Resonances for r-normally hyperbolic trapped sets

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- Motivated by decay of linear waves on black holes, we study resonances for r-normally hyperbolic trapped sets

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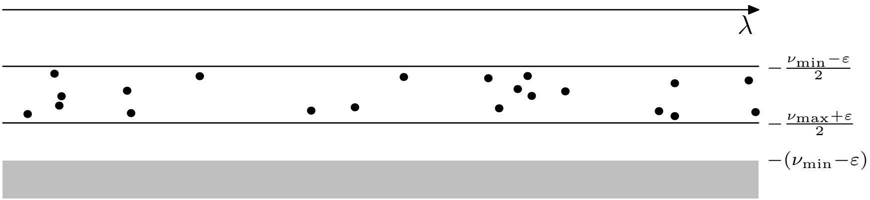

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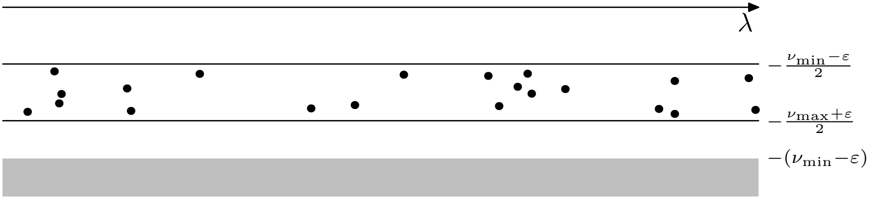

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- Our result relies on the construction of a microlocal projector corresponding to resonances in the band, a Fourier integral operator
- We also get new information on microlocalization of resonant states

Motivation: ringdown on black holes

Chandrasekhar '83:

...we may expect that any initial perturbation will, during its last stages, decay in a manner characteristic of the black hole itself and independent of the cause. In other words, we may expect that during these last stages, the black hole emits gravitational waves with frequencies and rates of damping that are characteristic of the black hole itself, in the manner of a bell sounding its last dying notes.

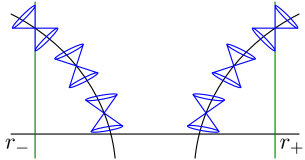
Motivation

Linear waves on (slowly) rotating Kerr–de Sitter black holes $(\widetilde{M}, \tilde{g})$

$$\Box_{\tilde{g}} u = f \in C_0^\infty, \quad u|_{t<0} = 0.$$

Open system: most energy escapes through the event horizons.

What is the behavior of $u(t)$ as $t \to +\infty$?



Two radial timelike geodesics, with light cones shown

Theorem [D '10, '11]

There is a discrete set of resonances $\{\lambda_j\} \subset \{\operatorname{Im} \lambda \le 0\}$ such that $\forall \nu > 0$, the following resonance expansion* holds

$$u(t, x) = \sum_{\operatorname{Im} \lambda_j > -\nu} e^{-it\lambda_j} \Pi_j f(x) + \mathcal{O}(e^{-\nu t}).$$

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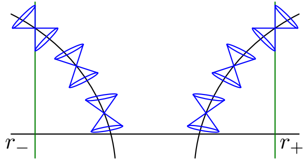

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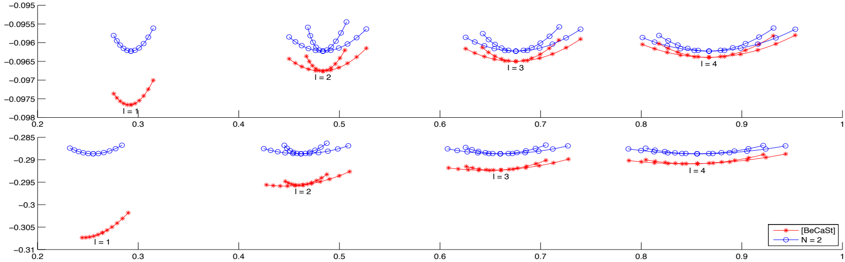
Moreover, resonances satisfy a quantization condition, that is they lie asymptotically on a distorted lattice.

The quantization condition agrees with the exact values computed in Berti–Cardoso–Starinets '09 (shown here for $a = 0, 0.05, \ldots, 0.25$)



Previous work: Sá Barreto–Zworski '97, Bony–Häfner '07, Dafermos–Rodnianski '07, Melrose–Sá Barreto–Vasy '08, Wunsch–Zworski '10, Vasy '10... + numerous results for $\Lambda = 0$

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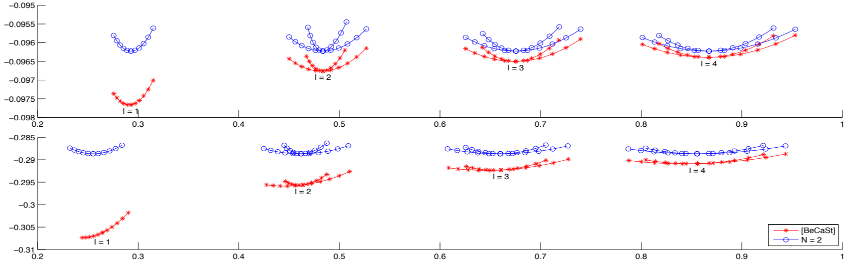

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Perturbations destroy symmetries and our ability to handle individual resonances (similarly to the compact case).

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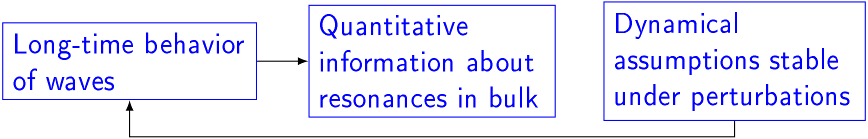


Setup

The structure of open ends/event horizons does not change our result (Vasy '10 for event horizons of K–dS) $\Longrightarrow$ use a simpler model at infinity:

$$(\widetilde{M}, \tilde{g}) = (M_x, g) \times \mathbb{R}_t, \quad (M_x, g) \simeq \mathbb{R}^n \text{ outside of a compact set}$$

Define resonances as poles of the meromorphic continuation of the resolvent of the Laplacian $\Delta \geq 0$:

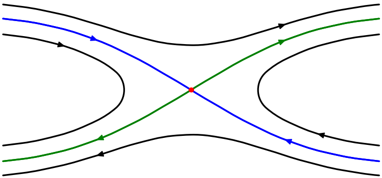
$$R(\lambda) = (\Delta - \lambda^2)^{-1} : \begin{cases} L^2(M) \to L^2(M), & \operatorname{Im} \lambda > 0; \\ L^2_{\mathrm{comp}}(M) \to L^2_{\mathrm{loc}}(M), & \operatorname{Im} \lambda \leq 0. \end{cases}$$

Long-living resonances: $\operatorname{Re} \lambda \to \infty$, $|\operatorname{Im} \lambda| \leq C$. Depend on trapping for the geodesic flow e^{tH_p}, $p(x, \xi) = |\xi|_g$:

$$\Gamma_\pm = \{\rho \in T^*M \mid e^{tH_p}(\rho) \not\to \infty \text{ as } t \to \mp\infty\}, \quad K = \Gamma_+ \cap \Gamma_-.$$

Dynamical assumptions

- $\Gamma_\pm$ are C^r hypersurfaces ($r \gg 1$)
- $K = \Gamma_+ \pitchfork \Gamma_-$ is symplectic
- $\nu_{\min} \leq \nu_{\max}$ are the expansion rates of e^{tH_p} along $\Gamma_\pm$ transversally to K
- $\mu_{\max} \geq 0$ is the maximal expansion rate of e^{tH_p} along K
- r-normal hyperbolicity: $\nu_{\min} > r \cdot \mu_{\max}$
- pinching: $\nu_{\min} > \frac{1}{2}\nu_{\max}$



Normally hyperbolic trapping, with Γ_+, Γ_-, and K.

- Stable under small smooth perturbations: Hirsch–Pugh–Shub '77
- True for slowly rotating K–dS (for S–dS, K is the photonsphere)
- More general normally hyperbolic trapping ($r = 0$) appears for Ruelle resonances for Anosov flows [Faure–Sjöstrand '11] and quantum chemistry [Goussev–Schubert–Waalkens–Wiggins '10]

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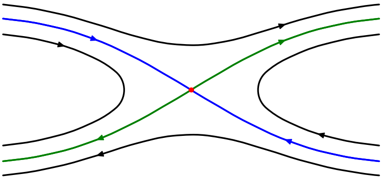

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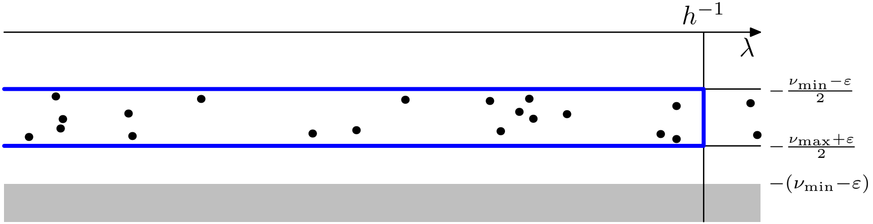
Main theorem [D '13]

For each fixed $\varepsilon > 0$, there exist two resonance free strips

$$\{\operatorname{Re}\lambda \gg 1, \operatorname{Im}\lambda \in [-(\nu_{\min} - \varepsilon), -\tfrac{1}{2}(\nu_{\max} + \varepsilon)] \cup [-\tfrac{1}{2}(\nu_{\min} - \varepsilon), \infty)\}$$

and in these strips, $\|\chi R(\lambda)\chi\|_{L^2 \to L^2} = \mathcal{O}(1)$ for $\chi \in C_0^\infty(M)$.
Moreover, resonances between the two strips satisfy a Weyl law as $h \to 0$:

$$\#\{0 \le \operatorname{Re}\lambda \le h^{-1}, \operatorname{Im}\lambda > -\tfrac{1}{2}(\nu_{\max} + \varepsilon)\} = \frac{\operatorname{Vol}_\sigma(K \cap B^*M)}{(2\pi h)^{n-1}} + o(h^{1-n}).$$



What about exact Kerr–de Sitter?

- $M > 0$ mass of the black hole
- a speed of rotation, $|a| < M$
- $\Lambda > 0$ cosmological constant, $9\Lambda M^2 < 1$

The metric is r-normally hyperbolic for $\Lambda \ll 1$ or $|a| \ll 1$: $\nu_{\min} > 0$, $\mu_{\max} = 0$.

However, the pinching condition $\nu_{\min} > \frac{1}{2}\nu_{\max}$ breaks down for $a = M - \varepsilon$, $\varepsilon \ll 1$: there exist two closed trajectories (equators) $E_\pm \subset K$ such that the local expansion rate ν satisfies as $\varepsilon \to 0$

$$\nu \sim \frac{3\sqrt{3}}{28M} \quad \text{on } E_+, \quad \nu \sim \frac{\sqrt{\varepsilon/2M}}{M} \quad \text{on } E_-.$$

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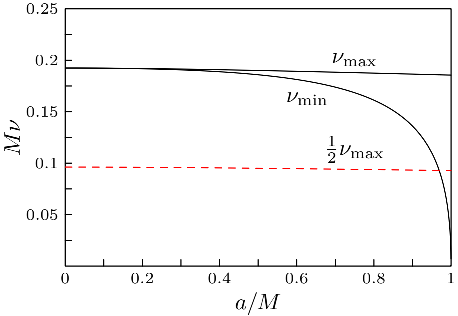

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Previous work

- Resonant free strips for NH trapped sets: Gérard–Sjöstrand '88, Dolgopyat '98, Liverani '04, Tsujii '10, Wunsch–Zworski '11, Nonnenmacher–Zworski '13. Our gap $\{\operatorname{Im}\lambda > -\frac{1}{2}(\nu_{\min} - \varepsilon)\}$ coincides with the one of N–Z '13 (valid in more general situations)
- Our Weyl law saturates general upper bounds in strips: Sjöstrand '90, Guillopé–Lin–Zworski '04, Sjöstrand–Zworski '07, Nonnenmacher –Sjöstrand–Zworski '11,'12, Datchev–D '12, Datchev–D–Zworski '12.
- Completely integrable cases: Gérard–Sjöstrand '87, Christianson '07, Sá Barreto–Zworski '97, D '11
- Bands of resonances/Weyl laws: Zworski '87, Sjöstrand–Vodev '97, Sjöstrand–Zworski '99, Sjöstrand '00,'11, Faure–Tsujii '12,'13

F–T '13 shares some ideas with presented work. It applies in settings with lower regularity, but does not recover microlocal structure away from K.

Ingredients of the proof

- Fourier integral operator Π projecting microlocally onto resonant states for the band of resonances
- Left/right ideals of pseudodifferential operators annihilating Π
- Local reduction to Taylor expansion
- Positive commutator estimates (replaced in this talk by wave propagation estimates)
- Grushin problems: resonances as zeros of a Fredholm determinant
- Trace formulas above and below the band of resonances, using microlocal analysis in λ
- Complex analysis (almost analytic continuation and argument principle) to obtain the Weyl law

Model case

$$M = \mathbb{R}, \quad P_0 = xD_x + \frac{1}{2i}, \quad U_0(t)f(x) = e^{-t/2}f(e^{-t}x)$$
$$p(x,\xi) = x\xi, \quad \nu_{\min} = \nu_{\max} = 1$$

The Taylor expansion

$$f(e^{-t}x) = f(0) + \mathcal{O}(e^{-t})_{L^2_{\mathrm{loc}}}$$

can be viewed as a resonance expansion:

$$U_0(t)f = U_0(t)\Pi_0 f + \mathcal{O}(e^{-3t/2}),$$
$$\Pi_0 f(x) := f(0),$$
$$U_0(t)\Pi_0 f = e^{-t/2}\Pi_0 f$$

A resonance at $\lambda = -\frac{i}{2}$ and $\Pi_0 = \delta_0 \otimes \mathbf{1}$ is the projector onto the corresponding resonant state.

General case

$$P = \sqrt{\Delta}, \quad U(t) = e^{-itP}$$

"Projector":

$$\Pi : C_0^\infty(M) \to C^\infty(M), \quad \text{Fourier integral operator}$$

"Resonance expansion": for $f \in C_0^\infty(M)$ living at frequencies $\sim h^{-1}$ and $\chi \in C_0^\infty(M)$, $\chi = 1$ near K, modulo $\mathcal{O}(h^\infty)$ errors,

$$\|\chi U(t)(1 - \Pi) f\|_{L^2} = \mathcal{O}(h^{-1} e^{-(\nu_{\min} - \varepsilon)t}) \|f\|_{L^2},$$

$$e^{-\frac{\nu_{\max} + \varepsilon}{2} t} \|\chi \Pi f\|_{L^2} \lesssim \|\chi U(t) \Pi f\|_{L^2} \lesssim e^{-\frac{\nu_{\min} - \varepsilon}{2} t} \|\chi \Pi f\|_{L^2}$$

This is enough for two resonance free strips
+ $u = \Pi u + \mathcal{O}(h^\infty)$ for resonant states in the band.

Also true for Kerr black holes, using Vasy–Zworski '00 (note: high frequency régime!)

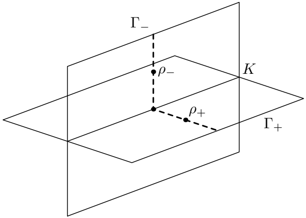
The projector Π

Construct unique h-Fourier integral operator Π, associated to a canonical relation $\Lambda \subset T^*M \times T^*M$, such that:

$$\Pi^2 = \Pi + \mathcal{O}(h^\infty), \qquad (1)$$

$$[\Delta, \Pi] = \mathcal{O}(h^\infty), \qquad (2)$$

microlocally near $K \cap S^*M$.



The canonical relation Λ (flow lines of $\mathcal{V}_\pm$ dashed).

If $\mathcal{V}_\pm \subset T\Gamma_\pm$ are the symplectic complements of $T\Gamma_\pm$ in $T(T^*M)$ and $\pi_\pm : \Gamma_\pm \to K$ are the projections along the flow lines of $\mathcal{V}_\pm$, then

$$\Lambda = \{(\rho_-, \rho_+) \mid \rho_\pm \in \Gamma_\pm,\ \pi_-(\rho_-) = \pi_+(\rho_+)\}.$$

Transport equations

To construct Π, we need to solve transport equations of the form

$$H_p a = f,\ \ a \in C^\infty(\Gamma_\pm),\ \ a|_K = 0$$

where $f \in C^\infty(\Gamma_\pm)$, $f|_K = 0$.
The unique solution is given by the integral

$$a = \pm \int_0^\infty f \circ e^{\mp t H_p}\, dt.$$

The integral converges: $|f \circ e^{\mp t H_p}| = \mathcal{O}(e^{-(\nu_{\min} - \varepsilon)t})$, but is a smooth?
Differentiate along K:

$$\partial^k a = \pm \int_0^\infty \partial^k f \circ e^{\mp t H_p} \cdot \partial^k e^{\mp t H_p}\, dt.$$

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Therefore, r-normal hyperbolicity guarantees that $a \in C^r$.

$$P=\sqrt{\Delta},\quad U(t)=e^{-itP},\quad \Pi\in I(\Lambda);$$
$$U_0(t)f(x)=e^{-t/2}f(e^{-t}x),\quad \Pi_0 f(x)=f(0).$$

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$$\|\chi U(t)(1-\Pi)f\|_{L^2}=\mathcal{O}(h^{-1}e^{-(\nu_{\min}-\varepsilon)t})\|f\|_{L^2},\tag{3}$$
$$e^{-\frac{\nu_{\max}+\varepsilon}{2}t}\|\chi\Pi f\|_{L^2}\lesssim\|\chi U(t)\Pi f\|_{L^2}\lesssim e^{-\frac{\nu_{\min}-\varepsilon}{2}t}\|\chi\Pi f\|_{L^2}.\tag{4}$$

To prove (3) for the model case, write

$$(1-\Pi_0)f = xf_1, \quad f_1(x) = \frac{f(x)-f(0)}{x}, \quad \|f_1\|_{L^2} \lesssim \|f\|_{H^1};$$
$$\chi U_0(t)(1-\Pi_0)f = \chi U_0(t)xU_0(-t)U_0(t)f_1, \quad U_0(t)xU_0(-t) = e^{-t}x.$$

To prove (4) for the model case, note that $hD_x\Pi f = 0$ and thus $\langle a(x)\Pi f, \Pi f\rangle$ depends only on $\int a(x)\,dx$. Then

$$\|\chi U_0(t)\Pi f\|_{L^2}^2 = \langle \chi_t^2(x)\Pi f, \Pi f\rangle, \quad \chi_t = U_0(-t)\chi U_0(t),$$

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The general case is handled using the same ideas, but replacing x and hD_x by pseudodifferential operators Θ_- and Θ_+ solving

$$\Pi\Theta_- = \mathcal{O}(h^\infty), \quad \Theta_+\Pi = \mathcal{O}(h^\infty). \tag{5}$$

Such $\Theta_\pm$ are not unique, but the sets of solutions to the equations form one-sided ideals invariant under the propagator $U(t)$.

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Open problems

- Show existence of further gaps and bands, below the first band of resonances. The main question is how to construct Fourier integral operators $\Pi_1, \Pi_2, \dots$ projecting onto resonant states for each band.
- Obtain a quantization condition if the flow is completely integrable on the trapped set, recovering Gérard–Sjöstrand '87, Sá Barreto–Zworski '97, and D '11.
- Get an optimal remainder $\mathcal{O}(h^{2-n})$ in the Weyl law (we prove $o(h^{1-n})$ and $\mathcal{O}(h^{2-n-})$ should not be much harder), perhaps using the work of Sjöstrand '00 on the damped wave equation.

Thank you for your attention!