.26) is satisfied. For each $v \in T\mathcal{U}$, define $\pi_0(v) \in \mathbb{R}$ by the formula

$$v = \pi^\perp(v) + \pi_0(v)X.$$

Since X is invariant under the flow, we have for some constant C,

$$|\pi_0(d\varphi^{(\ell+1)T}(x_j)\cdot v_j) - \pi_0(d\varphi^{\ell T}(x_j)\cdot v_j)| \le C|\pi^\perp(d\varphi^{\ell T}(x_j)\cdot v_j)|.$$

Summing these up and using that $\pi_0(v_j) = 0$, we get

$$|\pi_0(d\varphi^{t_j}(x_j)\cdot v_j)| \le C\sum_{\ell=0}^{L}|\pi^\perp(d\varphi^{\ell T}(x_j)\cdot v_j)|.$$

Denote the sum on the right-hand side by Σ. Using (2.24) for $v = d\varphi^{\ell T}(x_j)\cdot v_j$ and all $\ell = 1, \dots, L-1$, we get

$$\Sigma \le |\pi^\perp(v_j)| + |\pi^\perp(d\varphi^{t_j}(x_j)\cdot v_j)| + \Sigma/2.$$

Since $\pi^\perp(w_j) \to 0$ and $v_j \to 0$, we know that $|\pi^\perp(v_j)| + |\pi^\perp(d\varphi^{t_j}(x_j)\cdot v_j)| \to 0$ and thus $\Sigma \to 0$. Then $\pi_0(d\varphi^{t_j}(x_j)\cdot v_j) \to 0$, which implies that $w_j \to 0$, as required. □

Arguing as in [DyZw13, Appendix A], we obtain from Lemma 2.15 the following analog of [DyZw13, Lemma 2.1]:

Lemma 2.16. *Define the following measure on $\mathcal{M}\times\mathbb{R}$: $\tilde\mu=\mu\times dt$, where μ is some smooth measure on $\mathcal{M}$. Fix $t_e>0$ and a compact subset $V\subset\mathcal{U}$. Then there exist constants C,L such that for each $\varepsilon>0$, $T>t_e$, and $n=\dim\mathcal{M}$,*

$$\tilde\mu(\{(x,t)\mid t_e\le t\le T,\ d(x,\varphi^t(x))<\varepsilon,\ x\in V\})\le C\varepsilon^n e^{nLT}.$$

Letting $\varepsilon\to 0$, we obtain the following analog of [DyZw13, Lemma 2.2]:

Lemma 2.17. *Let $N(T)$ be the number of closed trajectories of φ^t on K of period no more than T. Then*

$$N(T)\le Ce^{(2n-1)LT}.$$

3. Semiclassical preliminaries

In this section, we discuss some general results from microlocal and semiclassical analysis, following the notation of [DyZw13, Section 2.3 and Appendix C]. While some of the facts mentioned here (such as Lemma 3.2) are standard, Lemma 3.7 below seems to be a new result.

3.1. Review of semiclassical notation.

Recall that we are working on a compact manifold $\mathcal{M}$ without boundary. We use the class $\Psi^k(\mathcal{M};\mathcal{E})$ of pseudodifferential operators of order k acting on sections of $\mathcal{E}$. The corresponding symbol class is denoted by $S^k(\mathcal{M})$, see [DyZw13, (C.1)]. The principal symbol

$$\sigma(\mathbf{A})\in S^k(\mathcal{M};\operatorname{End}(\mathcal{E}))/S^{k-1}(\mathcal{M};\operatorname{End}(\mathcal{E}))$$

of $\mathbf{A}\in\Psi^k(\mathcal{M};\mathcal{E})$ is in general a section of the endomorphism bundle $\operatorname{End}(\mathcal{E})$ pulled back to $T^*\mathcal{M}$, however in this paper we mostly work with *principally scalar* operators, whose principal symbols are products of functions on $T^*\mathcal{M}$ and the identity homomorphism on $\mathcal{E}$. The *wavefront set* $\operatorname{WF}(A)$ is a closed conic subset of $T^*\mathcal{M}\setminus 0$ which measures the concentration of A in the phase space, and the *elliptic set* $\operatorname{ell}(A)$ is an open conic subset of $T^*\mathcal{M}\setminus 0$ which measures where the principal symbol of A is invertible.

We also use the class of *semiclassical* pseudodifferential operators $\Psi_h^k(\mathcal{M};\mathcal{E})$, which depend on a positive parameter h tending to zero. Quantizing a symbol $a(x,\xi)$ in the h-sense is equivalent to quantizing the rescaled symbol $a(x,h\xi)$ in the nonsemiclassical sense. We use the notion of the semiclassical principal symbol

$$\sigma_h(A)\in S^k(\mathcal{M})/hS^{k-1}(\mathcal{M})$$

of a principally scalar $A\in\Psi_h^k(\mathcal{M};\mathcal{E})$. We also use the *fiber-radially compactified* cotangent bundle $\overline{T}^*\mathcal{M}$; the interior of this bundle is diffeomorphic to $T^*\mathcal{M}$ and the boundary $\partial\overline{T}^*\mathcal{M}$, called the *fiber infinity*, is diffeomorphic to the cosphere bundle $S^*\mathcal{M}$. The h-wavefront set $\operatorname{WF}_h(A)$ and the h-elliptic set $\operatorname{ell}_h(A)$ are now subsets of

$\overline{T}^*\mathcal{M}$. We use the symbol Ψ_h^{comp} to denote the class of operators in Ψ_h^k whose wavefront sets are compactly contained in $T^*\mathcal{M}$ (that is, do not intersect the fiber infinity).

We use the concept of the wavefront set $\mathrm{WF}(u) \subset T^*\mathcal{M} \setminus 0$ of any distribution $u \in \mathcal{D}'(\mathcal{M})$. We also consider wavefront sets $\mathrm{WF}'(B) \subset T^*(\mathcal{M}\times\mathcal{M})\setminus 0$ of operators $B : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$, defined as follows:

$$\mathrm{WF}'(B) = \{(x,\xi,y,-\eta) \mid (x,\xi,y,\eta) \in \mathrm{WF}(K_B)\} \tag{3.1}$$

where the Schwartz kernel $K_B \in \mathcal{D}'(\mathcal{M}\times\mathcal{M})$ is given by the formula (where we use any smooth density dy on $\mathcal{M}$)

$$Bf(x) = \int_{\mathcal{M}} K_B(x,y) f(y)\, dy, \quad f \in C^\infty(\mathcal{M}). \tag{3.2}$$

For distributions $u = u(h)$ and operators $B = B(h)$ which are h-tempered (in the sense that $\|u(h)\|_{H_h^{-N}} = \mathcal{O}(h^{-N})$ for some N), we consider the semiclassical wavefront sets $\mathrm{WF}_h(u) \subset \overline{T}^*\mathcal{M}$, $\mathrm{WF}_h'(B) \subset \overline{T}^*(\mathcal{M}\times\mathcal{M})$. By taking the union of the wavefront sets of all components, we can extend these notions to distributions and operators valued in smooth vector bundles.

We will use the following multiplicative property of h-wavefront sets away from fiber infinity: assume that $A(h), B(h) : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ are h-tempered and $Q \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$. Using [DyZw13, Lemma 2.3], we obtain

$$\begin{aligned} &(x,\xi,z,\zeta) \in \mathrm{WF}_h(AQB) \cap T^*(\mathcal{M}\times\mathcal{M}) \\ \Longrightarrow\ &\exists (y,\eta) \in \mathrm{WF}_h(Q) : (x,\xi,y,\eta) \in \mathrm{WF}_h(A),\ (y,\eta,z,\zeta) \in \mathrm{WF}_h(B). \end{aligned} \tag{3.3}$$

Finally, if $u \in \mathcal{D}'(\mathcal{V})$, where $\mathcal{V} \subset \mathcal{M}$ is an open set, then $\mathrm{WF}(u) \subset T^*\mathcal{V}\setminus 0$ is defined as the union of all $\mathrm{WF}(\chi u)$ for $\chi \in C_0^\infty(\mathcal{V})$; here χu is naturally embedded into $\mathcal{D}'(\mathcal{M})$. Similarly one can define $\mathrm{WF}(B) \subset T^*(\mathcal{U}\times\mathcal{U})\setminus 0$, where $B : C_0^\infty(\mathcal{U}) \to \mathcal{D}'(\mathcal{U})$ and $\mathcal{U} \subset \mathcal{M}$ is open, by using (3.1) and the previous definition with $\mathcal{V} := \mathcal{U}\times\mathcal{U}$.

3.2. **Semiclassical propagation estimates.** We start with several semiclassical estimates which form the basis of our proofs. To simplify their statements, we say for $p \in C^\infty(T^*\mathcal{M})$ that

$$p \in \mathrm{Hom}^k(T^*\mathcal{M};\mathbb{R})$$

if p is real-valued and homogeneous of degree k in ξ for $|\xi|$ large enough. If $p \in \mathrm{Hom}^1(T^*\mathcal{M};\mathbb{R})$, then the Hamiltonian field H_p extends to a smooth vector field on $\overline{T}^*\mathcal{M}$ which is tangent to $\partial\overline{T}^*\mathcal{M}$. For later use in this section, we recall the notation

$$\mathrm{Re}\,\mathbf{A} := \frac{\mathbf{A}+\mathbf{A}^*}{2}, \quad \mathrm{Im}\,\mathbf{A} := \frac{\mathbf{A}-\mathbf{A}^*}{2i},$$

where $\mathbf{A}$ is an operator $C^\infty(\mathcal{M};\mathcal{E}) \to \mathcal{D}'(\mathcal{M};\mathcal{E})$ and we fix a volume form on $\mathcal{M}$ and an inner product on $\mathcal{E}$ to define the adjoint operator $\mathbf{A}^*$.

First of all, we review the classical Duistermaat–Hörmander *propagation of singularities*, formulated using the following

Definition 3.1. *Assume that* $p \in \mathrm{Hom}^1(T^*\mathcal{M};\mathbb{R})$. *Let* $V, W \subset \overline{T}^*\mathcal{M}$ *be open sets. We say that a point* $(x,\xi) \in \overline{T}^*M$ *is* controlled by V inside of W, *if there exists* $T \geq 0$ *such that* $e^{-TH_p}(x,\xi) \in V$ *and* $e^{-tH_p}(x,\xi) \in W$ *for* $t \in [0,T]$. *Denote by*

$$\mathrm{Con}_p(V;W) \subset \overline{T}^*\mathcal{M} \tag{3.4}$$

the set of all such points. Note that $\mathrm{Con}_p(V;W)$ *is an open subset of* $\overline{T}^*\mathcal{M}$.

Propagation of singularities (see for instance [DyZw13, Proposition 2.5]) is then formulated as follows:

Lemma 3.2. *Assume that* $\mathbf{P} \in \Psi_h^1(\mathcal{M};\mathcal{E})$ *is principally scalar and* $\sigma_h(\mathbf{P}) = p - iq$ *where*[1] $p \in \mathrm{Hom}^1(T^*\mathcal{M};\mathbb{R})$ *and* q *is real-valued. Let* $A, B, B_1 \in \Psi_h^0(\mathcal{M})$ *be such that*

$$q \geq 0 \quad \text{near } \mathrm{WF}_h(B_1), \qquad \mathrm{WF}_h(A) \subset \mathrm{Con}_p(\mathrm{ell}_h(B);\mathrm{ell}_h(B_1)).$$

Then for each s, N *and* $\mathbf{u} \in C^\infty(\mathcal{M};\mathcal{E})$, *we have*

$$\|A\mathbf{u}\|_{H_h^s} \leq C\|B\mathbf{u}\|_{H_h^s} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^s} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{3.5}$$

In this subsection, we give a more general propagation estimate (Lemma 3.7) under the weaker assumption that the trajectories of e^{-tH_p} starting on $\mathrm{WF}_h(A)$ either pass through $\mathrm{ell}_h(B)$ or converge to some closed set L, while staying in $\mathrm{ell}_h(B_1)$. For this estimate, we need to additionally restrict the sign of the imaginary part of the subprincipal symbol of $\mathbf{P}$ on L, which is achieved by the following

Definition 3.3. *Let* $\mathbf{P} \in \Psi_h^1(\mathcal{M};\mathcal{E})$ *and* $L \subset \overline{T}^*\mathcal{M}$ *be a closed set. Fix a volume form on* $\mathcal{M}$ *and an inner product on the fibers of* $\mathcal{E}$; *this defines an inner product on* $L^2(\mathcal{M};\mathcal{E})$. *Fix also* $s \in \mathbb{R}$. *We say that*

$$\mathrm{Im}\,\mathbf{P} \lesssim -h \quad \text{on } H_h^s \quad \text{microlocally near } L \tag{3.6}$$

if there exist operators

$$\mathbf{Y}_1 \in \Psi_h^s(\mathcal{M};\mathcal{E}), \quad \mathbf{Y}_2 \in \Psi_h^{-s}(\mathcal{M};\mathcal{E}), \quad \mathbf{Z} \in \Psi_h^0(\mathcal{M};\mathcal{E});$$
$$\mathbf{Y}_1\mathbf{Y}_2 = 1 + \mathcal{O}(h^\infty) \quad \text{near } L, \quad L \subset \mathrm{ell}_h(\mathbf{Z}),$$

such that for each N, h *small enough, and each* $\mathbf{u} \in H_h^{1/2}(\mathcal{M};\mathcal{E})$,

$$\mathrm{Im}\langle \mathbf{Y}_1\mathbf{P}\mathbf{Y}_2\mathbf{u}, \mathbf{u}\rangle_{L^2} \leq -h\|\mathbf{Z}\mathbf{u}\|_{L^2}^2 + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}^2. \tag{3.7}$$

[1]Strictly speaking, this means that $\sigma_h(\mathbf{P}) = p - iq + \mathcal{O}(h)_{S^0}$. In particular, the real part of $\sigma_h(\mathbf{P})$ is independent of h.

Remarks. (i) The above definition does not actually depend on the choice of the volume form on M and the metric on the fibers of $\mathcal{E}$. Indeed, any other choice yields the inner product $\langle \mathbf{u}, \mathbf{v}\rangle' = \langle \mathbf{W}\mathbf{u}, \mathbf{W}\mathbf{v}\rangle$ for some invertible $\mathbf{W} \in C^\infty(\mathcal{M}; \operatorname{End}(\mathcal{E}))$. Applying (3.7) for the inner product $\langle \cdot, \cdot \rangle$ to $\mathbf{W}\mathbf{u}$, we obtain (3.7) for $\langle \cdot, \cdot \rangle'$ with the operators $\mathbf{Y}_1' = \mathbf{W}^{-1}\mathbf{Y}_1$, $\mathbf{Y}_2' = \mathbf{Y}_2\mathbf{W}$, and $\mathbf{Z}' = \mathbf{W}^{-1}\mathbf{Z}\mathbf{W}$.

(ii) If $L \cap \partial\overline{T}^*\mathcal{M} = \emptyset$, then Definition 3.3 also does not depend on the value of s. Indeed, for each $B \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$ such that $B = 1 + \mathcal{O}(h^\infty)$ microlocally near L, we can apply (3.7) to $B\mathbf{u}$ to get the same inequality with the operators $\mathbf{Y}_1' = B^*\mathbf{Y}_1$, $\mathbf{Y}_2' = \mathbf{Y}_2 B$, $\mathbf{Z}' = \mathbf{Z}B$ which lie in Ψ_h^{comp}, and thus in $\Psi_h^s(\mathcal{M})$ for all s.

(iii) The presence of the operators $\mathbf{Y}_1, \mathbf{Y}_2$ (which is inevitable in the case $s \neq 0$ as there is no canonical elliptic operator in Ψ_h^s, unlike the identity operator for $s = 0$) makes the definition (3.7) subtle. For instance, the sum of two operators satisfying (3.6) does not necessarily satisfy the same condition. Moreover, the real part $\operatorname{Re}\mathbf{P}$ enters the definition in a nontrivial way. In fact, the statement (3.6) does not change if $\mathbf{P}$ is replaced by

$$\mathbf{P}' := Y\mathbf{P}Y^{-1} = \mathbf{P} + [Y, \mathbf{P}]Y^{-1}, \quad \sigma_h\big(h^{-1}(\mathbf{P}' - \mathbf{P})\big) = iH_{\sigma_h(P)}f,$$

for any $Y \in \Psi_h^0(\mathcal{M})$ with $\sigma_h(Y) = e^f$. In particular, one can add functions of the form $H_{\operatorname{Re}\sigma_h(P)}f$ to the imaginary part of the subprincipal symbol of $\mathbf{P}$, which means that (3.7) is a really a statement about the ergodic averages of this symbol along the flow $\exp(tH_{\operatorname{Re}\sigma_h(P)})$. These subtleties do not play a role in our analysis because we will always enforce (3.7) by either adding a large term or taking sufficiently large $|s|$ – see the following two lemmas.

We will use the following formulation of the sharp Gårding inequality:

Lemma 3.4. *Assume that $\mathbf{P} \in \Psi_h^{2m+1}(\mathcal{M}; \mathcal{E})$ is principally scalar, $A \in \Psi_h^0(\mathcal{M})$, and*[2] *$\operatorname{Re}\sigma_h(\mathbf{P}) \leq 0$ in a neighborhood $U \subset \overline{T}^*\mathcal{M}$ of $\mathrm{WF}_h(A)$. Then there exists a constant C such that for each N and $\mathbf{u} \in H_h^{m+1/2}(\mathcal{M}; \mathcal{E})$,*

$$\operatorname{Re}\langle \mathbf{P}A\mathbf{u}, A\mathbf{u}\rangle_{L^2} \leq Ch\|A\mathbf{u}\|^2_{H_h^m} + \mathcal{O}(h^\infty)\|\mathbf{u}\|^2_{H_h^{-N}}.$$

Proof. Since $\mathbf{P}$ is principally scalar, we can write it as a sum of a scalar operator in $\Psi_h^{2m+1}(\mathcal{M})$ and an $h\Psi_h^{2m}(\mathcal{M}; \mathcal{E})$ remainder. Therefore, we may assume that $\mathbf{P}$ is scalar, which reduces us to the case when $\mathcal{E}$ is trivial.

Take $B \in \Psi_h^0(\mathcal{M})$ such that $B = 1 + \mathcal{O}(h^\infty)$ near $\mathrm{WF}_h(A)$, $\sigma_h(B) \geq 0$ everywhere, and $\mathrm{WF}_h(B) \subset U$. Then $\sigma_h(\operatorname{Re}(\mathbf{P}B)) \leq 0$ everywhere. By the standard sharp Gårding inequality [Zw, Theorem 9.11], there exists a constant C such that for $\mathbf{u} \in H_h^{m+1/2}(\mathcal{M})$

$$\operatorname{Re}\langle \mathbf{P}BA\mathbf{u}, A\mathbf{u}\rangle_{L^2} \leq Ch\|A\mathbf{u}\|^2_{H_h^m}.$$

[2]Since $\sigma_h(\mathbf{P}) \in S^{2m+1}/hS^{2m}$, the following inequality needs to be satisfied for some representative of this equivalence class.

Here we use a partition of unity and coordinate charts to reduce to the case $\mathcal{M} = \mathbb{R}^n$. It remains to note that $A = BA + \mathcal{O}(h^\infty)$ and thus

$$\operatorname{Re}\langle \mathbf{P}A\mathbf{u}, A\mathbf{u}\rangle_{L^2} = \operatorname{Re}\langle \mathbf{P}BA\mathbf{u}, A\mathbf{u}\rangle_{L^2} + \mathcal{O}(h^\infty)\|\mathbf{u}\|^2_{H_h^{-N}}$$

for each N. □

We now provide several situations in which (3.6) is satisfied:

Lemma 3.5. *Let $L \subset T^*\mathcal{M}$ be a closed subset, $\mathbf{P} \in \Psi_h^1(\mathcal{M};\mathcal{E})$ be principally scalar, $Q \in \Psi_h^0(\mathcal{M})$, and*

$$\operatorname{Im}\sigma_h(\mathbf{P}) \le 0 \quad \text{near } L, \quad \operatorname{Re}\sigma_h(Q) > 0 \quad \text{on } L. \tag{3.8}$$

Then $\operatorname{Im}(\mathbf{P} - iQ) \lesssim -h$ on H_h^s near L, for all s.

Proof. Take $Y_1 \in \Psi_h^s(\mathcal{M})$, $Y_2 \in \Psi_h^{-s}(\mathcal{M})$ such that $Y_1Y_2 = 1 + \mathcal{O}(h^\infty)$ near L. Take also $Z \in \Psi_h^0(\mathcal{M})$ such that near L, $Z = 1 + \mathcal{O}(h^\infty)$ and near $\mathrm{WF}_h(Z)$,

$$Y_1Y_2 = 1 + \mathcal{O}(h^\infty), \quad \operatorname{Im}\sigma_h(\mathbf{P}) \le 0, \quad \operatorname{Re}\sigma_h(Q) \ge \varepsilon > 0.$$

Then $\operatorname{Im}\sigma_h(Y_1\mathbf{P}Y_2) \le 0$ and $\operatorname{Re}\sigma_h(Y_1QY_2) \ge \varepsilon$ near $\mathrm{WF}_h(Z)$; therefore, by Lemma 3.4, there exists a constant C such that for each N and each $\mathbf{u} \in H_h^{1/2}(\mathcal{M};\mathcal{E})$,

$$\begin{aligned}
&\operatorname{Im}\langle Y_1\mathbf{P}Y_2Z\mathbf{u}, Z\mathbf{u}\rangle_{L^2} \le Ch\|Z\mathbf{u}\|^2_{L^2} + \mathcal{O}(h^\infty)\|\mathbf{u}\|^2_{H_h^{-N}};\\
&\operatorname{Re}\langle Y_1QY_2Z\mathbf{u}, Z\mathbf{u}\rangle_{L^2} \ge \varepsilon\|Z\mathbf{u}\|^2_{L^2} - Ch\|Z\mathbf{u}\|^2_{H_h^{-1/2}} - \mathcal{O}(h^\infty)\|u\|^2_{H_h^{-N}}.
\end{aligned}$$

This implies that for h small enough,

$$\operatorname{Im}\langle Y_1(\mathbf{P} - iQ)Y_2Z\mathbf{u}, Z\mathbf{u}\rangle_{L^2} \le (2Ch - \varepsilon)\|Z\mathbf{u}\|^2_{L^2} + \mathcal{O}(h^\infty)\|u\|_{H_h^{-N}}.$$

Therefore, (3.6) holds for small h with $\mathbf{Y}_1 := Z^*Y_1$, $\mathbf{Y}_2 := Y_2Z$, and $\mathbf{Z} := Z$. □

Lemma 3.6. *Let $L, \mathbf{P}$ satisfy the assumptions of Lemma 3.5 and additionally $p := \operatorname{Re}\sigma_h(\mathbf{P}) \in \operatorname{Hom}^1(T^*\mathcal{M};\mathbb{R})$. Assume next that $L \subset \partial\overline{T}^*\mathcal{M}$ and L is invariant under e^{tH_p}. Fix a metric $|\cdot|$ on the fibers of $T^*\mathcal{M}$. Then:*

1. *Assume that there exist $c, \gamma > 0$ such that*

$$\frac{|e^{tH_p}(x,\xi)|}{|\xi|} \ge ce^{\gamma|t|} \quad \text{for } (x,\xi) \in L,\ t \le 0. \tag{3.9}$$

(Note that the left-hand side of (3.9) extends to a smooth function on $\overline{T}^*\mathcal{M}$.) Then there exists s_0 such that for all $s > s_0$, $\operatorname{Im}\mathbf{P} \lesssim -h$ near L on H_h^s.

2. *Assume that there exist $c, \gamma > 0$ such that*

$$\frac{|e^{tH_p}(x,\xi)|}{|\xi|} \ge ce^{\gamma|t|} \quad \text{for } (x,\xi) \in L,\ t \ge 0. \tag{3.10}$$

Then there exists s_0 such that for all $s < s_0$, $\operatorname{Im}\mathbf{P} \lesssim -h$ near L on H_h^s.

Proof. 1. We first find $f \in \mathrm{Hom}^1(T^*\mathcal{M};\mathbb{R})$ such that $\langle\xi\rangle^{-1} f > 0$ on $\overline{T}^*\mathcal{M}$ and

$$\frac{H_p f}{f} < -\frac{\gamma}{2} < 0 \quad \text{on } L.$$

For that, we fix $f_0 \in \mathrm{Hom}^1(T^*\mathcal{M};\mathbb{R})$ such that $\langle\xi\rangle^{-1} f_0 > 0$ on $\overline{T}^*\mathcal{M}$. Then for T large enough, (3.9) implies that

$$\frac{f_0 \circ e^{-TH_p}}{f_0} > e^{\gamma T/2} \quad \text{on } L.$$

Using that $\log(f_0 \circ e^{-tH_p}) - \log f_0 \in \mathrm{Hom}^0(T^*\mathcal{M};\mathbb{R})$, we then define f by

$$\log f = \frac{1}{T}\int_0^T \log(f_0 \circ e^{-tH_p})\,dt, \quad \frac{H_p f}{f} = -\frac{1}{T}\log\Big(\frac{f_0 \circ e^{-TH_p}}{f_0}\Big) < -\frac{\gamma}{2} \quad \text{on } L.$$

Having constructed f, we take $Y_1 \in \Psi_h^s(\mathcal{M}), Y_2 \in \Psi_h^{-s}(\mathcal{M}), Z \in \Psi_h^0(\mathcal{M})$ such that

$$Z = 1 + \mathcal{O}(h^\infty) \quad \text{near } L, \quad Y_1 Y_2 = 1 + \mathcal{O}(h^\infty) \quad \text{near } \mathrm{WF}_h(Z);$$
$$\mathrm{Im}\,\sigma_h(\mathbf{P}) \le 0, \quad \sigma_h(Y_1) = f^s, \quad \text{and} \quad \frac{H_p f}{f} < -\frac{\gamma}{2} \quad \text{near } \mathrm{WF}_h(Z).$$

Then we have microlocally near $\mathrm{WF}_h(Z)$,

$$Y_1 \mathbf{P} Y_2 = \mathbf{P} + [Y_1, \mathbf{P}]Y_2, \quad [Y_1,\mathbf{P}]Y_2 \in h\Psi_h^0(\mathcal{M};\mathcal{E}), \quad \mathrm{Im}\,\sigma_h(h^{-1}[Y_1,\mathbf{P}]Y_2) = s\,\frac{H_p f}{f}.$$

Similarly to the proof of Lemma 3.5, by applying the sharp Gårding inequality twice we get for some constant C independent of $s > 0$, all N, and all $\mathbf{u} \in H_h^{1/2}(\mathcal{M};\mathcal{E})$

$$\begin{aligned}\mathrm{Im}\langle Y_1 \mathbf{P} Y_2 Z\mathbf{u}, Z\mathbf{u}\rangle_{L^2} &= \mathrm{Im}\langle \mathbf{P} Z\mathbf{u}, Z\mathbf{u}\rangle_{L^2} + \mathrm{Im}\langle [Y_1,\mathbf{P}]Y_2 Z\mathbf{u}, Z\mathbf{u}\rangle_{L^2} + \mathcal{O}(h^\infty)\|\mathbf{u}\|^2_{H_h^{-N}} \\ &\le \Big(C - \frac{s\gamma}{2}\Big) h\|Z\mathbf{u}\|^2_{L^2} + \mathcal{O}(h^\infty)\|u\|^2_{H_h^{-N}}.\end{aligned}$$

It remains to choose s large enough so that $\frac{s\gamma}{2} - C \ge 1$; then (3.6) holds with $\mathbf{Y}_1 := Z^* Y_1$, $\mathbf{Y}_2 := Y_2 Z$, and $\mathbf{Z} := Z$.

2. We argue similarly to part 1. First of all, we construct $f \in \mathrm{Hom}^1(T^*\mathcal{M};\mathbb{R})$ such that $\langle\xi\rangle^{-1} f > 0$ on $\overline{T}^*\mathcal{M}$ and

$$\frac{H_p f}{f} > \frac{\gamma}{2} > 0 \quad \text{on } L.$$

This is done as in part 1, reversing the direction of the flow. We next argue as before, replacing $C - \frac{s\gamma}{2}$ by $C + \frac{s\gamma}{2}$ and choosing $s < 0$ large enough in absolute value. □

We now formulate the main propagation estimate; see Figure 3.

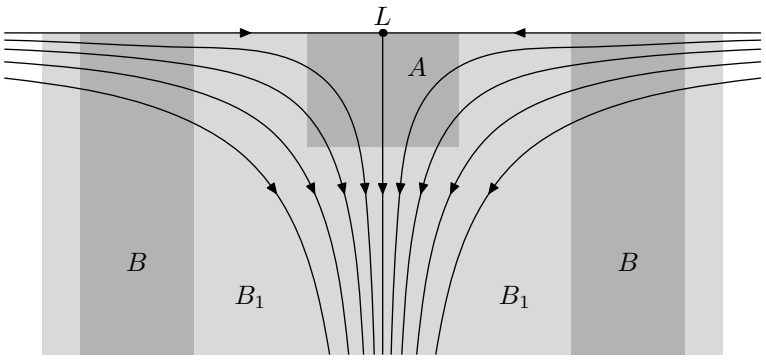


FIGURE 3. An illustration of Lemma 3.7 in the case where L lies inside the fiber infinity, the latter depicted by the horizontal line at the top. The lighter shaded region is the wavefront set of B_1, while the darker shaded regions are the wavefront sets of A and B. Several trajectories of the flow are displayed; Ω is the vertical trajectory converging to L.

Lemma 3.7. *Assume that $\mathbf{P} \in \Psi_h^1(\mathcal{M};\mathcal{E})$ is principally scalar with $\sigma_h(\mathbf{P}) = p - iq$, where p, q are real-valued and $p \in \mathrm{Hom}^1(T^*\mathcal{M};\mathbb{R})$. Let $L \subset \overline{T}^*\mathcal{M}$ be compact and invariant under e^{tH_p}. Assume that $A, B, B_1 \in \Psi_h^0(\mathcal{M})$ and $s \in \mathbb{R}$ are such that*

$$\mathrm{WF}_h(A) \subset \mathrm{ell}_h(B_1), \quad L \subset \mathrm{ell}_h(A), \quad L \cap \mathrm{WF}_h(B) = \emptyset,$$
$$q \geq 0 \quad \textit{near } \mathrm{WF}_h(B_1), \quad \mathrm{Im}\,\mathbf{P} \lesssim -h \quad \textit{on } H_h^s \quad \textit{near } L.$$

Consider the closed subset set of $\overline{T}^*\mathcal{M}$ (see (3.4))

$$\Omega := \{\langle\xi\rangle^{-1} p = 0\} \setminus \mathrm{Con}_p(\mathrm{ell}_h(B); \mathrm{ell}_h(B_1))$$

and assume that uniformly in $(x,\xi) \in \Omega \cap \mathrm{WF}_h(A)$,

$$e^{tH_p}(x,\xi) \to L \quad \textit{as } t \to -\infty; \quad e^{tH_p}(x,\xi) \in \mathrm{ell}_h(B_1) \quad \textit{for } t \leq 0. \tag{3.11}$$

Then for each N, for h small enough, and for each $\mathbf{u} \in C^\infty(\mathcal{M};\mathcal{E})$,

$$\|A\mathbf{u}\|_{H_h^s} \leq C\|B\mathbf{u}\|_{H_h^s} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^s} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{3.12}$$

Remarks. (i) The condition $\mathbf{u} \in C^\infty(\mathcal{M})$ can be relaxed as follows: let $m < s$ and $\mathrm{Im}\,\mathbf{P} \lesssim -h$ on $H_h^{s'}$ near L for all $s' \in [m, s]$, and the symbol of the corresponding operators $\mathbf{Z}$ is invertible on L uniformly in s'. Then the conditions $A\mathbf{u} \in H_h^m$, $B\mathbf{u} \in H_h^s$, $B_1\mathbf{P}u \in H_h^s$ imply that $A\mathbf{u} \in H_h^s$, and (3.12) holds. The proof works by improving the Sobolev regularity of $\mathbf{u}$ in small steps $\delta > 0$ (depending on the operators in (3.7)) by an approximation argument similar to the one in the proofs of [Va, Propositions 2.3–2.4]. For our purposes, it suffices to show (3.12) for $\mathbf{u} \in C^\infty$, so we avoid this approximation argument.

(ii) Lemma 3.7 implies several other semiclassical estimates:

- propagation of singularities (Lemma 3.2), by taking $L = \emptyset$;

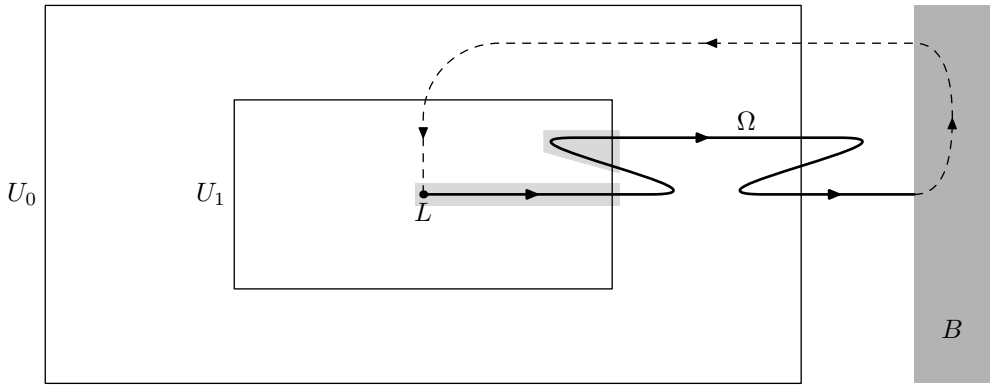


FIGURE 4. An illustration of the proof of Lemma 3.8. The darker shaded region is $\mathrm{WF}_h(B)$ and the lighter shaded region is $\operatorname{supp}\chi_1$. A possible trajectory of H_p is shown; Ω is the undashed part of this trajectory.

- radial points estimate [DyZw13, Proposition 2.6], by taking L to be a radial source, $B=0$, $\Omega=\{\langle\xi\rangle^{-1}p=0\}$, and using part 1 of Lemma 3.6;
- dual radial points estimate [DyZw13, Proposition 2.7], by taking L to be a radial sink, B microlocalized inside a punctured neighborhood of L, $\Omega\cap\mathrm{WF}_h(A)=L\cap\{\langle\xi\rangle^{-1}p=0\}$, and using part 2 of Lemma 3.6.

The first implication is circular, since the proof uses propagation of singularities.

The proof of Lemma 3.7 relies on the construction of a special escape function:

Lemma 3.8. *Under the assumptions of Lemma 3.7, let $U\subset\overline{T}^*\mathcal{M}$ be an open neighborhood of L. Then there exists a function $\chi\in C^\infty(\overline{T}^*\mathcal{M};[0,1])$ such that:*

(1) $\operatorname{supp}\chi\subset U$;
(2) $\chi=1$ *near* L;
(3) $H_p\chi\le 0$ *in some neighborhood of* Ω.

Proof. We take open neighborhoods (see Figure 4)

$$U_1\ \subset\ U_0\ \subset\ \mathrm{ell}_h(A)\setminus\mathrm{WF}_h(B)$$

of L such that

$$e^{-tH_p}(\Omega\cap\overline{U_0})\ \subset\ U\quad\text{and}\quad e^{-tH_p}(\Omega\cap\overline{U_1})\ \subset\ U_0\quad\text{for all } t\ge 0. \tag{3.13}$$

The first equation in (3.13) follows from (3.11) for t large enough independently of U_0; for bounded t, it suffices to use the fact that L is invariant under the flow and take U_0 small enough. The set U_1 is constructed in the same way.

By (3.11), there exists $T>0$ such that

$$e^{-TH_p}(\Omega\cap\overline{U_0})\ \subset\ U_1. \tag{3.14}$$

Take a function $\chi_1 \in C_0^\infty(U_0;[0,1])$ such that $\chi_1 = 1$ near $(\Omega \cap \overline{U_1}) \cup L$ and

$$e^{-tH_p}(\operatorname{supp}\chi_1) \ \subset \ U_0 \quad \text{for all } t \in [0,T]. \tag{3.15}$$

The existence of such function follows from the second equation in (3.13) and the invariance of L under the flow.

We have

$$e^{tH_p}(\Omega) \cap \operatorname{supp}\chi_1 \ \subset \ \Omega \quad \text{for all } t \in [0,T]. \tag{3.16}$$

Indeed, assume that $(x,\xi) \in \operatorname{supp}\chi_1$ and $e^{-tH_p}(x,\xi) \in \Omega$, but $(x,\xi) \notin \Omega$. Since the set $\{\langle\xi\rangle^{-1}p = 0\}$ is invariant under the flow and contains $e^{-tH_p}(x,\xi)$, we have $(x,\xi) \in \operatorname{Con}_p(\operatorname{ell}_h(B);\operatorname{ell}_h(B_1))$. By Definition 3.1, there exists $T' \geq 0$ such that $e^{-T'H_p}(x,\xi) \in \operatorname{ell}_h(B)$ and $e^{-t'H_p}(x,\xi) \in \operatorname{ell}_h(B_1)$ for all $t' \in [0,T']$. By (3.15), we have $e^{-sH_p}(x,\xi) \notin \operatorname{ell}_h(B)$ for $s \in [0,T]$, which implies that $T' > T \geq t$. Then $e^{-tH_p}(x,\xi) \in \operatorname{Con}_p(\operatorname{ell}_h(B);\operatorname{ell}_h(B_1))$, which contradicts the fact that $e^{-tH_p}(x,\xi) \in \Omega$.

Combining (3.14) and (3.16), we get

$$\Omega \cap e^{-TH_p}(\operatorname{supp}\chi_1) \ \subset \ \Omega \cap e^{-TH_p}(\Omega \cap \operatorname{supp}\chi_1) \ \subset \ \Omega \cap U_1. \tag{3.17}$$

Combining (3.16) with the first part of (3.13), we get

$$\Omega \cap e^{-tH_p}(\operatorname{supp}\chi_1) \ \subset \ e^{-tH_p}(\Omega \cap \operatorname{supp}\chi_1) \ \subset U \quad \text{for all } t \in [0,T]. \tag{3.18}$$

It follows from (3.17) that for (x,ξ) near $\Omega \cap \operatorname{supp}(\chi_1 \circ e^{TH_p})$, we have $\chi_1(x,\xi) = 1$. Since $0 \leq \chi_1 \leq 1$, we have for (x,ξ) in some neighborhood of Ω,

$$\chi_1(x,\xi) \geq \chi_1(e^{TH_p}(x,\xi)). \tag{3.19}$$

Put

$$\widetilde{\chi} = \frac{1}{T}\int_0^T \chi_1 \circ e^{tH_p}\,dt, \quad H_p\widetilde{\chi} = \frac{\chi_1 \circ e^{TH_p} - \chi_1}{T},$$

then $H_p\widetilde{\chi} \leq 0$ in some neighborhood of Ω. By (3.18), $\Omega \cap \operatorname{supp}\widetilde{\chi} \subset U$. Since $\chi_1 = 1$ near L, we also have $\widetilde{\chi} = 1$ near L. It remains to put $\chi := \chi_2\widetilde{\chi}$, where $\chi_2 \in C_0^\infty(U;[0,1])$ satisfies $\chi_2 = 1$ near $(\Omega \cap \operatorname{supp}\widetilde{\chi}) \cup L$. □

We now give

Proof of Lemma 3.7. We start with the estimate

$$\|\mathbf{A}_1\mathbf{u}\|_{H_h^s} \leq C\|B\mathbf{u}\|_{H_h^s} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^s} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}} \tag{3.20}$$

valid for all $\mathbf{A}_1 \in \Psi_h^0(\mathcal{M};\mathcal{E})$ such that $\operatorname{WF}_h(\mathbf{A}_1) \subset \operatorname{ell}_h(B_1) \setminus \Omega$. Indeed, we have

$$\operatorname{WF}_h(\mathbf{A}_1) \ \subset \ (\operatorname{ell}_h(B_1) \cap \operatorname{ell}_h(\mathbf{P})) \cup \operatorname{Con}_p(\operatorname{ell}_h(B);\operatorname{ell}_h(B_1)),$$

therefore by a partition of unity we may reduce to the situation when $\operatorname{WF}_h(\mathbf{A}_1)$ is contained either inside $\operatorname{ell}_h(B_1) \setminus \{\langle\xi\rangle^{-1}p = 0\}$ or inside $\operatorname{Con}_p(\operatorname{ell}_h(B);\operatorname{ell}_h(B_1))$. The first case is handled by the elliptic estimate [DyZw13, Proposition 2.4] and the second one, by propagation of singularities (Lemma 3.2).

Similarly we have the estimate

$$\|A\mathbf{u}\|_{H_h^s} \le C\|B\mathbf{u}\|_{H_h^s} + C\|\mathbf{A}_2\mathbf{u}\|_{H_h^s} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^s} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}} \tag{3.21}$$

valid for all $\mathbf{A}_2 \in \Psi_h^0(\mathcal{M};\mathcal{E})$ such that $L \subset \operatorname{ell}_h(\mathbf{A}_2)$, where we use the following corollary of (3.11):

$$\operatorname{WF}_h(A) \subset (\operatorname{ell}_h(B_1) \cap \operatorname{ell}_h(\mathbf{P})) \cup \operatorname{Con}_p(\operatorname{ell}_h(B);\operatorname{ell}_h(B_1)) \cup \operatorname{Con}_p(\operatorname{ell}_h(\mathbf{A}_2);\operatorname{ell}_h(B_1)).$$

Next, using Definition 3.3, choose $\mathbf{Y}_1 \in \Psi_h^s(\mathcal{M};\mathcal{E})$, $\mathbf{Y}_2 \in \Psi_h^{-s}(\mathcal{M};\mathcal{E})$, and $\mathbf{Z} \in \Psi_h^0(\mathcal{M};\mathcal{E})$ such that

$$\mathbf{Y}_1\mathbf{Y}_2 = 1 + \mathcal{O}(h^\infty) \quad \text{near } \overline{U}, \quad \overline{U} \subset \operatorname{ell}_h(\mathbf{Z}),$$

for some neighborhood $U \subset \operatorname{ell}_h(A) \cap \operatorname{ell}_h(B_1)$ of L, and for each $\mathbf{v} \in C^\infty(\mathcal{M};\mathcal{E})$,

$$\operatorname{Im}\langle \mathbf{P}'\mathbf{v}, \mathbf{v}\rangle_{L^2} \le -h\|\mathbf{Z}\mathbf{v}\|_{L^2}^2 + \mathcal{O}(h^\infty)\|\mathbf{v}\|_{H_h^{-N}}^2, \tag{3.22}$$

where $\mathbf{P}' := \mathbf{Y}_1\mathbf{P}\mathbf{Y}_2 \in \Psi_h^1(\mathcal{M};\mathcal{E})$. Note that $\sigma_h(\mathbf{P}') = \sigma_h(\mathbf{P})$ on U.

We now claim that it suffices to show that there exist operators

$$A_1, A_2, B_2 \in \Psi_h^0(\mathcal{M}), \quad \operatorname{WF}_h(A_1) \subset U \setminus \Omega, \quad L \subset \operatorname{ell}_h(A_2), \quad \operatorname{WF}_h(B_2) \subset U,$$

such that for each $\mathbf{v} \in C^\infty(\mathcal{M};\mathcal{E})$,

$$\|A_2\mathbf{v}\|_{L^2} \le C\|A_1\mathbf{v}\|_{L^2} + Ch^{-1}\|B_2\mathbf{P}'\mathbf{v}\|_{L^2} + Ch^{1/2}\|B_2\mathbf{v}\|_{H_h^{-1/2}} + \mathcal{O}(h^\infty)\|\mathbf{v}\|_{H_h^{-N}}. \tag{3.23}$$

Indeed, applying (3.23) to $\mathbf{v} := \mathbf{Y}_1\mathbf{u}$ and assuming that $\operatorname{WF}_h(A_2) \subset U$, we get

$$\|\mathbf{A}_2\mathbf{u}\|_{H_h^s} \le C\|\mathbf{A}_1\mathbf{u}\|_{H_h^s} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^s} + Ch^{1/2}\|A\mathbf{u}\|_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{3.24}$$

where $\mathbf{A}_j := \mathbf{Y}_2 A_j \mathbf{Y}_1$ satisfy $\operatorname{WF}_h(\mathbf{A}_1) \subset U \setminus \Omega$, $L \subset \operatorname{ell}_h(\mathbf{A}_2)$. Here we used the fact that $\operatorname{WF}_h(B_2) \subset \operatorname{ell}_h(A)$ and the elliptic estimate to bound $\|B_2\mathbf{v}\|_{H_h^{-1/2}}$ in terms of $\|A\mathbf{u}\|_{H_h^{s-1/2}}$. To obtain the required estimate (3.12), it remains to combine this with (3.20) and (3.21), and take h small enough to eliminate the $Ch^{1/2}\|A\mathbf{u}\|_{H_h^{s-1/2}}$ remainder.

We now prove (3.23) using a positive commutator argument. Let χ be the function constructed in Lemma 3.8. Fix $F \in \Psi_h^0(\mathcal{M})$ such that

$$\sigma_h(F) = \chi, \quad \operatorname{WF}_h(F) \subset U.$$

Then

$$\sigma_h(ih^{-1}[\operatorname{Re}\mathbf{P}', F^*F]) = 2\chi H_p\chi \le 0 \quad \text{near } \Omega.$$

Therefore

$$[\operatorname{Re}\mathbf{P}', F^*F] = -ih(\mathbf{G}_1 + \mathbf{G}_2), \tag{3.25}$$

where $\mathbf{G}_j \in \Psi_h^0(\mathcal{M};\mathcal{E})$ are self-adjoint and principally scalar, and

$$\operatorname{WF}_h(\mathbf{G}_j) \subset U; \quad \sigma_h(\mathbf{G}_1) \le 0; \quad \operatorname{WF}_h(\mathbf{G}_2) \cap \Omega = \emptyset.$$

For each $\mathbf{v} \in C^\infty(\mathcal{M}; \mathcal{E})$, we have

$$\operatorname{Im}\langle \mathbf{P}'\mathbf{v}, F^*F\mathbf{v}\rangle_{L^2} = \frac{i}{2}\langle [\operatorname{Re}\mathbf{P}', F^*F]\mathbf{v}, \mathbf{v}\rangle_{L^2} + \operatorname{Re}\langle F^*F(\operatorname{Im}\mathbf{P}')\mathbf{v}, \mathbf{v}\rangle_{L^2}. \tag{3.26}$$

Take $A_1, B_2 \in \Psi_h^0(\mathcal{M})$ such that

$$\begin{gathered}\operatorname{WF}_h(A_1) \subset U \setminus \Omega, \quad A_1 = 1 + \mathcal{O}(h^\infty) \quad \text{near } \operatorname{WF}_h(\mathbf{G}_2);\\ \operatorname{WF}_h(B_2) \subset U, \quad B_2 = 1 + \mathcal{O}(h^\infty) \quad \text{near } \operatorname{WF}_h(F) \cup \operatorname{WF}_h(\mathbf{G}_1) \cup \operatorname{WF}_h(\mathbf{G}_2).\end{gathered}$$

By the sharp Gårding inequality (Lemma 3.4) applied to $\mathbf{G}_1$ and the elliptic estimate applied to $\mathbf{G}_2$, the product $\frac{i}{2}\langle [\operatorname{Re}\mathbf{P}', F^*F]\mathbf{v}, \mathbf{v}\rangle_{L^2}$ is equal to

$$\begin{aligned}&\frac{h}{2}\big(\langle \mathbf{G}_1 B_2\mathbf{v}, B_2\mathbf{v}\rangle_{L^2} + \langle \mathbf{G}_2 A_1\mathbf{v}, A_1\mathbf{v}\rangle_{L^2}\big) + \mathcal{O}(h^\infty)\|\mathbf{v}\|^2_{H_h^{-N}}\\ &\leq Ch^2\|B_2\mathbf{v}\|^2_{H_h^{-1/2}} + Ch\|A_1\mathbf{v}\|^2_{L^2} + \mathcal{O}(h^\infty)\|\mathbf{v}\|^2_{H_h^{-N}}.\end{aligned} \tag{3.27}$$

Next, $h^{-1}F^*[F, \operatorname{Im}\mathbf{P}'] \in \Psi_h^0$ and its principal symbol is imaginary valued, therefore $\operatorname{Re}(F^*[F, \operatorname{Im}\mathbf{P}']) \in h^2\Psi_h^{-1}$. It then follows from (3.22) that

$$\begin{aligned}\operatorname{Re}\langle F^*F(\operatorname{Im}\mathbf{P}')\mathbf{v}, \mathbf{v}\rangle_{L^2} &= \operatorname{Im}\langle \mathbf{P}'F\mathbf{v}, F\mathbf{v}\rangle_{L^2} + \langle \operatorname{Re}(F^*[F, \operatorname{Im}\mathbf{P}'])\mathbf{v}, \mathbf{v}\rangle_{L^2}\\ &\leq -h\|\mathbf{Z}F\mathbf{v}\|^2_{L^2} + Ch^2\|B_2\mathbf{v}\|^2_{H_h^{-1/2}} + \mathcal{O}(h^\infty)\|\mathbf{v}\|^2_{H_h^{-N}}.\end{aligned} \tag{3.28}$$

Since $\mathbf{Z}$ is elliptic on $U \supset \operatorname{WF}_h(F)$, we have

$$\|F\mathbf{v}\|^2_{L^2} \leq C\|\mathbf{Z}F\mathbf{v}\|^2_{L^2} + \mathcal{O}(h^\infty)\|\mathbf{v}\|^2_{H_h^{-N}}.$$

Combining this with (3.26)–(3.28), we get

$$\begin{aligned}&C^{-1}h\|F\mathbf{v}\|^2_{L^2} - Ch\|A_1\mathbf{v}\|^2_{L^2} - Ch^2\|B_2\mathbf{v}\|^2_{H_h^{-1/2}}\\ &\leq -\operatorname{Im}\langle \mathbf{P}'\mathbf{v}, F^*F\mathbf{v}\rangle + \mathcal{O}(h^\infty)\|\mathbf{v}\|^2_{H_h^{-N}}\\ &\leq C\|B_2\mathbf{P}'\mathbf{v}\|_{L^2}\cdot\|F\mathbf{v}\|_{L^2} + \mathcal{O}(h^\infty)\|\mathbf{v}\|^2_{H_h^{-N}},\end{aligned}$$

which implies (3.23) with $A_2 := F$, finishing the proof. □

4. Properties of the resolvent

In this section, we prove Theorems 1 and 2, and show microlocalization statements for the resolvent that form the basis of the proof of Theorem 3 in the next section. We follow in part the argument of [DyZw13], based on the strategy of [FaSj].

4.1. **Auxiliary resolvent.** In this section, we introduce an auxiliary resolvent depending on the semiclassical parameter $h > 0$. Recall the function ρ, the constant ε, and the vector field X_1 used in Lemma 2.1, and let p be defined in (2.8).

Anisotropic spaces. We first construct the *anisotropic Sobolev spaces* on which the auxiliary resolvent will be defined. The order function of these spaces is given by the following (see Figure 5(a))

Lemma 4.1. *There exists $m \in C^\infty(S^*\mathcal{M};\mathbb{R})$ such that, with H_p pulled back to $S^*\mathcal{M}$ by the projection $\kappa : T^*\mathcal{M}\setminus 0 \to S^*\mathcal{M}$, and $E_\pm^*$ defined in Lemma 2.10,*

(1) *$m = 1$ in a neighborhood of $\kappa(E_-^*) \supset \kappa(E_s^*)$;*
(2) *$m = -1$ in a neighborhood of $\kappa(E_+^*) \supset \kappa(E_u^*)$;*
(3) *$H_p m \le 0$ in a neighborhood of $\{p = 0\}$;*
(4) *$\operatorname{supp} m \subset \{\rho > -2\varepsilon\}$ and $\operatorname{supp} m \cap \{\rho = -\varepsilon\} \cap \{X_1\rho = 0\} = \emptyset$.*

Proof. Let $m_\pm$ be the functions constructed in Lemma 2.12, then

(1) $m_- = 1$ and $m_+ = 0$ in a neighborhood of $\kappa(E_-^*)$;
(2) $m_+ = 1$ and $m_- = 0$ in a neighborhood of $\kappa(E_+^*)$;
(3) $\pm H_p m_\pm \ge 0$ on $V \setminus \pi^{-1}(V_\pm)$, where V is a neighborhood of $\{p = 0\}$ and $V_\pm := \pi(\operatorname{supp} m_\pm)\setminus \mathcal{U} \subset \Sigma_\pm \setminus \mathcal{U}$ are compact. Here $\pi : S^*\mathcal{M} \to \mathcal{M}$ is the projection map and $\Sigma_\pm$ are defined in (2.4).

Next, take the functions $\chi_\pm$ constructed in Lemma 2.8 (with the sets $V_\pm$ defined in (3) above). We have $\pm H_p(\chi_\pm \circ \pi) \ge 0$ everywhere, $\chi_\pm = 0$ near K, and $\pm H_p(\chi_\pm\circ\pi) > \delta > 0$ on $\pi^{-1}(V_\pm)$. Then for a large enough constant $R > 0$, the function

$$m := m_- - m_+ + R(\chi_-\circ\pi - \chi_+\circ\pi)$$

satisfies conditions (1)–(3). Condition (4) follows immediately from the fact that $\operatorname{supp} m_\pm \subset \pi^{-1}(\Sigma_\pm) \subset \{\rho > -\varepsilon\}$ and $\operatorname{supp}\chi_\pm \subset \{\rho > -2\varepsilon\}\cap\{\pm X_1\rho < 0\}$. □

We now consider m as a homogeneous function of degree 0 on $T^*\mathcal{M}$ and define the weight $\widetilde{m} \in C^\infty(T^*\mathcal{M})$ by

$$\widetilde{m}(x,\xi) = (1 - \chi_m(x,\xi))m(x,\xi)\log|\xi|, \tag{4.1}$$

where $\chi_m \in C_0^\infty(T^*\mathcal{M};[0,1])$ is equal to 1 near the zero section and supported in $\{|\xi| < 1\}$. Take an operator

$$G = G(h) \in \bigcap_{\delta>0}\Psi_h^\delta(\mathcal{M}), \quad \sigma_h(G) = \widetilde{m}, \quad \mathrm{WF}_h(G)\cap\{\xi = 0\} = \emptyset. \tag{4.2}$$

We moreover require that

$$\mathrm{WF}_h(G) \ \subset\ \{\rho > -2\varepsilon\}, \quad \mathrm{WF}_h(G)\cap\{\rho = -\varepsilon\}\cap\{X_1\rho = 0\} = \emptyset. \tag{4.3}$$

For each $r \in \mathbb{R}$, we define the *anisotropic Sobolev space* $\mathcal{H}_h^r$ as follows:

$$\mathcal{H}_h^r := \exp(-rG(h))(L^2(\mathcal{M};\mathcal{E})), \quad \|\mathbf{u}\|_{\mathcal{H}_h^r} := \|\exp(rG(h))\mathbf{u}\|_{L^2(\mathcal{M};\mathcal{E})}. \tag{4.4}$$

As explained for instance in [DyZw13, Sections 3.1 and 3.3], we have

$$H_h^{C_m|r|}(\mathcal{M};\mathcal{E}) \subset \mathcal{H}_h^r \subset H_h^{-C_m|r|}(\mathcal{M};\mathcal{E}) \tag{4.5}$$

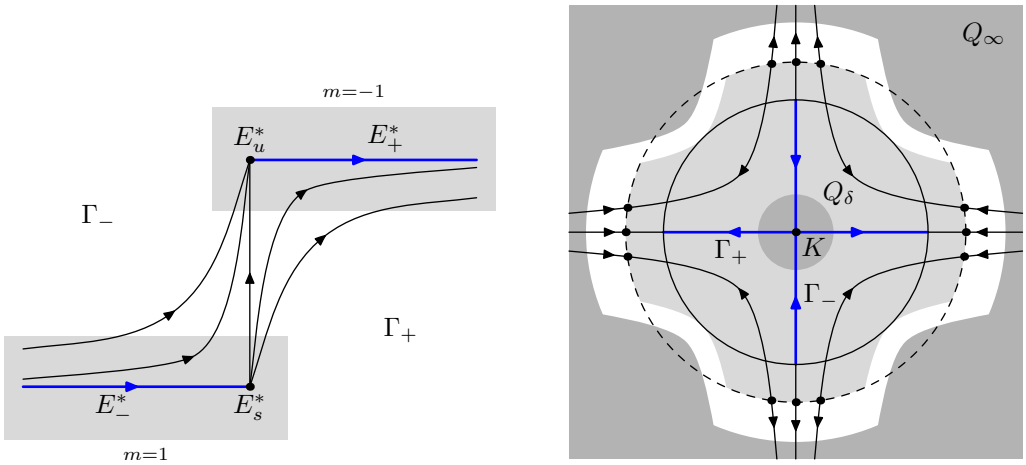


FIGURE 5. (a) The flow e^{tH_p} on $S^*\mathcal{M} \cap \{p = 0\}$. The left half of the figure represents the sphere bundle over Γ_-, the right half represents Γ_+, and the vertical midline represents K. (To obtain this from Figure 2, glue the rear halves of the two vertical planes.) The thick blue lines are $E_\pm^*$ and the shaded boxes represent the regions where $m = \pm 1$. (b) The flow φ^t near $\mathcal{U}$ (pictured by the solid circle). The dashed circle is $\{\rho = -\varepsilon\}$, consisting of fixed points; q_1 is supported near this circle. The lighter shaded region denotes the set Σ from (2.4), while the darker shaded regions denote the supports of Q_∞ and Q_δ (the latter microlocalized near zero frequency).

where H_h^r stands for the standard semiclassical Sobolev space [Zw, Section 14.2.4] and $C_m = \sup_{S^*\mathcal{M}} |m|$; we can take $C_m = 1$ for distributions supported inside $\mathcal{U}$. Moreover, the norms of $\mathcal{H}_h^r$ for different h are all equivalent with constants depending on h.

Since $m = 1$ near $\kappa(E_-^*)$, we have $\widetilde{m} = \log|\xi|$ near $\kappa(E_-^*) \subset \partial\overline{T}^*\mathcal{M}$, where

$$\kappa : T^*\mathcal{M} \setminus 0 \ \to \ S^*\mathcal{M} = \partial\overline{T}^*\mathcal{M} \tag{4.6}$$

is the projection map. It follows from [Zw, Theorems 8.6 and 8.10] that $\mathcal{H}_h^r$ is microlocally equivalent to the standard semiclassical Sobolev space H_h^r near $\kappa(E_-^*)$ in the following sense: for each $A \in \Psi_h^0(\mathcal{M})$ such that $\mathrm{WF}_h(A)$ is contained in a small neighborhood of $\kappa(E_-^*)$ and all $\mathbf{u} \in C^\infty(\mathcal{M}; \mathcal{E})$, we have

$$\|A\mathbf{u}\|_{\mathcal{H}_h^r} \le C\|\mathbf{u}\|_{H_h^r}, \quad \|A\mathbf{u}\|_{H_h^r} \le C\|\mathbf{u}\|_{\mathcal{H}_h^r}. \tag{4.7}$$

Since $m = -1$ near $\kappa(E_+^*)$, we similarly have for each $A \in \Psi_h^0(\mathcal{M})$ with $\mathrm{WF}_h(A)$ contained in a small neighborhood of $\kappa(E_+^*)$,

$$\|A\mathbf{u}\|_{\mathcal{H}_h^r} \le C\|\mathbf{u}\|_{H_h^{-r}}, \quad \|A\mathbf{u}\|_{H_h^{-r}} \le C\|\mathbf{u}\|_{\mathcal{H}_h^r}. \tag{4.8}$$

Complex absorbing operators. Take small $\delta > 0$ and choose

$$Q_\infty \in \Psi_h^1(\mathcal{M}), \quad Q_\delta \in \Psi_h^{\mathrm{comp}}(\mathcal{M}), \quad q_1 \in C^\infty(\mathcal{M}) \tag{4.9}$$

such that $\sigma_h(Q_\infty), \sigma_h(Q_\delta), q_1 \geq 0$ everywhere and

(1) $\mathbb{1}_{\Sigma'} Q_\infty = Q_\infty \mathbb{1}_{\Sigma'} = 0$, where Σ' is a neighborhood of $\overline{\Sigma}$ and Σ is defined in (2.4);
(2) $\{\rho \leq -2\varepsilon\} \cup \big(\{\rho = -\varepsilon\} \cap \{X_1 \rho = 0\}\big) \subset \mathrm{ell}_h(Q_\infty)$;
(3) $Q_\delta = \chi_\delta Q_\delta \chi_\delta$ for some $\chi_\delta \in C^\infty(\mathcal{M})$ supported in a δ-neighborhood of K, $\mathrm{WF}_h(Q_\delta) \subset \{|\xi| < \delta\}$, and $\{x \in K,\ \xi = 0\} \subset \mathrm{ell}_h(Q_\delta)$;
(4) $\operatorname{supp} q_1 \cap \overline{\mathcal{U}} = \emptyset$ and $q_1 > 0$ on $\{\rho = -\varepsilon\}$;
(5) $\mathrm{WF}_h(G) \cap (\mathrm{WF}_h(Q_\delta) \cup \mathrm{WF}_h(Q_\infty)) = \emptyset$.

The existence of Q_∞ is guaranteed by Lemma 2.7 and the fact that $\Sigma \subset \{\rho > -\varepsilon\}$. Condition (5) can be satisfied by (4.2) and (4.3). See Figure 5(b).

The use of the absorbing operator Q_δ goes back to [FaSj]; we will follow closely the later argument of [DyZw13]. By contrast, the operator Q_∞ is something specific to open systems; such complex absorbing operators have been previously used in scattering theory, see for instance [St, NoZw09, Va]. The absorbing potential q_1 guarantees invertibility on $\{\rho = -\varepsilon\}$; making Q_∞ elliptic there would destroy the propagation of support property, meaning that Lemma 4.4 below would no longer be true.

Existence of the auxiliary resolvent. Introduce the modified operator

$$\mathbf{P}_\delta = \mathbf{P}_\delta(h) = \frac{h}{i}\mathbf{X} - i(Q_\infty + q_1 + Q_\delta) \in \Psi_h^1(\mathcal{M}; \mathcal{E}) \tag{4.10}$$

which acts $\mathcal{D}_h^r \to \mathcal{H}_h^r$, where

$$\mathcal{D}_h^r := \{\mathbf{u} \in \mathcal{H}_h^r \mid \mathbf{P}_\delta \mathbf{u} \in \mathcal{H}_h^r\}, \quad \|\mathbf{u}\|_{\mathcal{D}_h^r} := \|\mathbf{u}\|_{\mathcal{H}_h^r} + \|\mathbf{P}_\delta \mathbf{u}\|_{\mathcal{H}_h^r}. \tag{4.11}$$

Note that, with p defined in (2.8),

$$\sigma_h(\mathbf{P}_\delta) = p - i(\sigma_h(Q_\infty) + \sigma_h(Q_\delta) + q_1).$$

The main result of this subsection is the following

Lemma 4.2. *Take $C_1, C_2 > 0$. Then there exists $r_0 = r_0(C_1) \geq 0$ such that for all $r \geq r_0$ and $0 < h < h_0(C_1, C_2, r)$, the inverse*

$$\mathbf{R}_\delta(z) = (\mathbf{P}_\delta - z)^{-1} : \mathcal{H}_h^r \to \mathcal{D}_h^r, \quad z \in [-C_2 h, C_2 h] + i[-C_1 h, 1] \tag{4.12}$$

exists and satisfies the bound

$$\|\mathbf{R}_\delta(z)\|_{\mathcal{H}_h^r \to \mathcal{H}_h^r} \leq C h^{-1}. \tag{4.13}$$

Furthermore, the h-wavefront set of $\mathbf{R}_\delta(z)$ satisfies

$$\mathrm{WF}_h'(\mathbf{R}_\delta(z)) \cap T^*(\mathcal{U} \times \mathcal{U}) \subset \Delta(T^*\mathcal{U}) \cup \Upsilon_+, \tag{4.14}$$

where $\Delta(T^*\mathcal{U})$ is the diagonal of $T^*\mathcal{U}$ and Υ_+ is the positive flow-out of e^{tH_p} on $\{p=0\}$ inside $\pi^{-1}(\mathcal{U})$ (here $\pi : \overline{T}^*\mathcal{M} \to \mathcal{M}$ is the projection map)

$$\Upsilon_+ = \{(e^{tH_p}(y,\eta), y, \eta) \mid t \geq 0,\ p(y,\eta) = 0,\ y \in \mathcal{U},\ \varphi^t(y) \in \mathcal{U}\}.$$

Remarks. (i) The proof of Lemma 4.2 can be summarized as follows:

- the anisotropic spaces give invertibility at the projections of the sets E_u^*, E_s^* to fiber infinity $\partial\overline{T}^*\mathcal{M}$;
- together, q_1 and Q_∞ give invertibility on $\{\rho \leq -\varepsilon\}$;
- the operator Q_δ gives invertibility on the set $\{(x,0) \mid x \in K\}$; and
- invertibility elsewhere is obtained by propagation of singularities.

(ii) One can specify the value of r_0 more precisely. Indeed, the condition $r \geq r_0$ is only needed to ensure that (4.20), (4.21) hold. Examining the proof of Lemma 3.6, we see immediately that we can take for some large fixed constant $\widetilde{C} > 0$,

$$r_0 = \widetilde{C}(1 + C_1). \tag{4.15}$$

Moreover, if $h^{-1} \operatorname{Im} z$ is large enough and positive, then we can take $r_0 = 0$.

(iii) If additionally $\mathbf{X}^* = -\mathbf{X}$ near K with respect to some smooth measure on $\mathcal{U}$ and some inner product on the fibers of $\mathcal{E}$ (e.g. when $\mathcal{E} = \mathbb{R}$, $\mathbf{X} = X$, and X admits a smooth invariant measure), then we can take for some $\widetilde{C} > 0$, any r_0 with

$$r_0 > \widetilde{C} C_1. \tag{4.16}$$

Furthermore, replacing $\gamma/2$ in the proof of Lemma 3.6 by a constant arbitrarily close to γ, we can put $\widetilde{C} := \gamma^{-1}$, where γ is the minimal expansion rate appearing in (1.5).

Proof. We use the strategy of the proof of [DyZw13, Proposition 3.4]. One could similarly adapt the construction of [FaSj, Section 3], however the method of [DyZw13] is more convenient for the wavefront set statements, needed for Theorem 3.

To reduce $\mathcal{H}_h^r$ estimates to L^2 estimates, we use the conjugated operator (see [DyZw13, Section 3.3] for details)

$$\mathbf{P}_{\delta,r} = e^{rG}\mathbf{P}_\delta e^{-rG} = \mathbf{P}_\delta + r[G, \mathbf{P}_\delta] + \mathcal{O}(h^2)_{\Psi_h^{-1+}}. \tag{4.17}$$

Since $\operatorname{WF}_h(G) \cap (\operatorname{WF}_h(Q_\delta) \cup \operatorname{WF}_h(Q_\infty)) = \emptyset$, $\sigma_h(Q_\infty), \sigma_h(Q_\delta), q_1$ are real-valued, and q_1 is a pseudodifferential operator of order 0, we get

$$\operatorname{Re} \sigma_h(\mathbf{P}_{\delta,r}) = p + \mathcal{O}(h)_{S^0}. \tag{4.18}$$

Since $H_p m \leq 0$ near $\{p = 0\}$, it follows from (4.1) that $H_p \widetilde{m} \leq 0$ modulo S^0 near $\{\langle\xi\rangle^{-1} p = 0\}$. Together with the fact that $\sigma_h(Q_\infty), \sigma_h(Q_\delta), q_1 \geq 0$, this implies

$$\operatorname{Im} \sigma_h(\mathbf{P}_{\delta,r}) \leq 0 \quad \text{near } \{\langle\xi\rangle^{-1} p = 0\}. \tag{4.19}$$

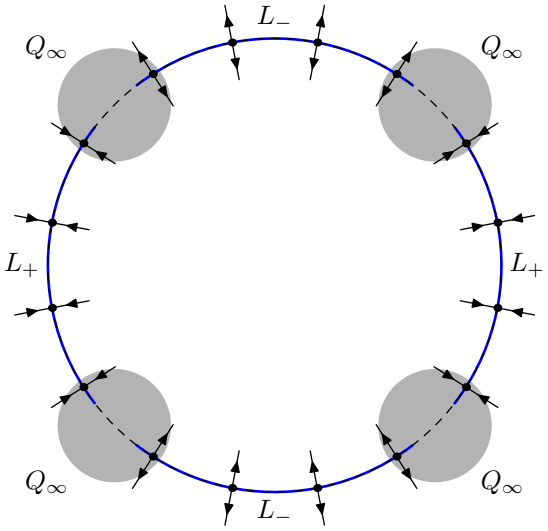


FIGURE 6. The sets $L_\pm$ (thick blue lines) and the elliptic set of Q_∞ (shaded) near $\{\rho = -\varepsilon\}$ (dashed circle). The arrows depict the flow φ^t.

Now, by (2.11), $L = \kappa(E_s^*)$ satisfies (3.9), where κ is defined in (4.6). By part 1 of Lemma 3.6, there exists $r_0 := r_0(C_1)$ such that

$$\operatorname{Im}(\mathbf{P}_\delta - z) \lesssim -h \quad \text{near } \kappa(E_s^*) \quad \text{on } H_h^r, \quad r \geq r_0, \quad \operatorname{Im} z \geq -C_1 h; \tag{4.20}$$

here we use Definition 3.3. Similarly, $L = \kappa(E_u^*)$ satisfies (3.10). By part 2 of Lemma 3.6,

$$\operatorname{Im}(\mathbf{P}_\delta - z) \lesssim -h \quad \text{near } \kappa(E_u^*) \quad \text{on } H_h^{-r}, \quad r \geq r_0, \quad \operatorname{Im} z \geq -C_1 h. \tag{4.21}$$

Since (4.19) is true when q_1 is removed from $\mathbf{P}_\delta$, and $\operatorname{Re}\sigma_h(e^{rG} q_1 e^{-rG}) = q_1(x) > 0$ on $\{\rho = -\varepsilon\} \subset \{\langle\xi\rangle^{-1} p = 0\} \subset \overline{T}^*\mathcal{M}$, we have by Lemma 3.5,

$$\operatorname{Im}(\mathbf{P}_{\delta,r} - z) \lesssim -h \quad \text{near } \{\rho = -\varepsilon\} \quad \text{on } L^2, \quad \operatorname{Im} z \geq -C_1 h. \tag{4.22}$$

Finally, since $\{\rho = -\varepsilon\} \cap \{X_1 \rho = 0\} \subset \operatorname{ell}_h(Q_\infty)$, there exist compact sets (see Figure 6)

$$L_\pm \ \subset \ \{\rho = -\varepsilon\} \cap \{\pm X_1 \rho < 0\} \ \subset \ \mathcal{M} \tag{4.23}$$

such that on $\overline{T}^*\mathcal{M}$, and with $L_\pm^\circ$ denoting the interior of $L_\pm$ inside $\{\rho = -\varepsilon\}$,

$$\{\rho = -\varepsilon\} \cap \{\pm X_1 \rho \leq 0\} \ \subset \ \pi^{-1}(L_\pm^\circ) \cup \operatorname{ell}_h(Q_\infty). \tag{4.24}$$

The proof of the lemma is based on the following bound similar to [DyZw13, (3.10)]:

$$\|\mathbf{u}\|_{\mathcal{H}_h^r} \leq C h^{-1} \|\mathbf{f}\|_{\mathcal{H}_h^r}, \quad \mathbf{u} \in \mathcal{D}_h^r, \quad \mathbf{f} = (\mathbf{P}_\delta - z)\mathbf{u}. \tag{4.25}$$

By [FaSj, Lemma A.1] applied to $e^{rG}\mathbf{u}$ and the operator $\mathbf{P}_{\delta,r} \in \Psi_h^1$, for each fixed h and each $\mathbf{u} \in \mathcal{D}_h^r$ there exists a sequence $\mathbf{u}_j \in C^\infty(\mathcal{M}; \mathcal{E})$ such that $\mathbf{u}_j \to \mathbf{u}$ in $\mathcal{H}_h^r$ and $(\mathbf{P}_\delta - z)\mathbf{u}_j \to \mathbf{f}$ in $\mathcal{H}_h^r$. Therefore, it suffices to prove (4.25) for the case $\mathbf{u} \in C^\infty(\mathcal{M}; \mathcal{E})$.

We now use semiclassical estimates to obtain bounds on $A\mathbf{u}$, where $A \in \Psi_h^0(\mathcal{M})$ falls into one of the following cases. We will typically arrive to a propagation estimate of the form

$$\|A\mathbf{u}\|_{\mathcal{H}_h^r} \leq C\|B\mathbf{u}\|_{\mathcal{H}_h^r} + Ch^{-1}\|B_1\mathbf{f}\|_{\mathcal{H}_h^r} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{\mathcal{H}_h^r}, \tag{4.26}$$

for some choice of operators $B, B_1 \in \Psi_h^0(\mathcal{M})$. The term $B\mathbf{u}$ will be controlled by previously considered cases and we keep track of the wavefront set of B_1 to show (4.14).

Case 1: $\mathrm{WF}_h(A) \cap \{\langle\xi\rangle^{-1}p = 0\} \subset \mathrm{ell}_h(Q_\infty) \cup \mathrm{ell}_h(Q_\delta)$. Then $\mathbf{P}_\delta - z$, and thus $\mathbf{P}_{\delta,r} - z$, is elliptic on $\mathrm{WF}_h(A)$. Similarly to [DyZw13, Proposition 3.4, Case 1], we find for some $B_1 \in \Psi_h^0(\mathcal{M})$ microlocalized in a small neighborhood of $\mathrm{WF}_h(A)$,

$$\|A\mathbf{u}\|_{\mathcal{H}_h^r} \leq C\|B_1\mathbf{f}\|_{\mathcal{H}_h^r} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{\mathcal{H}_h^r}. \tag{4.27}$$

Note that the $\mathcal{H}_h^r$ bound on the operator $\mathbf{P}_\delta - z$ is equivalent to the L^2 bound on the operator $\mathbf{P}_{\delta,r} - z$; we will use this fact in the next cases.

Case 2: $\mathrm{WF}_h(A)$ is contained in a small neighborhood of $\pi^{-1}(L_-) \subset \overline{T}^*\mathcal{M}$, where L_- is defined in (4.23) and $\pi : \overline{T}^*\mathcal{M} \to \mathcal{M}$ is the projection map; moreover, $\pi^{-1}(L_-) \subset \mathrm{ell}_h(A)$.

For each $(x,\xi) \in \mathrm{WF}_h(A)$, $\varphi^t(x)$ uniformly converges to $\{\rho = -\varepsilon\} \cap \{X_1\rho > 0\}$ as $t \to -\infty$. Here we used that $L_- \subset \{X_1\rho > 0\}$, $\varphi^t(x) = e^{tX}$, $X = \psi(\rho)X_1$, and $\operatorname{sgn}\psi(\rho) = \operatorname{sgn}(\rho + \varepsilon)$ (see Lemma 2.1 and Figures 6 and 7).

Take $B \in \Psi_h^0$ such that

$$\{\rho = -\varepsilon\} \cap \{X_1\rho \geq 0\} \setminus \pi^{-1}(L_-^\circ) \subset \mathrm{ell}_h(B), \quad \mathrm{WF}_h(B) \subset \mathrm{ell}_h(Q_\infty).$$

We apply Lemma 3.7, with $\mathbf{P} = \mathbf{P}_{\delta,r} - z$, $L = \pi^{-1}(L_-)$, $s = 0$, and B_1 elliptic in a sufficiently large neighborhood of L depending on $\mathrm{WF}_h(A)$. All assumptions of this lemma are satisfied, except for the condition $L \cap \mathrm{WF}_h(B) = \emptyset$. Indeed, L is invariant under e^{TH_p} since L_- consists of fixed points of X and thus is invariant under φ_t. The condition (4.22) implies that $\operatorname{Im}\mathbf{P} \lesssim -h$ on L^2 near L. Moreover, for each $(x,\xi) \in \Omega \cap \mathrm{WF}_h(A)$, the point $\lim_{t\to-\infty}\varphi^t(x)$ lies in L_-°.

Finally, the condition $L \cap \mathrm{WF}_h(B) = \emptyset$ can be waived as it is only used in Lemma 3.8 and we can instead construct the required function χ directly. In fact, using the coordinates (2.6), we see that there exists $\chi = \chi(x) \in C^\infty(\mathcal{M}; [0,1])$ supported in an arbitrarily small neighborhood of L_- such that $\chi = 1$ near L_- and $H_p\chi \leq 0$ everywhere.

Now, the estimate (3.12) gives (4.26) for some $B_1 \in \Psi_h^0(\mathcal{M})$ microlocalized in a small neighborhood of $\pi^{-1}(L_-)$. The term $B\mathbf{u}$ is controlled by Case 1.

Case 3: $\mathrm{WF}_h(A)$ is contained in a small neighborhood of some $(x,\xi) \in \{\langle\xi\rangle^{-1}p = 0\}$, where $x \in \overline{\mathcal{U}} \setminus \Gamma_+$. By (1.3), there exists $T > 0$ such that $\varphi^{-T}(x) \notin \overline{\mathcal{U}}$. Similarly to the proof of Lemma 2.8, we use part 2 of Lemma 2.6 to see that $X_1\rho(\varphi^{-T}(x)) > 0$. We

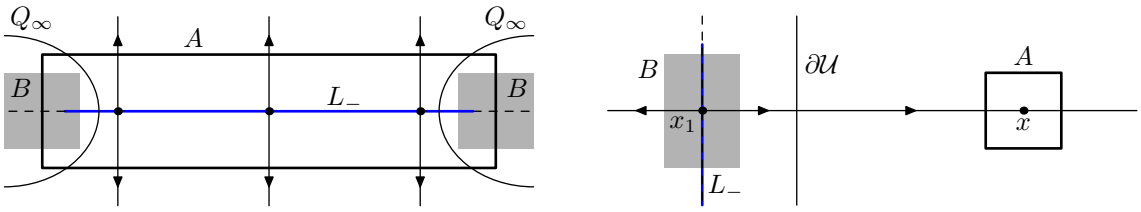


FIGURE 7. An illustration of Cases 2 and 3, with the flow lines of φ^t drawn. The solid blue lines are L_-, the dashed lines containing them are $\{\rho = -\varepsilon\}$, and the semicircles denote $\mathrm{ell}_h(Q_\infty)$. In this and the following figures, the elliptic sets of B are shaded and $\mathrm{WF}_h(A)$ are pictured by the rectangles.

apply part 2 of Lemma 2.6 (with $[\alpha, \beta] = [-\varepsilon, 0]$) again to see that there exists $T' > 0$ such that $x_1 := e^{-T'X_1}(\varphi^{-T}(x)) \in \{\rho = -\varepsilon\} \cap \{X_1\rho > 0\}$ and $e^{-tX_1}(\varphi^{-T}(x)) \in \{-\varepsilon < \rho < 0\}$ for all $t \in [0, T']$. Since $X = \psi(\rho)X_1$, it follows that (see Figure 7)

$$\varphi^{-t}(x) \ \to \ x_1 \in \{\rho = -\varepsilon\} \cap \{X_1\rho > 0\} \quad \text{as } t \to +\infty.$$

By (4.24), there exists $B \in \Psi_h^0$ such that $\pi^{-1}(x_1) \subset \mathrm{ell}_h(B)$ and $B\mathbf{u}$ is controlled either by Case 1 (if $\pi^{-1}(x_1) \subset \mathrm{ell}_h(Q_\infty)$) or by Case 2 (if $x_1 \in L_-$). By propagation of singularities (Lemma 3.2) applied to $\mathbf{P}_{\delta,r}$, the estimate (4.26) holds for some $B_1 \in \Psi_h^0$ microlocalized in a small neighborhood of $\{e^{-tH_p}(x,\xi) \mid t \geq 0\}$.

Case 4: $\mathrm{WF}_h(A)$ is contained in a small neighborhood U of $\kappa(E_s^*)$, where κ is defined in (4.6); moreover, $\kappa(E_s^*) \subset \mathrm{ell}_h(A)$. Take $B, B_1 \in \Psi_h^0$ such that for some arbitrarily small fixed open sets $V \supset \kappa(E_-^*) \cap \{\rho = \varepsilon\}$ and $W \supset \kappa(E_-^*)$

$$\begin{aligned} \kappa(E_-^*) \cap \{\rho = \varepsilon\} \ &\subset \ \mathrm{ell}_h(B), \quad \mathrm{WF}_h(B) \ \subset \ V; \\ \kappa(E_-^*) \ &\subset \ \mathrm{ell}_h(B_1), \quad \mathrm{WF}_h(B_1) \ \subset \ W; \end{aligned}$$

see Figure 8. We also assume that (4.7) holds for the operators A, B, B_1.

We claim that for some choice of U depending on B, B_1,

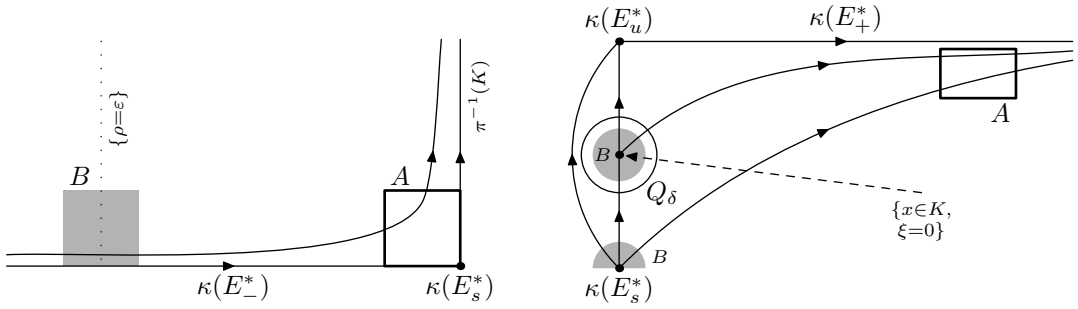
$$U \cap \{\langle\xi\rangle^{-1}p = 0\} \setminus \pi^{-1}(\Gamma_+) \ \subset \ \mathrm{Con}_p(\mathrm{ell}_h(B); \mathrm{ell}_h(B_1)), \tag{4.28}$$

see Definition 3.1 for the notation on the right-hand side. To see (4.28), we first note that by Lemma 2.11, there exists $T' \geq 0$ (depending on B, B_1, but not on U, as long as U lies inside a fixed small neighborhood of $\kappa(E_s^*)$) such that for each $T \geq T'$ and each

$$(x,\xi) \in U \cap \{\langle\xi\rangle^{-1}p = 0\}, \quad \varphi^{-T}(x) \in \{\rho = \varepsilon\}, \tag{4.29}$$

we have

$$e^{-TH_p}(x,\xi) \in \mathrm{ell}_h(B); \quad e^{-tH_p}(x,\xi) \in \mathrm{ell}_h(B_1) \quad \text{for all } t \in [T', T]. \tag{4.30}$$



Case 4 Case 5

FIGURE 8. An illustration of Cases 4 and 5, with the flow lines of e^{tH_p} drawn on $\overline{T}^*_{\Gamma_-}\mathcal{M}$ (Case 4) and $\overline{T}^*_{\Gamma_+}\mathcal{M}$ (Case 5). The right (Case 4) and left (Case 5) side of the pictures is $\pi^{-1}(K)$.

Since $\kappa(E_s^*)$ is invariant under the flow and lies inside $\mathrm{ell}_h(B_1)\setminus\{\rho=\varepsilon\}$, we can make sure that (4.29) never holds for $T\in[0,T')$ and (4.30) holds for all $t\in[0,T]$, as long as U is chosen small enough depending on B,B_1,T'. Now, for each $(x,\xi)\in U\cap\{\langle\xi\rangle^{-1}p=0\}\setminus\pi^{-1}(\Gamma_+)$, there exists $T\geq 0$ such that (4.29) holds. Then (4.30) implies that $(x,\xi)\in\mathrm{Con}_p(\mathrm{ell}_h(B);\mathrm{ell}_h(B_1))$, which proves (4.28).

We now apply Lemma 3.7, with $\mathbf{P}=\mathbf{P}_\delta-z$, $L=\kappa(E_s^*)$, $s=r$. To verify that $\operatorname{Im}\mathbf{P}\lesssim -h$ near L on H_h^r, we use (4.20). By (4.28), we have $\Omega\cap\mathrm{WF}_h(A)\subset\pi^{-1}(\Gamma_+)$. By Lemmas 2.3 and 2.11, we have $e^{-tH_p}(x,\xi)\to L$ as $t\to+\infty$ uniformly in $(x,\xi)\in\Omega\cap\mathrm{WF}_h(A)$. Finally, by (4.7), the space H_h^r can be replaced by $\mathcal{H}_h^r$ in the estimate.

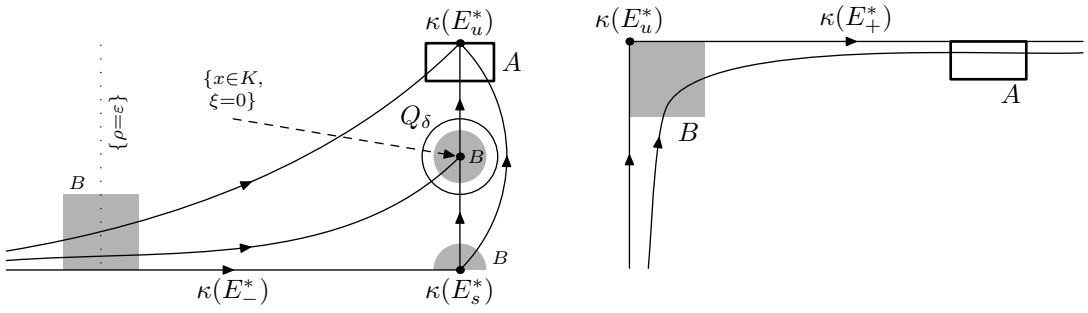
We see that (3.12) gives the estimate (4.26). By Lemma 2.2, $K\cap\{\rho=\varepsilon\}=\emptyset$ for ε small enough; therefore we can choose V so that $\pi(\mathrm{WF}_h(B))\subset\mathcal{U}\setminus\Gamma_+$. Then the term $B\mathbf{u}$ is controlled by Case 3.

Case 5: $\mathrm{WF}_h(A)$ is contained in a small neighborhood of some $(x,\xi)\in\{\langle\xi\rangle^{-1}p=0\}$, where $x\in\Gamma_+$ and $(x,\xi)\notin\kappa(E_+^*)$. If $\xi\notin\overline{E_+^*(x)}$, then by part 4 of Lemma 2.10, we have $e^{-tH_p}(x,\xi)\to\kappa(E_s^*)$ as $t\to+\infty$. Otherwise $\xi\in E_+^*(x)$ does not lie on the fiber infinity; by part 3 of Lemma 2.10, we have $e^{-tH_p}(x,\xi)\to\{x\in K,\ \xi=0\}$ as $t\to+\infty$.

Similarly to Case 3, we use propagation of singularities to obtain the estimate (4.26), where B_1 is microlocalized in a small neighborhood of $\{e^{-tH_p}(x,\xi)\mid t\geq 0\}$ and $\mathrm{WF}_h(B)$ lies either in a small neighborhood of $\kappa(E_s^*)$ or in a small neighborhood of $\{x\in K,\ \xi=0\}$. In the first case, $B\mathbf{u}$ is controlled by Case 4; in the second case, $\mathrm{WF}_h(B)\subset\mathrm{ell}_h(Q_\delta)$ and $B\mathbf{u}$ is controlled by Case 1. See Figure 8.

Case 6: $\mathrm{WF}_h(A)$ is contained in a small neighborhood U of $\kappa(E_u^*)$; moreover, $\kappa(E_u^*)\subset\mathrm{ell}_h(A)$. Take $B\in\Psi_h^0$ such that (see Figure 9)

$$\kappa(E_s^*)\cup\{x\in K,\ \xi=0\}\cup(\{\rho=\varepsilon\}\cap\pi^{-1}(\Gamma_-))\ \subset\ \mathrm{ell}_h(B)$$



Case 6 Case 7

FIGURE 9. An illustration of Cases 6 and 7, with the flow lines of e^{tH_p} drawn on $\overline{T}^*_{\Gamma_-}\mathcal{M}$ (Case 6) and $\overline{T}^*_{\Gamma_+}\mathcal{M}$ (Case 7).

and $\mathrm{WF}_h(B)$ lies in a small neighborhood of the above set. Let $B_1 \in \Psi_h^0$ satisfy $\{\langle\xi\rangle^{-1}p = 0\} \subset \mathrm{ell}_h(B_1)$ and (4.19) hold near $\mathrm{WF}_h(B_1)$. We claim that for U small enough,

$$U \cap \{\langle\xi\rangle^{-1}p = 0\} \setminus \kappa(E_+^*) \ \subset \ \mathrm{Con}_p(\mathrm{ell}_h(B); \mathrm{ell}_h(B_1)). \tag{4.31}$$

To show (4.31), take $(x,\xi) \in U \cap \{\langle\xi\rangle^{-1}p = 0\} \setminus \kappa(E_+^*)$. If $x \in \Gamma_+$, then by the analysis of Case 5, $(x,\xi) \in \mathrm{Con}_p(\mathrm{ell}_h(B); \mathrm{ell}_h(B_1))$. If $x \notin \Gamma_+$, then there exists $T > 0$ such that $\varphi^{-T}(x) \in \{\rho = \varepsilon\}$; we claim that $\varphi^{-T}(x) \in \mathrm{ell}_h(B)$. Indeed, otherwise $\varphi^{-T}(x)$ does not lie in some fixed closed subset of $\overline{\mathcal{U}}$ which does not intersect Γ_-, which implies that $\varphi^{t-T}(x) \notin \pi(U)$ for $\pi(U)$ a small enough neighborhood of K and all $t \geq 0$; putting $t := T$, we get a contradiction.

We now apply Lemma 3.7 with $\mathbf{P} = \mathbf{P}_\delta - z$, $L = \kappa(E_u^*)$, $s = -r$. To see that $\mathrm{Im}\,\mathbf{P} \lesssim -h$ near L on H_h^{-r}, we use (4.21). By (4.31), $\Omega \cap \mathrm{WF}_h(A) \subset \kappa(E_+^*)$. Then by Lemma 2.3 and the invariance of E_+^* under the flow, $e^{-tH_p}(x,\xi) \to \kappa(E_+^*) \cap \pi^{-1}(K) = \kappa(E_u^*)$ as $t \to +\infty$ uniformly in $(x,\xi) \in \Omega \cap \mathrm{WF}_h(A)$.

By (3.12), we obtain (4.26). The space H_h^{-r} can be replaced in (3.12) by $\mathcal{H}_h^r$; indeed, (3.24) still holds by (4.8) and (3.20), (3.21) follow by propagation of singularities for the conjugated operator $\mathbf{P}_{\delta,r} - z$. The term $B\mathbf{u}$ is controlled by Cases 1, 3, and 4, corresponding to the parts of B lying near $\{x \in K,\ \xi = 0\}$, $\{\rho = \varepsilon\} \cap \pi^{-1}(\Gamma_-)$, and $\kappa(E_s^*)$ respectively.

Case 7: $\mathrm{WF}_h(A)$ is contained in a small neighborhood of some $(x,\xi) \in \kappa(E_+^*)$. Take $B \in \Psi_h^0$ which is microlocalized in a small neighborhood of $\kappa(E_u^*)$ and $\kappa(E_u^*) \subset \mathrm{ell}_h(B)$. Let B_1 be as in Case 6. By Lemma 2.3 and the invariance of E_+^*, we see that $\mathrm{WF}_h(A) \subset \mathrm{Con}_p(\mathrm{ell}_h(B); \mathrm{ell}_h(B_1))$. Similarly to Case 3, propagation of singularities gives (4.26). The term $B\mathbf{u}$ is controlled by Case 6. See Figure 9.

Case 8: $\mathrm{WF}_h(A)$ is contained in a small neighborhood of some $(x,\xi) \in \{\langle\xi\rangle^{-1}p = 0\}$ and $(x,\xi) \notin \mathrm{ell}_h(Q_\infty) \cup \pi^{-1}(\overline{\mathcal{U}} \cup L_- \cup L_+)$. Here $L_\pm$ are defined in (4.23). Then

$x \in \{-2\varepsilon < \rho < 0\} \setminus \{\rho = -\varepsilon\}$. By part 1 of Lemma 2.6 (with $[\alpha,\beta] = [-2\varepsilon,\varepsilon]$ or $[\alpha,\beta] = [-\varepsilon,0]$), and since $X = \psi(\rho)X_1$, $\operatorname{sgn}\psi(\rho) = \operatorname{sgn}(\rho+\varepsilon)$, we see that one of the following holds (see Figure 10)

(1) there exists $T \geq 0$ such that $x_1 := \varphi^{-T}(x) \in \partial\overline{\mathcal{U}}$, or
(2) there exists $T \geq 0$ such that $x_1 := \varphi^{-T}(x) \in \{\rho = -2\varepsilon\}$, or
(3) there exists $x_1 \in \{\rho = -\varepsilon\} \cap \{X_1\rho \geq 0\}$ such that $\varphi^{-t}(x) \to x_1$ as $t \to +\infty$.

Take $B \in \Psi_h^0$ such that $\pi^{-1}(x_1) \in \operatorname{ell}_h(B)$, but $\pi(\operatorname{WF}_h(B))$ lies in a small neighborhood of x_1. Let B_1 be as in Case 6. Similarly to Case 3, by propagation of singularities we get (4.26). The term $B\mathbf{u}$ can be estimated in each of the situations above as follows:

(1) by Cases 1, 3, 5, and 7;
(2) by Case 1, since $\pi^{-1}(x_1) \subset \operatorname{ell}_h(Q_\infty)$;
(3) by Case 2 if $x_1 \in L_-$, and by Case 1 otherwise (as then $\pi^{-1}(x_1) \subset \operatorname{ell}_h(Q_\infty)$).

Case 9: $\operatorname{WF}_h(A)$ is contained in a small neighborhood of $\pi^{-1}(L_+)$ and $\pi^{-1}(L_+) \subset \operatorname{ell}_h(A)$, where L_+ is defined in (4.23). We in particular require that

$$\operatorname{WF}_h(A) \subset \{-\tfrac{3}{2}\varepsilon < \rho < -\tfrac{1}{2}\varepsilon\} \cap \{X_1\rho < 0\}.$$

Take $B \in \Psi_h^0$ such that (see Figure 10)

$$\begin{aligned}
\operatorname{WF}_h(B) \ &\subset \ \big(\{-2\varepsilon < \rho < 0\} \cap \{\rho \neq -\varepsilon\}\big) \cup \operatorname{ell}_h(Q_\infty),\\
\pi^{-1}\big(\{\rho = -\varepsilon\} \cap \{X_1\rho \leq 0\} \setminus L_+^\circ\big) \ &\subset \ \operatorname{ell}_h(B),\\
\pi^{-1}\big(\{\rho = -\tfrac{3}{2}\varepsilon\} \cup \{\rho = -\tfrac{1}{2}\varepsilon\}\big) \ &\subset \ \operatorname{ell}_h(B).
\end{aligned}$$

We apply Lemma 3.7, with $\mathbf{P} = \mathbf{P}_{\delta,r} - z$, $L = \pi^{-1}(L_+)$, $s = 0$, and B_1 chosen as in Case 6. To verify that $\operatorname{Im}\mathbf{P} \lesssim -h$ near L on L^2, we use (4.22). The condition $L \cap \operatorname{WF}_h(B) = \emptyset$ does not hold, but similarly to Case 2 it can be waived by taking a function $\chi \in C^\infty(\mathcal{M};[0,1])$ which is supported in a small enough neighborhood of L_+, but $\chi = 1$ near L_+. As follows from the next paragraph, this function satisfies the conclusions of Lemma 3.8, in fact $H_p\chi = 0$ near Ω.

To finish verifying the assumptions of Lemma 3.7, note that $\operatorname{WF}_h(A) \cap \Omega \subset \pi^{-1}(L_+^\circ)$. Indeed, let $(x,\xi) \in \operatorname{WF}_h(A)$. If $x \notin \{\rho = -\varepsilon\}$, then by part 1 of Lemma 2.6 (with $[\alpha,\beta] = [-\frac{3}{2}\varepsilon,-\varepsilon]$ or $[\alpha,\beta] = [-\varepsilon,-\frac{1}{2}\varepsilon]$), either $\varphi^{-T}(x) \in \{\rho = -\frac{3}{2}\varepsilon\} \cup \{\rho = -\frac{1}{2}\varepsilon\}$ for some $T \geq 0$, or $\varphi^{-t}(x) \to x_1 \in \{\rho = -\varepsilon\}$ as $t \to +\infty$. The latter option is impossible if $\pi(\operatorname{WF}_h(A))$ is sufficiently close to $L_+ \subset \{X_1\rho < 0\}$, and the former option gives $(x,\xi) \notin \Omega$. If $x \in \{\rho = -\varepsilon\}$, then we also have $x \in \{X_1\rho < 0\}$. Therefore, either $x \in L_+^\circ$ or $(x,\xi) \in \operatorname{ell}_h(B)$; in the latter case, $(x,\xi) \notin \Omega$.

Now, the estimate (3.12) gives (4.26). The term $B\mathbf{u}$ can be estimated by Cases 1 and 8.

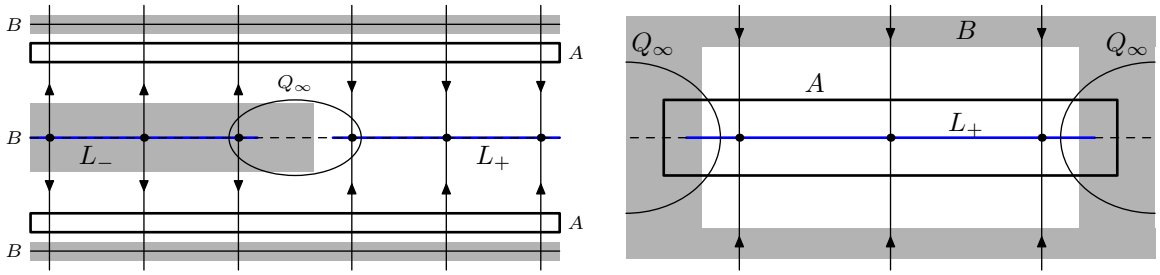


FIGURE 10. An illustration of Cases 8 and 9, with the flow lines of φ^t drawn. The solid blue lines are $L_\pm$ and the dashed lines containing them are $\{\rho = -\varepsilon\}$; the solid lines on the top and bottom of Case 8 are $\partial\mathcal{U}$ and $\{\rho = -2\varepsilon\}$. The (semi)circles denote $\mathrm{ell}_h(Q_\infty)$.

Combining the above cases and using a pseudodifferential partition of unity, we get the estimate (4.25). More precisely, if $A \in \Psi_h^0$ and $\mathrm{WF}_h(A)$ lies in a small neighborhood of $\pi^{-1}(\overline{\mathcal{U}})$, then $A\mathbf{u}$ is estimated by a combination of Cases 1, 3, 5, and 7. If $\pi^{-1}(\mathrm{WF}_h(A)) \cap \overline{\mathcal{U}} = \emptyset$, then $A\mathbf{u}$ is estimated by a combination of Cases 1, 2, 8, and 9.

Reversing the direction of propagation (replacing X_1 by $-X_1$, $\mathbf{X}$ by $-\mathbf{X}$, m by $-m$, and switching E_s^* with E_u^*, E_+^* with E_-^*, and L_+ with L_-), we repeat the above reasoning to get the adjoint estimate similar to [DyZw13, (3.17)]

$$\|\mathbf{u}\|_{\mathcal{H}_h^{-r}} \le Ch^{-1}\|\mathbf{f}\|_{\mathcal{H}_h^{-r}}, \quad \mathbf{u} \in \mathcal{H}_h^{-r}, \quad \mathbf{f} := (\mathbf{P}_\delta^* - \bar{z})\mathbf{u} \in \mathcal{H}_h^{-r}. \tag{4.32}$$

Note that $\mathcal{H}_h^{-r}$ is dual to $\mathcal{H}_h^r$ with respect to the L^2 pairing. The functional analytic argument given at the end of the proof of [DyZw13, Proposition 3.4] shows that together, (4.25) and (4.32) imply invertibility of $\mathbf{P}_\delta - z : \mathcal{D}_h^r \to \mathcal{H}_h^r$ and the bound (4.13).

It remains to verify the wavefront set condition (4.14). By [DyZw13, Lemma 2.3], it suffices to show that for each $(x,\xi),(y,\eta) \in T^*\mathcal{U}$ such that $(y,\eta) \ne (x,\xi)$ and either $p(x,\xi) \ne 0$ or $(x,\xi) \ne e^{tH_p}(y,\eta)$ for all $t \ge 0$, there exist $A \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$ and $B_2 \in \Psi_h^0(\mathcal{M})$ such that

$$(x,\xi) \in \mathrm{ell}_h(A), \ (y,\eta) \notin \mathrm{WF}_h(B_2),$$

and for each $\mathbf{u} \in \mathcal{D}_h^r$ with $\mathbf{f} := (\mathbf{P}_\delta - z)\mathbf{u}$,

$$\|A\mathbf{u}\|_{\mathcal{H}_h^r} \le Ch^{-1}\|B_2\mathbf{f}\|_{\mathcal{H}_h^r} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{\mathcal{H}_h^r}. \tag{4.33}$$

As remarked after (4.25), an approximation argument reduces us to the case $\mathbf{u} \in C^\infty$. Then (4.33) follows by a combination of Cases 1, 3, and 5. Here we use that the operator B_1 from Case 2 is microlocalized in a small neighborhood of $\pi^{-1}(L_-) \subset \pi^{-1}(\mathcal{M}\backslash\mathcal{U})$ and the same operator from Case 4 is microlocalized in a small neighborhood of $\kappa(E_-^*) \subset \partial\overline{T}^*\mathcal{M}$; thus their wavefront sets do not contain (y,η). □

4.2. **Proofs of Theorems 1 and 2.** In this section, we show the meromorphic continuation of the resolvent $\mathbf{R}(\lambda)$ defined in (1.11). We start with the following corollary of Lemma 4.2:

Lemma 4.3. *Let $Q_\infty \in \Psi_h^1(\mathcal{M}), Q_\delta \in \Psi_h^{\mathrm{comp}}(\mathcal{M}), q_1 \in C^\infty(\mathcal{M})$ be introduced in (4.9), and $\mathcal{H}_h^r, \mathcal{D}_h^r$ be given by (4.4), (4.11). Fix $C_1, C_2 > 0$ and $r > r_0 = r_0(C_1)$. Define*

$$\mathbf{P}_0 = \mathbf{P}_0(h) := \frac{h}{i}\mathbf{X} - i(Q_\infty + q_1) : \mathcal{D}_h^r \to \mathcal{H}_h^r. \tag{4.34}$$

Then for $0 < h < h_0(C_1, C_2, r)$,

1. *$\mathbf{P}_0 - z : \mathcal{D}_h^r \to \mathcal{H}_h^r$ is a Fredholm operator of index zero for $z \in [-C_2 h, C_2 h] + i[-C_1 h, 1]$.*
2. *The inverse*

$$\mathbf{R}_0(z) := (\mathbf{P}_0 - z)^{-1} : \mathcal{H}_h^r \to \mathcal{D}_h^r, \quad z \in [-C_2 h, C_2 h] + i[-C_1 h, 1] \tag{4.35}$$

is a meromorphic family of operators with poles of finite rank.

Proof. 1. Take $z \in [-C_2 h, C_2 h] + i[-C_1 h, 1]$. By Lemma 4.2, $\mathbf{P}_\delta - z : \mathcal{D}_h^r \to \mathcal{H}_h^r$ is invertible, where $\mathbf{P}_\delta$ is defined in (4.10). We write

$$\mathbf{P}_\delta - z = \mathbf{P}_0 - z - iQ_\delta.$$

Now, Q_δ is compactly microlocalized (that is, $\mathrm{WF}_h(Q_\delta) \Subset T^*\mathcal{M}$) so it is smoothing; that is, Q_δ is bounded $H^{-N}(\mathcal{M}) \to H^N(\mathcal{M})$ for all N. By Rellich's Theorem (using the fact that $\mathcal{M}$ is compact and $e^{\pm rG}$ are pseudodifferential operators), we see that Q_δ is a compact operator $\mathcal{H}_h^r \to \mathcal{H}_h^r$ and thus $\mathcal{D}_h^r \to \mathcal{H}_h^r$. It follows that $\mathbf{P}_0 - z : \mathcal{D}_h^r \to \mathcal{H}_h^r$ is a Fredholm operator of index zero.

2. The meromorphy of $\mathbf{R}_0(z)$ follows by analytic Fredholm theory [Zw, Proposition D.4], as long as $\mathbf{P}_0 - z$ is known to be invertible for at least one value of z. We take $z = i$; it suffices to prove the estimate

$$\|\mathbf{u}\|_{\mathcal{H}_h^r} \le Ch^{-1}\|\mathbf{f}\|_{\mathcal{H}_h^r}, \quad \mathbf{u} \in \mathcal{D}_h^r, \quad \mathbf{f} = (\mathbf{P}_0 - i)\mathbf{u}. \tag{4.36}$$

Similarly to (4.17), let $\mathbf{P}_{0,r} := e^{rG}\mathbf{P}_0 e^{-rG} \in \Psi_h^1(\mathcal{M}; \mathcal{E})$. Note that $\operatorname{Im}\sigma_h(\mathbf{P}_{0,r}) \le 0$ near $\{\langle\xi\rangle^{-1}p = 0\}$ and $\operatorname{Re}\sigma_h(\mathbf{P}_{0,r}) = p$ similarly to (4.18), (4.19). By (4.4) and the approximation argument following (4.25), we reduce (4.36) to

$$\|\mathbf{v}\|_{L^2} \le Ch^{-1}\|\mathbf{g}\|_{L^2}, \quad \mathbf{v} \in C^\infty(\mathcal{M}; \mathcal{E}), \quad \mathbf{g} = (\mathbf{P}_{0,r} - i)\mathbf{v}. \tag{4.37}$$

We now apply Lemma 3.7, with $\mathbf{P} = \mathbf{P}_{0,r} - i$, $L = \{\langle\xi\rangle^{-1}p = 0\}$, $s = 0$, and $B = 0$. Note that $\operatorname{Im}\mathbf{P} \lesssim -h$ on L^2 near L by Lemma 3.5, with $Q := 1$. By (3.12), we get for some $A, B_1 \in \Psi^0$ such that $\{\langle\xi\rangle^{-1}p = 0\} \subset \mathrm{ell}_h(A)$ and B_1 is microlocalized in a neighborhood of $\{\langle\xi\rangle^{-1}p = 0\}$,

$$\|A\mathbf{v}\|_{L^2} \le Ch^{-1}\|B_1\mathbf{g}\|_{L^2} + \mathcal{O}(h^\infty)\|\mathbf{v}\|_{L^2}.$$

Combining this with the elliptic estimate (4.27) valid for $\mathrm{WF}_h(A)\cap\{\langle\xi\rangle^{-1}p=0\}=\emptyset$, we get (4.37). □

The operator $\mathbf{R}_0(z)$ depends on the choice of Q_∞, q_1 (and thus on h). It is independent of the choice of r, but proving this would require a separate argument. However, the restriction of this operator to $\mathcal{U}$ is independent of Q_∞, q_1, r. This is a byproduct of the following

Lemma 4.4. *In the notation of Lemma 4.3, let $\lambda\in[-C_1,h^{-1}]+i[-C_2,C_2]$ and put $z:=ih\lambda$. Assume also that $\operatorname{Re}\lambda>C_0$, where C_0 is defined in (1.9). Then*

$$\mathbf{R}(\lambda)\mathbf{f}=-ih\mathbf{R}_0(z)\mathbf{f}|_{\mathcal{U}}\quad\text{for all}\quad\mathbf{f}\in C_0^\infty(\mathcal{U};\mathcal{E}).\tag{4.38}$$

Proof. By analyticity and since h can be chosen arbitrarily small, it suffices to prove (4.38) in the case $\operatorname{Re}\lambda>C_3$, where $C_3>C_0$ is a large enough constant depending on r, but not on h. As discussed after (4.4), the anisotropic Sobolev space $\mathcal{H}_h^r$ contains the standard Sobolev space $H_h^N(\mathcal{M};\mathcal{E})$, for N large enough depending on r.

We consider an extension of $\mathbf{X}$ to $\mathcal{M}$ such that (1.6) holds on $\mathcal{M}$. Note that (1.6) holds also for the operator $\mathbf{X}+h^{-1}q_1$, since $q_1\in C^\infty(\mathcal{M})$ is a multiplication operator. We claim that for some C_3 depending on N, but not on h,

$$\|e^{-t(\mathbf{X}+h^{-1}q_1)}\mathbf{f}\|_{H_h^N}\le C(h)e^{C_3t},\quad t\ge0.\tag{4.39}$$

This follows by writing the transfer operator $e^{-t(\mathbf{X}+h^{-1}q_1)}$ in the form similar to (1.7) using a local trivialization of the bundle $\mathcal{E}$, with V now a matrix. Here we use the fact that each derivative of φ^{-t} is bounded exponentially in t. The term $h^{-1}q_1$ does not change the value of C_3, as $q_1\ge0$ everywhere and $t\ge0$, see (1.7).

Now, for $\operatorname{Re}\lambda>C_3$ and $z:=ih\lambda$ not a pole of $\mathbf{R}_0$, consider the function

$$\mathbf{v}:=\int_0^\infty e^{-t(\mathbf{X}+h^{-1}q_1+\lambda)}\mathbf{f}\,dt\ \in\ H_h^N(\mathcal{M};\mathcal{E})\ \subset\ \mathcal{H}_h^r.$$

Since $\operatorname{supp}\mathbf{f}\subset\mathcal{U}$, $\operatorname{supp}q_1\cap\overline{\mathcal{U}}=\emptyset$, and $\mathcal{U}$ is convex, it follows that (see (1.7))

$$\mathbf{R}(\lambda)\mathbf{f}=\mathbf{v}\quad\text{on }\mathcal{U}.\tag{4.40}$$

We also have $\operatorname{supp}\mathbf{v}\subset\overline{\Sigma}$, where Σ is defined in (2.4). (In fact, (1.8) implies that $\operatorname{supp}\mathbf{v}\subset\overline{\Sigma_+}$.) Therefore, $Q_\infty\mathbf{v}=0$. It follows that

$$(\mathbf{P}_0-z)\mathbf{v}=-ih\mathbf{f},\quad\mathbf{v}\in\mathcal{D}_h^r.$$

Since $\mathbf{P}_0-z$ is invertible $\mathcal{D}_h^r\to\mathcal{H}_h^r$, we have

$$-ih\mathbf{R}_0(z)\mathbf{f}=\mathbf{v}.\tag{4.41}$$

Combining (4.40) and (4.41), we get (4.38). □

Proof of Theorem 1. By Lemmas 4.3 and 4.4, the operator $-ih\,\mathbb{1}_{\mathcal{U}}\,\mathbf{R}_0(ih\lambda)\,\mathbb{1}_{\mathcal{U}}$ gives the meromorphic continuation of $\mathbf{R}(\lambda)$ in the region $[-C_1,h^{-1}]+i[-C_2,C_2]$ for h small enough. Since C_1,C_2 can be chosen arbitrarily and h can be arbitrarily small, we obtain the continuation to the entire complex plane. □

Note that for all $\lambda\in\mathbb{C}$,

$$(\mathbf{X}+\lambda)\mathbf{R}(\lambda)=\mathbf{R}(\lambda)(\mathbf{X}+\lambda)=1\ :\ C_0^\infty(\mathcal{U})\to\mathcal{D}'(\mathcal{U}). \tag{4.42}$$

Indeed, by analytic continuation it suffices to consider the case $\operatorname{Re}\lambda>C_0$; in this case, (4.42) follows from (1.10).

The following microlocalization statement is used in the proofs of Theorems 2 and 3. See (3.1) and (3.2) for the notation used below.

Lemma 4.5. *Let $\lambda_0\in\mathbb{C}$. Then the expansion (1.13) holds for $\mathbf{R}_H(\lambda):C_0^\infty(\mathcal{U};\mathcal{E})\to\mathcal{D}'(\mathcal{U};\mathcal{E})$ holomorphic near λ_0 and a finite rank operator $\Pi=\Pi_{\lambda_0}:C_0^\infty(\mathcal{U};\mathcal{E})\to\mathcal{D}'(\mathcal{U};\mathcal{E})$. Moreover,* $\operatorname{supp}K_\Pi\subset\Gamma_+\times\Gamma_-$ *and*

$$\operatorname{WF}'(\mathbf{R}_H(\lambda))\ \subset\ \Delta(T^*\mathcal{U})\cup\Upsilon_+\cup(E_+^*\times E_-^*),\quad \operatorname{WF}'(\Pi)\ \subset\ E_+^*\times E_-^*, \tag{4.43}$$

where $\Delta(T^*\mathcal{U})$ is the diagonal, $E_\pm^*\subset T^*\overline{\mathcal{U}}$ are defined in Lemma 2.10, and

$$\Upsilon_+=\{(e^{tH_p}(y,\eta),y,\eta)\mid t\geq 0,\ p(y,\eta)=0,\ y\in\mathcal{U},\ \varphi^t(y)\in\mathcal{U}\}.$$

Proof. We argue similarly to the proof of [DyZw13, Proposition 3.3]. By Theorem 1,

$$\mathbf{R}(\lambda)=\mathbf{R}_H(\lambda)+\sum_{j=1}^{J(\lambda_0)}\frac{\mathbf{A}_j}{(\lambda-\lambda_0)^j} \tag{4.44}$$

where $\mathbf{R}_H(\lambda):C_0^\infty(\mathcal{U};\mathcal{E})\to\mathcal{D}'(\mathcal{U};\mathcal{E})$ is holomorphic near λ_0 and $\mathbf{A}_j:C_0^\infty(\mathcal{U};\mathcal{E})\to\mathcal{D}'(\mathcal{U};\mathcal{E})$ are finite rank operators. Plugging this expansion into (4.42), we get

$$\mathbf{A}_{j+1}=-(\mathbf{X}+\lambda_0)\mathbf{A}_j,\ 1\leq j<J(\lambda_0);\quad (\mathbf{X}+\lambda_0)\mathbf{A}_{J(\lambda_0)}=0. \tag{4.45}$$

The expansion (1.13) follows from here by putting $\Pi:=\mathbf{A}_1$.

If $\psi_1,\psi_2\in C_0^\infty(\mathcal{U})$ satisfy $\operatorname{supp}\psi_1\cap\Gamma_-=\operatorname{supp}\psi_2\cap\Gamma_+=\emptyset$, then Lemma 2.9 shows that $\mathbf{R}(\lambda)\psi_1,\psi_2\mathbf{R}(\lambda)$ are holomorphic for all $\lambda\in\mathbb{C}$. Therefore, $\Pi\psi_1=\psi_2\Pi=0$; this implies that $\operatorname{supp}K_\Pi\subset\Gamma_+\times\Gamma_-$.

We finally prove (4.43). We start by writing the following identity relating the auxiliary resolvents defined by (4.12) and (4.35) (we put $z:=ih\lambda$):

$$\mathbf{R}_0(z)=\mathbf{R}_\delta(z)-i\mathbf{R}_\delta(z)Q_\delta\mathbf{R}_\delta(z)-\mathbf{R}_\delta(z)Q_\delta\mathbf{R}_0(z)Q_\delta\mathbf{R}_\delta(z).$$

Since Q_δ is supported inside $\mathcal{U}$, by (4.38) this gives

$$\mathbf{R}(\lambda)=-ih\,\mathbb{1}_{\mathcal{U}}\left(\mathbf{R}_\delta(z)-i\mathbf{R}_\delta(z)Q_\delta\mathbf{R}_\delta(z)\right)\mathbb{1}_{\mathcal{U}}-\mathbb{1}_{\mathcal{U}}\,\mathbf{R}_\delta(z)Q_\delta\mathbf{R}(\lambda)Q_\delta\mathbf{R}_\delta(z)\,\mathbb{1}_{\mathcal{U}}\,. \tag{4.46}$$

We analyse each of the terms on the right-hand side separately. By (4.14), we have

$$\mathrm{WF}'_h(\mathbf{R}_\delta(z))\cap T^*(\mathcal{U}\times\mathcal{U}) \ \subset\ \Delta(T^*\mathcal{U})\cup\Upsilon_+. \tag{4.47}$$

By (3.3), and since $\mathrm{WF}_h(Q_\delta)\subset T^*\mathcal{U}$, we get

$$\mathrm{WF}'_h(\mathbf{R}_\delta(z)Q_\delta\mathbf{R}_\delta(z))\cap T^*(\mathcal{U}\times\mathcal{U}) \ \subset\ \Delta(T^*\mathcal{U})\cup\Upsilon_+.$$

To handle the third term in (4.46), note that for each family of operators $\mathbf{T}(\lambda):C_0^\infty(\mathcal{U};\mathcal{E})\to\mathcal{D}'(\mathcal{U};\mathcal{E})$ which is holomorphic in λ and independent of h, we have by (3.3)

$$\mathrm{WF}'_h(\mathbf{R}_\delta(z)Q_\delta\mathbf{T}(\lambda)Q_\delta\mathbf{R}_\delta(z))\cap T^*(\mathcal{U}\times\mathcal{U}) \ \subset\ \Theta_\delta^+\times\Theta_\delta^-,$$
$$\Theta_\delta^\pm = T^*\mathcal{U}\cap\bigcup_{\pm t\geq 0} e^{tH_p}(\mathrm{WF}_h(Q_\delta)).$$

Plugging the expansion (1.13) into the third term in (4.46) and using that the terms in this expansion are h-independent and $\mathbf{R}(\lambda)$ does not depend on δ, we get

$$\mathrm{WF}'_h(\mathbf{R}_H(\lambda))\cap T^*(\mathcal{U}\times\mathcal{U}) \ \subset\ \Delta(T^*\mathcal{U})\cup\Upsilon_+\cup\Big(\bigcap_\delta\Theta_\delta^+\times\bigcap_\delta\Theta_\delta^-\Big),$$
$$\mathrm{WF}'_h(\Pi)\cap T^*(\mathcal{U}\times\mathcal{U}) \ \subset\ \bigcap_\delta\Theta_\delta^+\times\bigcap_\delta\Theta_\delta^-.$$

Since $\mathbf{R}_H(\lambda)$ is independent of h, by [DyZw13, (2.6)] we have

$$\mathrm{WF}'(\mathbf{R}_H(\lambda)) = \mathrm{WF}'_h(\mathbf{R}_H(\lambda))\cap(T^*(\mathcal{U}\times\mathcal{U})\setminus 0),$$

and same is true for Π. To show (4.43), it remains to prove that

$$\bigcap_\delta\Theta_\delta^\pm\subset E_\pm^*.$$

Take $(y,\eta)\in\bigcap_\delta\Theta_\delta^\pm$. By taking a sequence of δ converging to 0, we see that there exists a sequence t_j such that $e^{\mp t_jH_p}(y,\eta)\to\{x\in K,\ \xi=0\}$. If $y\notin\Gamma_\pm$, then the trajectory $\{\varphi^{\mp t}(y)\mid t\geq 0\}$ never passes through some neighborhood of K; therefore, we have $y\in\Gamma_\pm$. Since p is preserved along the trajectories of e^{tH_p}, we have $p(y,\eta)=0$. Finally, if $\eta\notin E_\pm^*(y)$, then by part 4 of Lemma 2.10 the trajectory $\{e^{\mp tH_p}(y,\eta)\mid t\geq 0\}$ never passes through some neighborhood of the zero section and we have a contradiction. It follows that $(y,\eta)\in E_\pm^*$ as required. □

For the proof of Theorem 2, we also need

Lemma 4.6. *Assume that $\mathbf{u}\in\mathcal{D}'(\mathcal{U};\mathcal{E})$ satisfies*

$$\operatorname{supp}\mathbf{u}\ \subset\ \Gamma_+,\quad \mathrm{WF}(\mathbf{u})\ \subset\ E_+^*. \tag{4.48}$$

For some $\lambda\in\mathbb{C}$, put $\mathbf{f}:=(\mathbf{X}+\lambda)\mathbf{u}$. Take $\chi,\chi'\in C_0^\infty(\mathcal{U})$ satisfying (2.3) and $\chi=1$ near K. Then:

1. For $r>0$ large enough, $\chi'\mathbf{u}$ and $\chi'\mathbf{f}$ lie in the space $\mathcal{H}_h^r$ from (4.4). By Lemmas 4.3 and 4.4, this makes it possible to define $\mathbf{R}(\lambda)\chi'\mathbf{f}\in\mathcal{D}'(\mathcal{U};\mathcal{E})$.

2. We have $\chi\mathbf{u} = \chi\mathbf{R}(\lambda)\chi'\mathbf{f}$.

Proof. 1. Take $r > 0$ large enough so that $\chi'\mathbf{u} \in H_h^{-r}(\mathcal{M};\mathcal{E})$. By Lemma 4.1, the order function m is equal to -1 near $E_+^* \supset \mathrm{WF}(\chi'\mathbf{u})$. Let $G = G(h)$ be the operator defined in (4.2). Then $e^{rG(h)}$ is a nonsemiclassical pseudodifferential operator of order $-r$ microlocally near $\mathrm{WF}(\chi'\mathbf{u})$. It follows that $e^{rG(h)}\chi'\mathbf{u} \in L^2$ and thus $\chi'\mathbf{u} \in \mathcal{H}_h^r$. Since $\mathbf{f}$ satisfies (4.48) as well, we similarly have $\chi'\mathbf{f} \in \mathcal{H}_h^r$.

2. Since $\chi'\mathbf{u} \in \mathcal{H}_h^r$ and $\mathbf{X}\chi'\mathbf{u} \in \mathcal{H}_h^r$ for r large enough, we have by Lemmas 4.3 and 4.4,

$$\chi\mathbf{u} = \chi\chi'\mathbf{u} = \chi\mathbf{R}(\lambda)(\mathbf{X}+\lambda)\chi'\mathbf{u}.$$

Now, by (1.6)

$$(\mathbf{X}+\lambda)\chi'\mathbf{u} = (X\chi')\mathbf{u} + \chi'\mathbf{f}.$$

Take $x \in \Gamma_+ \cap \operatorname{supp}(X\chi')$. By Lemma 2.3, there exists $t' > 0$ such that $\varphi^{-t'}(x) \in \operatorname{supp}\chi$. Then by (2.3), $\varphi^t(x) \notin \operatorname{supp}\chi$ for all $t \geq 0$. In particular, $x \notin \Gamma_-$. Since $\operatorname{supp}\mathbf{u} \subset \Gamma_+$, there exists $\psi_1 \in C_0^\infty(\mathcal{M})$ such that $(X\chi')\mathbf{u} = \psi_1(X\chi')\mathbf{u}$ and $\operatorname{supp}\psi_1 \cap \Gamma_- = \emptyset$. Then $\mathbf{R}(\lambda)\psi_1$ is given by (2.7). It follows from (1.8) that

$$\chi\mathbf{R}(\lambda)(X\chi')\mathbf{u} = 0$$

and thus $\chi\mathbf{u} = \chi\mathbf{R}(\lambda)\chi'\mathbf{f}$ as needed. □

Proof of Theorem 2. The expansion (1.13) and the properties (1.14) have already been established in Lemma 4.5. Therefore, it remains to prove (1.15). The property $\mathbf{X}\Pi = \Pi\mathbf{X}$ follows from (1.13) and (4.42). By (1.14) and (4.45), we know that

$$\operatorname{Ran}\Pi \ \subset\ \operatorname{Res}_{\mathbf{X}}^{(J(\lambda_0))}(\lambda_0),$$

therefore it remains to prove that for each N,

$$\mathbf{u} \in \operatorname{Res}_{\mathbf{X}}^{(N)}(\lambda_0) \implies \mathbf{u} = \Pi\mathbf{u}. \tag{4.49}$$

Take $\chi \in C_0^\infty(\mathcal{U})$ such that $\chi = 1$ near K and let $\chi' \in C_0^\infty(\mathcal{U})$ be constructed in Lemma 2.5. We claim that for each $j = 0, \dots, N$,

$$\chi\mathbf{R}(\lambda)\chi'(\mathbf{X}+\lambda_0)^j\mathbf{u} = \sum_{k=j}^{N-1} \frac{(-1)^{k-j}\chi(\mathbf{X}+\lambda_0)^k\mathbf{u}}{(\lambda-\lambda_0)^{k-j+1}}. \tag{4.50}$$

We argue by induction on $j = N, \dots, 0$. For $j = N$, we have $(\mathbf{X}+\lambda_0)^j\mathbf{u} = 0$ and (4.50) is trivial. Now, assume that (4.50) is true for $j+1$. Using the identity

$$(\mathbf{X}+\lambda_0)^j\mathbf{u} = \frac{(\mathbf{X}+\lambda)(\mathbf{X}+\lambda_0)^j\mathbf{u} - (\mathbf{X}+\lambda_0)^{j+1}\mathbf{u}}{\lambda-\lambda_0}$$

and Lemma 4.6 for the first term on the right-hand side, we obtain (4.50) for j, finishing its proof.

Now, take $j = 0$ in (4.50) and use (1.13). Equating the terms next to $(\lambda - \lambda_0)^{-1}$, we obtain $\chi \mathbf{u} = \chi \Pi \chi' \mathbf{u}$. Moreover, $\Pi \chi' \mathbf{u} = \Pi \mathbf{u}$ since $\operatorname{supp} K_\Pi \subset \Gamma_+ \times \Gamma_-$, $\operatorname{supp} \mathbf{u} \subset \Gamma_+$, and $\chi' = 1$ near K. Since χ could be chosen arbitrarily, this gives (4.49). □

5. Dynamical traces and zeta functions

In this section we prove Theorem 3. More generally, we prove in Theorem 4 below that the dynamical trace $F_{\mathbf{X}}(\lambda)$ associated to $\mathbf{X}$ is equal to the flat trace of a certain operator featuring the resolvent $\mathbf{R}(\lambda)$; this flat trace gives the meromorphic extension of $F_{\mathbf{X}}(\lambda)$. The key ingredient of the proof is the wavefront set condition (4.43) on the meromorphic extension of the resolvent. We follow the strategy of [DyZw13] and refer the reader to that paper for the parts of the proof that remain unchanged in our more general case.

5.1. **Meromorphic extension of traces.** We first show how to express Pollicott–Ruelle resonances of $\mathbf{X}$ as the poles of a certain trace expression featuring closed geodesics. To write down this expression, we need to introduce some notation. Define the vector bundle $\mathcal{E}_0$ over $\overline{\mathcal{U}}$ by

$$\mathcal{E}_0(x) = \{\eta \in T^*_x \mathcal{M} \mid \langle X(x), \eta \rangle = 0\}, \quad x \in \overline{\mathcal{U}}. \tag{5.1}$$

Assume that $x, \varphi^t(x) \in \mathcal{U}$ for some t. Define the *linearized Poincaré map*

$$\mathcal{P}_{x,t} : \mathcal{E}_0(x) \to \mathcal{E}_0(\varphi^t(x)), \quad \mathcal{P}_{x,t} = (d\varphi^t(x))^{-T}|_{\mathcal{E}_0(x)}.$$

Next, the parallel transport

$$\alpha_{x,t} : \mathcal{E}(x) \to \mathcal{E}(\varphi^t(x))$$

is defined as follows: for each $\mathbf{u} \in C^\infty(\mathcal{M}; \mathcal{E})$, we put $\alpha_{x,t}(\mathbf{u}(x)) = e^{-t\mathbf{X}} \mathbf{u}(\varphi^t(x))$. This definition only depends on the value of $\mathbf{u}$ at x; indeed, (1.8) shows that if $\mathbf{u}(x) = 0$, then $e^{-t\mathbf{X}} \mathbf{u}(\varphi^t(x)) = 0$ as well (by writing $\mathbf{u}$ as a sum of expressions of the form $f\mathbf{v}$, where $f \in C^\infty(\mathcal{M})$ vanish at x).

Now, assume that $\gamma(t) = \varphi^t(x_0)$ is a closed trajectory, that is $\gamma(T) = \gamma(0)$ for some $T > 0$. (We call T the period of γ, and regard the same γ with two different values of T as two different closed trajectories. The minimal positive $T^\sharp$ such that $\gamma(T^\sharp) = \gamma(0)$ is called the *primitive period*.) Assume also that $x_0 \in \mathcal{U}$; this implies immediately that γ lies inside K. The operators $\alpha_{\gamma(t),T} : \mathcal{E}(\gamma(t)) \to \mathcal{E}(\gamma(t))$, as well as $\mathcal{P}_{\gamma(t),T} : \mathcal{E}_0(\gamma(t)) \to \mathcal{E}_0(\gamma(t))$, are conjugate to each other for different t, therefore the trace and the determinant

$$\operatorname{tr} \alpha_\gamma := \operatorname{tr} \alpha_{\gamma(t),T}, \quad \det(I - \mathcal{P}_\gamma) := \det(I - \mathcal{P}_{\gamma(t),T}) \tag{5.2}$$

do not depend on t. Note that by (2.11),

$$\det(I - \mathcal{P}_\gamma) \neq 0. \tag{5.3}$$

The main result of this subsection, and the key ingredient for showing meromorphic continuation of dynamical zeta functions, is

Theorem 4. *Define for* $\operatorname{Re}\lambda \gg 1$,

$$F_{\mathbf{X}}(\lambda) := \sum_{\gamma} \frac{e^{-\lambda T_\gamma}\, T_\gamma^\sharp \operatorname{tr}\alpha_\gamma}{|\det(I-\mathcal{P}_\gamma)|} \tag{5.4}$$

where the sum is over all closed trajectories γ *inside* K, $T_\gamma > 0$ *is the period of* γ, *and* $T_\gamma^\sharp$ *is the primitive period. Then* $F(\lambda)$ *extends meromorphically to* $\lambda \in \mathbb{C}$. *The poles of* $F(\lambda)$ *are the Pollicott–Ruelle resonances of* $\mathbf{X}$ *and the residue at a pole* λ_0 *is equal to the rank of* Π_{λ_0} *(see Theorem 2).*

Remark. The sum (5.4) converges for large $\operatorname{Re}\lambda$, since $|\det(I-\mathcal{P}_\gamma)|$ is bounded away from zero, α_γ grows at most exponentially in T_γ, and the number of closed trajectories grows at most exponentially by Lemma 2.17.

Proof. We use the concept of the flat trace of an operator $\mathbf{A} : C^\infty(\mathcal{M};\mathcal{U}) \to \mathcal{D}'(\mathcal{M};\mathcal{U})$ satisfying the condition

$$\operatorname{WF}'(\mathbf{A}) \cap \Delta(T^*\mathcal{M}\setminus 0) = \emptyset, \quad \Delta(T^*\mathcal{M}) = \{(x,\xi,x,\xi) \mid (x,\xi)\in T^*\mathcal{M}\setminus 0\}. \tag{5.5}$$

The flat trace is defined as the integral of the restriction of the Schwartz kernel $K_{\mathbf{A}} \in \mathcal{D}'(\mathcal{M}\times\mathcal{M}; \operatorname{End}(\mathcal{E}))$ to the diagonal:

$$\operatorname{tr}^\flat \mathbf{A} := \int_{\mathcal{M}} \operatorname{tr}_{\operatorname{End}(\mathcal{E})} K_{\mathbf{A}}(x,x)\, dx$$

and $\operatorname{tr}_{\operatorname{End}(\mathcal{E})} K_{\mathbf{A}}(x,x)$ is a well-defined distribution on $\mathcal{M}$ due to (5.5) – see [DyZw13, §2.4].

The starting point of the proof is the Atiyah–Bott–Guillemin trace formula [Gu]

$$\operatorname{tr}^\flat \int_0^\infty \varphi(t)\chi e^{-t\mathbf{X}}\chi\, dt = \sum_\gamma \frac{\varphi(T_\gamma) T_\gamma^\sharp \operatorname{tr}\alpha_\gamma}{|\det(I-\mathcal{P}_\gamma)|}, \quad \varphi\in C_0^\infty(0,\infty), \tag{5.6}$$

where $\chi \in C_0^\infty(\mathcal{U})$ is any function such that $\chi = 1$ near K. Note that the Schwartz kernel $\chi(x)K_{e^{-t\mathbf{X}}}(x,y)\chi(y)$ is a smooth function times the delta function of the submanifold $\{y=\varphi^{-t}(x)\}\subset \mathbb{R}_t\times\mathcal{M}_x\times\mathcal{M}_y$, so the wavefront set of this kernel is contained in the conormal bundle to this surface [HöI, Example 8.2.5]

$$\{y=\varphi^{-t}(x),\ \xi = -(d\varphi^{-t}(x))^T\cdot\eta,\ \tau = \langle X(y),\eta\rangle,\ \eta\neq 0,\ x,y\in\operatorname{supp}\chi\}.$$

Here τ is the momentum dual to t. If $\mathbf{A}_\varphi$ is the integral on the left-hand side of (5.6), then its Schwartz kernel is the pushforward of $\varphi(t)\chi(x)K_{e^{-t\mathbf{X}}}(x,y)\chi(y)$ under the map $(t,x,y)\mapsto(x,y)$, therefore [HöI, Example 8.2.5 and Theorem 8.2.13]

$$\operatorname{WF}'(\mathbf{A}_\varphi) \subset \{x=\varphi^t(y),\ \xi = \mathcal{P}_{y,t}\cdot\eta,\ \eta\in\mathcal{E}_0(y),\ t\in\operatorname{supp}\varphi,\ x,y\in\operatorname{supp}\chi\}.$$

By (5.3), $\mathbf{A}_\varphi$ satisfies (5.5) and thus the left-hand side of (5.6) is well-defined. See [DyZw13, Appendix B] for a detailed proof of (5.6), which generalizes directly to our situation. Note that the Poincaré map defined in [DyZw13, (B.1)] is the transpose of the one used in this paper, which does not change the determinant (5.3).

As in [DyZw13, §4], using (5.6), Lemma 2.16, and the fact that the right-hand side is well-defined by the wavefront set condition (see below), we get for some $C_1 > 0$,

$$F_{\mathbf{X}}(\lambda) = \operatorname{tr}^\flat\left(\chi e^{-t_0(\mathbf{X}+\lambda)}\mathbf{R}(\lambda)\chi\right), \quad \operatorname{Re}\lambda > C_1 \tag{5.7}$$

where $t_0 > 0$ is small enough so that $t_0 < T_\gamma$ for all γ. We also make t_0 small enough so that $\varphi^{-t_0}(\operatorname{supp}\chi) \subset \mathcal{U}$; then $\chi e^{-t_0(\mathbf{X}+\lambda)}\mathbf{R}(\lambda)\chi$ is a well-defined compactly supported operator on $\mathcal{U}$.

Note that by (1.8) and by [HöI, Example 8.2.5], the wavefront set of the operator $e^{-t_0\mathbf{X}}$ is contained in the graph of $e^{t_0 H_p}$. Then by (4.43) and multiplicativity of wavefront sets [HöI, Theorem 8.2.14], we have for each $\lambda \in \mathbb{C}$ which is not a resonance,

$$\begin{aligned}\operatorname{WF}'(\chi e^{-t_0(\mathbf{X}+\lambda)}\mathbf{R}(\lambda)\chi) \ \subset\ & \{(e^{t_0H_p}(y,\eta),y,\eta) \mid (y,\eta)\in T^*\mathcal{U}\setminus 0\} \ \cup\ (E_+^*\times E_-^*)\\ \cup\ & \{(e^{tH_p}(y,\eta),y,\eta) \mid (y,\eta)\in T^*\mathcal{U}\setminus 0,\ \eta\in\mathcal{E}_0(y),\ t\geq t_0\},\end{aligned}$$

and a similar statement is true for the regular and the singular parts of this operator when λ is a resonance – see Lemma 4.5. It follows from (2.9) and (5.3) that the operator $\chi e^{-t_0(\mathbf{X}+\lambda)}\mathbf{R}(\lambda)\chi$ satisfies (5.5); therefore, the right-hand side of (5.7) is defined as a meromorphic function of $\lambda\in\mathbb{C}$, and its poles are the resonances of $\mathbf{X}$.

It remains to show that for each resonance λ_0, the meromorphic continuation of $F_{\mathbf{X}}(\lambda)$ has a simple pole at λ_0 with residue equal to the rank of Π_{λ_0}. By (5.7) and recalling the expansion (1.13), it suffices to show that

$$\operatorname{tr}^\flat\sum_{j=1}^{J(\lambda_0)}(-1)^{j-1}\frac{\chi e^{-t_0(\mathbf{X}+\lambda)}(\mathbf{X}+\lambda_0)^{j-1}\Pi_{\lambda_0}\chi}{(\lambda-\lambda_0)^j} = \frac{\operatorname{rank}\Pi_{\lambda_0}}{\lambda-\lambda_0}+\operatorname{Hol}(\lambda)$$

where $\operatorname{Hol}(\lambda)$ stands for a function which is holomorphic near λ_0. Expanding $e^{-t_0(\mathbf{X}+\lambda)}$ at $\lambda=\lambda_0$, we see that it is enough to prove that

$$\begin{aligned}\operatorname{tr}^\flat(\chi e^{-t_0(\mathbf{X}+\lambda_0)}\Pi_{\lambda_0}\chi) &= \operatorname{rank}\Pi_{\lambda_0};\\ \operatorname{tr}^\flat(\chi e^{-t_0(\mathbf{X}+\lambda_0)}(\mathbf{X}+\lambda_0)^j\Pi_{\lambda_0}\chi) &= 0, \quad j\geq 1.\end{aligned} \tag{5.8}$$

By (1.14), each operator on the left-hand side can be written as a finite sum $\sum_\ell \mathbf{u}_\ell\otimes\mathbf{v}_\ell$, where $\otimes$ denotes the Hilbert tensor product and

$$\begin{aligned}\mathbf{u}_\ell &\in \mathcal{D}'(\mathcal{U};\mathcal{E}), \quad \operatorname{supp}\mathbf{u}_\ell\subset\Gamma_+, \quad \operatorname{WF}(\mathbf{u}_\ell)\subset E_+^*;\\ \mathbf{v}_\ell &\in \mathcal{D}'(\mathcal{U};\mathcal{E}^*\otimes|\Omega|^1), \quad \operatorname{supp}\mathbf{v}_\ell\subset\Gamma_-, \quad \operatorname{WF}(\mathbf{v}_\ell)\subset E_-^*.\end{aligned}$$

Since $\Gamma_+\cap\Gamma_- = K$ is compactly contained in $\mathcal{U}$ and $E_+^*\cap E_-^*\cap(T^*\mathcal{U}\setminus 0)=\emptyset$, by [HöI, Theorem 8.2.13] we can define the inner product $\langle\mathbf{u}_\ell,\mathbf{v}_{\ell'}\rangle$ for each ℓ,ℓ'. This implies

that the operators of the form $\sum_\ell \mathbf{u}_\ell \otimes \mathbf{v}_\ell$ can be multiplied (and form an algebra), they satisfy (5.5), and their flat traces are given by $\sum_\ell \langle \mathbf{u}_\ell, \mathbf{v}_\ell \rangle$.

Recall the spaces $\mathrm{Res}^{(k)} := \mathrm{Res}^{(k)}_{\mathbf{X}}(\lambda_0)$ defined in (1.12). By Theorem 2, and since $\chi = 1$ near K, the operators

$$\chi(e^{-t_0(\mathbf{X}+\lambda_0)}\Pi_{\lambda_0} - \Pi_{\lambda_0})\chi, \quad \chi e^{-t_0(\mathbf{X}+\lambda_0)}(\mathbf{X}+\lambda_0)^j \Pi_{\lambda_0}\chi, \quad j \geq 1 \tag{5.9}$$

are nilpotent; more precisely, they map for $1 < k \leq J(\lambda_0)$,

$$C_0^\infty(\mathcal{U}) \to \chi\,\mathrm{Res}^{(J(\lambda_0))}, \quad \chi\,\mathrm{Res}^{(k)} \to \chi\,\mathrm{Res}^{(k-1)}, \quad \chi\,\mathrm{Res}^{(1)} \to 0. \tag{5.10}$$

(The propagation operator $e^{-t_0(\mathbf{X}+\lambda_0)}$ does not cause any trouble since $\Gamma_+ \cap \varphi^{-t_0}(\mathcal{U}) \subset \mathcal{U}$ by (1.2) and thus each element of $e^{-t_0(\mathbf{X}+\lambda)_0}\,\mathrm{Res}^{(k)}$ can be restricted to $\mathcal{U}$ to yield another element of $\mathrm{Res}^{(k)}$.) It follows from (5.10) that the operators in (5.9) have zero flat trace. Since $\Pi_{\lambda_0}^2 = \Pi_{\lambda_0}$ and $\chi = 1$ near K, we have $\mathrm{tr}^\flat(\chi\Pi_{\lambda_0}\chi) = \operatorname{rank}\Pi_{\lambda_0}$; (5.8) follows. □

5.2. **Meromorphic extension of zeta functions.** In this section, we prove Theorem 3. Using the Taylor series of $\log(1-x)$, we get for $\operatorname{Re}\lambda \gg 1$,

$$\log\zeta_V(\lambda) = -\sum_{\gamma^\sharp}\sum_{k=1}^\infty \frac{\exp(-kT_{\gamma^\sharp}(\lambda + V_{\gamma^\sharp}))}{k} = -\sum_\gamma \frac{T_\gamma^\sharp \exp(-T_\gamma(\lambda+V_\gamma))}{T_\gamma}$$

where the last sum is over all closed trajectories γ of φ^t, with periods $T_\gamma > 0$ and primitive periods $T_\gamma^\sharp$, and V_γ is defined in (1.16). It follows that for $\operatorname{Re}\lambda \gg 1$,

$$\frac{\zeta_V'(\lambda)}{\zeta_V(\lambda)} = \sum_\gamma T_\gamma^\sharp e^{-\lambda T_\gamma} e^{-T_\gamma V_\gamma}.$$

To reduce the right-hand side to an expression that can be handled by Theorem 4, we make the assumption that, for the Poincaré determinants defined in (5.2),

$$|\det(I - \mathcal{P}_\gamma)| = (-1)^\beta \det(I-\mathcal{P}_\gamma) \quad \text{with } \beta \text{ independent of } \gamma. \tag{5.11}$$

This condition holds when E_s is orientable, with $\beta = \dim E_s$, see [DyZw13, §2.2]. See [GLP, Appendix B] for methods which can be used to eliminate the orientability assumption.

Similarly to (5.2), for each $\ell = 0, \ldots, n-1$, with $n = \dim\mathcal{U}$, the trace

$$\mathrm{tr}\wedge^\ell \mathcal{P}_\gamma = \mathrm{tr}\wedge^\ell \mathcal{P}_{\gamma(t),T_\gamma}, \quad \wedge^\ell \mathcal{P}_{\gamma(t),T_\gamma} : \wedge^\ell \mathcal{E}_0(\gamma(t)) \to \wedge^\ell \mathcal{E}_0(\gamma(t))$$

does not depend on t; here $\wedge^\ell$ denotes ℓth antisymmetric power. Using the identity $\det(I-\mathcal{P}_\gamma) = \sum_{\ell=0}^{n-1}(-1)^\ell \mathrm{tr}\wedge^\ell\mathcal{P}_\gamma$, we get for $\operatorname{Re}\lambda \gg 1$,

$$\frac{\zeta_V'(\lambda)}{\zeta_V(\lambda)} = \sum_{\ell=0}^{n-1}(-1)^{\ell+\beta}F_\ell(\lambda), \quad F_\ell(\lambda) = \sum_\gamma \frac{T_\gamma^\sharp e^{-\lambda T_\gamma}e^{-T_\gamma V_\gamma}\,\mathrm{tr}\wedge^\ell\mathcal{P}_\gamma}{|\det(I-\mathcal{P}_\gamma)|}.$$

To show that $\zeta_V(\lambda)$ continues meromorphically to $\lambda \in \mathbb{C}$, it is enough to show that for each ℓ, the function $F_\ell(\lambda)$ continues meromorphically to $\lambda \in \mathbb{C}$ with simple poles and integer residues. This follows from Theorem 4, applied to the operator

$$\mathbf{X}_\ell := \mathcal{L}_X + V : C^\infty(\mathcal{U}; \wedge^\ell \mathcal{E}_0) \to C^\infty(\mathcal{U}; \wedge^\ell \mathcal{E}_0),$$

where $\mathcal{E}_0$ is defined in (5.1), $\wedge^\ell \mathcal{E}_0$ is embedded into the bundle Ω^ℓ of differential ℓ-forms on $\mathcal{U}$ as the kernel of the interior product operator ι_X, and $\mathcal{L}_X$ is the Lie derivative along X on Ω^ℓ, restricted to $\wedge^\ell \mathcal{E}_0$.

6. Examples

6.1. **A basic example.** We start with the following basic example:

$$\mathcal{U} = \{x_1^2 + x_2^2 < 1\} \times \mathbb{S}^1_{x_3} \subset \mathbb{R}^2 \times \mathbb{S}^1, \quad \mathcal{E} = \mathbb{C}, \quad X = \mathbf{X} = x_1 \partial_{x_1} - x_2 \partial_{x_2} + \partial_{x_3}.$$

It is straightforward to verify that assumptions (A1)–(A5) from the introduction are satisfied, with

$$\begin{gathered} \varphi^t(x_1, x_2, x_3) = (e^t x_1, e^{-t} x_2, x_3) \quad \text{if } e^{2t} x_1^2 + e^{-2t} x_2^2 \le 1; \\ \Gamma_+ = [-1,1]_{x_1} \times \{0\}_{x_2} \times \mathbb{S}^1_{x_3}, \ \Gamma_- = \{0\}_{x_1} \times [-1,1]_{x_2} \times \mathbb{S}^1_{x_3}, \ K = \{(0,0)\}_{x_1} \times \mathbb{S}^1_{x_3}; \\ E_u(0,0,x_3) = \mathbb{R}\partial_{x_1}, \quad E_s(0,0,x_3) = \mathbb{R}\partial_{x_2}, \end{gathered}$$

and the extended dual stable/unstable bundles from Lemma 2.10 given by

$$E_+^*(x_1, 0, x_3) = \mathbb{R} dx_2, \quad E_-^*(0, x_2, x_3) = \mathbb{R} dx_1.$$

Then $u \in \mathcal{D}'(\mathcal{U})$ satisfies $\operatorname{supp} u \subset \Gamma_+$ and $\operatorname{WF}(u) \subset E_+^*$ if and only if it has the form

$$u = \sum_{j=0}^N u_\ell(x_1, x_3) \partial_{x_2}^\ell \delta(x_2), \quad u_\ell \in C^\infty((-1,1)_{x_1} \times \mathbb{S}^1_{x_3}),$$

here the fact that u_ℓ are smooth follows from the wavefront set condition. A direct calculation shows that the space of resonant states $\operatorname{Res}_X^{(1)}(\lambda)$ defined in (1.12) is nontrivial if and only if

$$\lambda = \lambda_{\ell,k} = -1 - \ell + ik, \quad \ell \in \mathbb{N}_0, \quad k \in \mathbb{Z}$$

and the spaces $\operatorname{Res}_X^{(j)}(\lambda_{\ell,k})$ are the same for all $j \ge 1$ and spanned by

$$x_1^m \partial_{x_2}^{\ell - m} \delta(x_2) e^{-ikx_3}, \quad m = 0, \dots, \ell.$$

By Theorem 2, the resonances of X are exactly $\lambda_{\ell,k}$, with $\operatorname{rank} \Pi_{\lambda_{\ell,k}} = \ell + 1$. Another way to see the same fact is to apply Theorem 4 from §5.1, with

$$F_X(\lambda) = \pi \sum_{m \in \mathbb{N}} \frac{e^{-2\pi m \lambda}}{\cosh(2\pi m) - 1};$$

we use that $F_X(\lambda)$ is holomorphic in $\{\operatorname{Re}\lambda > -1\}$ and satisfies the functional equation

$$F_X(\lambda+1) + F_X(\lambda-1) - 2F_X(\lambda) = 2\pi \sum_{m\in\mathbb{N}} e^{-2\pi m\lambda} = \frac{2\pi}{e^{2\pi\lambda}-1}.$$

We remark that the assumptions from the introduction are satisfied for the vector fields $\pm(x_1\partial_{x_1} + x_2\partial_{x_2}) + \partial_{x_3}$; we leave the details to the reader.

6.2. **Riemannian manifolds with boundary.** Consider a smooth m-dimensional compact Riemannian manifold (M, g) with strictly convex boundary; that is, the second fundamental form at ∂M with respect to the inward pointing normal is positive definite. Let $\overline{\mathcal{U}} = SM$ be its unit tangent bundle, and consider the vector field X generating the geodesic flow on SM. One can equivalently consider the geodesic flow on the unit cotangent bundle S^*M, which is naturally a contact flow.

The vector field X satisfies assumptions (A1)–(A3) in the introduction. To see (1.1), choose a coordinate system x on M such that M locally has the form $\{x_1 \geq 0\}$. Let $x(t) \in M$ be a geodesic (on an extension of M past the boundary) such that $\dot{x}_1(t) = 0$. By the geodesic equation

$$\ddot{x}_1(t) + \sum_{i,j} \Gamma^1_{ij}(x(t))\dot{x}_i(t)\dot{x}_j(t) = 0$$

and since the matrix $\Gamma^1_{ij}(x(0))_{i,j=2}^m$ is positive definite by the strict convexity of the boundary, we get $\ddot{x}_1(t) < 0$ as required.

We assume that the flow $\varphi^t = e^{tX}$ is hyperbolic on the trapped set $K \subset \mathcal{U}$ in the sense of assumption (A4) in the introduction. This is in particular true if g has negative sectional curvature in a neighborhood of K, see for instance [Kl, §3.9 and Theorem 3.2.17].

We now discuss an application of the results of this paper to boundary problems for the geodesic flow. For each $(x, v) \in \mathcal{U}$, define

$$\ell_\pm(x, v) = \pm \sup\{t > 0 \mid \varphi^{\pm t}(x, v) \in \mathcal{U}\} \in [-\infty, \infty]$$

as the time of escape to ∂M in forward $(+)$ and backward $(-)$ time. Note that

$$\Gamma_\pm \cap \mathcal{U} = \{(x, v) \mid \ell_\mp(x, v) = \mp\infty\}, \quad K = \{(x, v) \mid \ell_+(x, v) = +\infty,\ \ell_-(x, v) = -\infty\}.$$

Define the incoming $(-)$, outgoing $(+)$, and tangent (0) boundary by

$$\partial_\pm SM := \{(x, v) \in \partial SM \mid \mp\langle d\rho, v\rangle > 0\}, \quad \partial_0 SM := \{(x, v) \in \partial SM \mid \langle d\rho, v\rangle = 0\},$$

where ρ is a defining function of the boundary in M. The Liouville measure $d\mu$ on SM is invariant by the flow and it is straightforward to check that the boundary value problem

$$(-X \pm \lambda)u_\pm = f \in L^2_{\text{comp}}(\mathcal{U}), \quad u_\pm = 0 \quad \text{near } \partial_\pm SM, \quad u_\pm \in L^2(SM), \tag{6.1}$$

is uniquely solvable for $\operatorname{Re}\lambda > 0$, and the solution is given by

$$u_\pm(x,v) = \int_0^{\ell_\pm(x,v)} e^{-\lambda|t|} f(\varphi^t(x,v))dt. \tag{6.2}$$

This defines a bounded map $R_\pm(\lambda) : L^2(\mathcal{U}) \to L^2(\mathcal{U})$ by putting $R_\pm(\lambda)f := u_\pm$.

Proposition 6.1. *Assume that (M,g) is a compact Riemannian manifold with strictly convex boundary and hyperbolic trapped set, and $\varphi^t : \mathcal{U} \to \mathcal{U}$, $\overline{\mathcal{U}} = SM$, be the geodesic flow. Let $E_\mp^* \subset T^*_{\Gamma_\mp}\overline{\mathcal{U}}$ be defined by Lemma 2.10. Then:*

1. The operators $R_\pm(\lambda)$ have meromorphic continuation to $\lambda \in \mathbb{C}$ as operators $R_\pm(\lambda) : C_0^\infty(\mathcal{U}) \to \mathcal{D}'(\mathcal{U})$, with poles of finite rank.

2. Assume that $\lambda \in \mathbb{C}$ is not a pole of $R_\pm$. Then for each $f \in C_0^\infty(\mathcal{U})$, $u_\pm = R_\pm(\lambda)f$ is the unique solution in $\mathcal{D}'(\mathcal{U})$ to the problem

$$(-X \pm \lambda)u_\pm = f, \quad u_\pm = 0 \text{ near } \partial_\pm SM \cup \partial_0 SM, \quad \operatorname{WF}(u_\pm) \subset E_\mp^*. \tag{6.3}$$

Moreover, $R_\pm(\lambda)$ acts $H_0^s(\mathcal{U}) \to H^{-s}(\mathcal{U})$ for all $s > \gamma^{-1}\max(0, -\operatorname{Re}\lambda)$, where $\gamma > 0$ is the constant in (1.5). Finally, there exist conic neighborhoods $U_\pm$ of $E_\pm^*$ such that for each compactly supported $A_\pm \in \Psi^0(\mathcal{U})$ with $\operatorname{WF}'(A_\pm) \subset U_\pm$, the operators $A_\pm R_\pm(\lambda)$ act $H_0^s(\mathcal{U}) \to H^s(\mathcal{U})$.

3. Assume that $\lambda \in \mathbb{C}$ is a pole of $R_\pm$. Then there exists a nonzero solution $u_\pm \in \mathcal{D}'(\mathcal{U})$ to the problem (6.3) with $f = 0$; in fact, $\operatorname{supp} u_\pm \subset \Gamma_\mp$.

Proof. We establish the properties of $R_-(\lambda)$; the properties of $R_+(\lambda)$ are obtained by flipping the sign of X.

1. Put $\mathcal{E} := \mathbb{C}$, $\mathbf{X} := X$ in assumption (A5) in the introduction. Comparing (6.2) with (1.10), we see that $R_-(\lambda)f = -\mathbf{R}(\lambda)f$ for all $f \in C_0^\infty(\mathcal{U})$ and $\operatorname{Re}\lambda > 0$. It remains to apply Theorem 1.

2. The fact that $R_-(\lambda)f$ is a solution to (6.3) follows by analytic continuation from (6.1); to see that $\operatorname{WF}(R_-(\lambda)f) \subset E_+^*$, we use (4.43). To see uniqueness, assume that $u_- \in \mathcal{D}'(\mathcal{U})$ solves (6.3) with $f = 0$. Using the equation $(X+\lambda)u_- = 0$ and the fact that u_- vanishes near $\partial_- SM \cup \partial_0 SM$, we see that $\operatorname{supp} u_- \subset \Gamma_+$. Then $u_- = 0$ by Theorem 2.

To see that $R_-(\lambda) : H_0^s(\mathcal{M}) \to H^{-s}(\mathcal{M})$ is bounded, we use Lemma 4.3 and Lemma 4.4, together with the properties of anisotropic spaces $\mathcal{H}_h^r$ given in (4.5), and the discussion on the admissible values of r in Remark (iii) following Lemma 4.2. In the latter step we use the fact that the Liouville measure is invariant under the flow. By (4.7), this also implies that $A_- R_-(\lambda)$ acts $H_0^s(\mathcal{U}) \to H^s(\mathcal{U})$.

3. This is a restatement of the characterization of resonant states in Theorem 2. □

6.3. **Complete Riemannian manifolds.** Another example which fits to our setting, which reduces to the one discussed in §6.2, is the case of a complete Riemannian manifold (M, g) satisfying:

(1) there exists a function $F \in C^\infty(M; \mathbb{R})$ such that for each $a \geq 0$, $M_a := \{F \leq a\}$ is a compact domain whose boundary $\{F = a\}$ is smooth and strictly convex with respect to g;
(2) the trapped set K of the geodesic flow $\varphi^t : SM \to SM$ is hyperbolic in the sense of assumption (A4) in the introduction.

A particular case of such manifold is given by negatively curved complete Riemannian manifolds (M, g) which admit a compact region M_0 with strictly convex smooth boundary ∂M_0 such that, if ν is the unit normal exterior pointing vector field to ∂M_0 and $\pi : SM \to M$ the projection on the base, the map

$$\psi : [0, \infty) \times \partial M_0 \to M \setminus M_0^\circ, \quad \psi(t, x) = \pi(\varphi^t(x, \nu(x)))$$

is a smooth diffeomorphism. In the coordinates (t, x) defined by ψ, the metric has the form $dt^2 + g_1(t, x, dx)$, therefore the function t is the geodesic distance to ∂M_0. Thus lifting everything to the universal cover and applying Theorem 4.1 (see in particular Remark 4.3 there; we use that M is negatively curved) in [BiO'N], we have that $F := t$ produces a strictly convex foliation, verifying assumption (1). Assumption (2) follows from [Kl, §3.9 and Theorem 3.2.17]. An asymptotically hyperbolic manifold in the sense of Mazzeo–Melrose [MaMe] with negative curvature satisfies these properties, and thus in particular any convex co-compact hyperbolic manifold (with constant negative curvature) does too.

We define the incoming/outgoing tails $\Gamma_\pm \subset SM$ on the entire M by

$$(x, v) \notin \Gamma_\pm \implies \varphi^t(x, v) \to \infty \quad \text{as } t \to \mp\infty.$$

Note that $K = \Gamma_+ \cap \Gamma_-$ is contained in $\{F \leq 0\}$. By Lemma 2.10, we define the vector bundles $E_\pm^*$ over $\Gamma_\pm$. For $f \in C_0^\infty(SM)$, define

$$R(\lambda) f = \int_0^\infty e^{-\lambda t} (f \circ \varphi^{-t})\, dt, \quad \operatorname{Re} \lambda > 0.$$

Proposition 6.2. *Under assumptions (1) and (2) above, the operator $R(\lambda)$ admits a meromorphic extension to $\lambda \in \mathbb{C}$ as an operator*

$$R(\lambda) : C_0^\infty(SM) \to \mathcal{D}'(SM)$$

with poles of finite multiplicity. Moreover, $\lambda \in \mathbb{C}$ is a pole of $R(\lambda)$ (that is, a Pollicott–Ruelle resonance) if and only if there exists a non-zero $u \in \mathcal{D}'(SM)$ satisfying

$$(X + \lambda)u = 0, \quad \operatorname{supp} u \subset \Gamma_+, \quad \operatorname{WF}(u) \subset E_+^*. \tag{6.4}$$

Proof. Applying Proposition 6.1 to the manifolds $(M_a, g|_{M_a})$, we continue meromorphically $R_a(\lambda) = \mathbb{1}_{SM_a} R(\lambda) \mathbb{1}_{SM_a}$ for all $a \geq 0$. By analytic continuation, we have $\mathbb{1}_{SM_a} R_b(\lambda) \mathbb{1}_{SM_a} = R_a(\lambda)$ for $0 \leq a \leq b$. To show that the family of operators $R_a(\lambda)$ can be pieced together to an operator $R(\lambda) : C_0^\infty(SM) \to \mathcal{D}'(SM)$, it suffices to show that each λ has the same multiplicity (i.e., the rank of the operator Π_λ from Theorem 2) as a resonance of $R_a(\lambda)$ and $R_b(\lambda)$, for $0 \leq a \leq b$. By Theorem 2, it is then enough to show that for each $j \geq 1$, the restriction operator

$$\begin{aligned} \mathbb{1}_{SM_a} : \{u \in \mathcal{D}'(SM_b) \mid (X+\lambda)^j u = 0,\ \operatorname{supp} u \subset \Gamma_+,\ \operatorname{WF}(u) \subset E_+^*\} \\ \to \{u \in \mathcal{D}'(SM_a) \mid (X+\lambda)^j u = 0,\ \operatorname{supp} u \subset \Gamma_+,\ \operatorname{WF}(u) \subset E_+^*\} \end{aligned} \tag{6.5}$$

is a linear isomorphism. (The case $j = 1$ also gives the characterization (6.4).) The fact that (6.5) is an isomorphism follows by using the equation $(X+\lambda)^j u = 0$ together with the fact that $\varphi^{-t}(SM_b \cap \Gamma_+) \subset SM_a$ for t large enough; the latter is a corollary of Lemma 2.3. □

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