QUANTUM ERGODICITY OF EISENSTEIN FUNCTIONS AT COMPLEX ENERGIES

SEMYON DYATLOV

ABSTRACT. We consider a surface M with constant curvature cusp ends and its Eisenstein functions $E_j(\lambda; z)$, $z \in M$. These are the plane waves associated to the jth cusp and the spectral parameter λ, $(\Delta - 1/4 - \lambda^2)E_j = 0$. We prove quantum unique ergodicity (QUE) of E_j's for $\operatorname{Re}\lambda \to \infty$ and $\operatorname{Im}\lambda \to \nu > 0$; the limiting measure is naturally defined and decays exponentially along the geodesic flow. In particular, taking a sequence of λ's corresponding to scattering resonances, we obtain QUE of resonant states with energies away from the real line. As an application, we also show that the scattering matrix tends to 0 in strips separated from the real line.

1. INTRODUCTION

Concentration of eigenfunctions of the Laplacian in phase space dates back to the papers of Schnirelman [Sch], Colin de Verdière [CdV], and Zelditch [Ze1]. Their quantum ergodicity (QE) result states that on a compact Riemannian manifold without boundary whose geodesic flow is ergodic with respect to the Liouville measure, a density one subsequence of eigenfunctions microlocally converges to this measure. In particular this result applies to compact hyperbolic surfaces. The papers [Ja, Li, LuSa, So, Ze2] studied the question for finite area hyperbolic surfaces, that is hyperbolic quotients with cusps. In particular, [Ze2] established QE for any such surface, if embedded eigenfunctions are augmented with *Eisenstein functions* on the unitarity axis; the latter parametrize continuous spectrum of the Laplacian arising from the presence of cusps. For the modular surface one has a stronger statement of quantum unique ergodicity (QUE): any sequence of Hecke–Maass forms [Li, So] or Eisenstein series on the unitarity axis [LuSa, Ja] converges to the Liouville measure. Guillarmou and Naud [GuiNa] have recently studied equidistribution of Eisenstein functions for convex co-compact hyperbolic manifolds; that is, in the presence of funnels, but not cusps. See [No, Sa, Ze3] for recent reviews of various other results.

The present paper considers an arbitrary surface with cusps and phase space concentration of Eisenstein functions for energies in the upper half-plane, away from the real line. Without utilising global properties of the geodesic flow, such as hyperbolicity or ergodicity, we establish a form of QUE in that case — see Theorem 1.

Our motivation comes from the natural question of quantum ergodicity of resonant states. These replace eigenfunctions on non-compact manifolds, but there are no rigorous results about their equidistribution in phase space. See however interesting physics papers by Keating et al. [KeNoNoSi] and Nonnenmacher–Rubin [NoRu].

In the case of manifolds with cusps and ergodic geodesic flows, resonant states close to the real axis are believed to satisfy QE; that is, most of them should converge to the Liouville measure in the sense of Definition 1.1. As stated in Theorem 3, QUE for Eisenstein series away from the real line yields a QUE result for resonant states with complex energies at a fixed distance from the real line. Although this does not answer the QE question for the more interesting case $\operatorname{Im}\lambda \to 0$, it seems to be the first result on quantum ergodicity for resonant states.

We proceed to a rigorous formulation of the results. Let (M, g) be a two-dimensional complete Riemannian manifold with cusp ends; that is, M is the union of a compact set and finitely many *cusp regions* $\mathcal{C}_1, \dots, \mathcal{C}_m$, where each $\mathcal{C}_j$ posesses a system of *canonical coordinates*

$$(r, \theta) \in (R_j, \infty) \times \mathbb{S}^1,\ \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z}),$$

with R_j some constant, such that the metric g on $\mathcal{C}_j$ has the form

$$g = dr^2 + e^{-2r} d\theta^2. \tag{1.1}$$

(To avoid confusing canonical coordinate systems for different cusps, we will always specify which cusp region we are considering.) A classical example of such M is a finite area hyperbolic surface without conic points. In fact, the present paper applies to surfaces with conic points as well, as one can get rid of these by passing to a finite covering space. The metric (1.1) has constant curvature -1; we could consider cusps of different constant curvatures, or even a slightly more general class of real analytic metrics on the cusps; we restrict our attention to (1.1) for simplicity.

Let Δ be the (nonnegative) Laplace–Beltrami operator corresponding to the metric g; this operator is self-adjoint, its spectrum is contained in $[0, \infty)$, and the spectrum in $[0, 1/4)$ consists of finitely many eigenvalues [Mü, Section 1]. The Eisenstein functions

$$E_j(\lambda; z),\ j = 1, \dots, m,\ z \in M,\ \operatorname{Im}\lambda > 0,\ \lambda \notin (0, i/2]$$

are unique solutions to the equation

$$(\Delta - 1/4 - \lambda^2)u = 0,\ u \in C^\infty(M), \tag{1.2}$$

that satisfy

$$u \in L^2(X \setminus \mathcal{C}_j),\ u(r, \theta) - e^{(1/2 - i\lambda)r} \in L^2(\mathcal{C}_j). \tag{1.3}$$

(See Section 3 for details.) To define $L^2(M)$, we use the volume form Vol induced by g. We study concentration of $E_j(\lambda)$ in the phase space as $\operatorname{Re}\lambda \to \infty$ in the upper half-plane using semiclassical measures. The definition needs to be more complicated than in the case of compact manifolds as E_j's do not lie in $L^2(M)$:

Definition 1.1. *Assume that the sequence λ_n satisfies*[1]

$$\operatorname{Re}\lambda_n \to +\infty,\ \operatorname{Im}\lambda_n \to \nu > 0 \ \text{as}\ n \to \infty. \tag{1.4}$$

Put

$$h_n = (\operatorname{Re}\lambda_n)^{-1},\ \omega_n = i\operatorname{Im}\lambda_n, \tag{1.5}$$

so that $h_n\lambda_n = 1+h_n\omega_n$, $h_n \to 0$, $\omega_n \to i\nu$ as $n \to \infty$. We say that $E_j(\lambda_n)$ converges to a Radon measure μ on T^*M, which is then called a semiclassical measure associated to *$\mathcal{C}_j$ and the constant ν, if for each $\chi \in C_0^\infty(M)$ and each semiclassical pseudodifferential operator $A(h) \in \Psi^0(M)$ with principal symbol $a \in C^\infty(T^*M)$,*

$$\langle A(h_n)\chi E_j(\lambda_n), \chi E_j(\lambda_n)\rangle_{L^2(M)} \to \int_{T^*M} a\,\chi^2 d\mu.$$

In particular, we can take as $A(h)$ the multiplication operator by $a(z) \in C_0^\infty(M)$, in which case we get

$$\int a(z)|E_j(\lambda_n)|^2\,d\operatorname{Vol} \to \int_{T^*M} a(z)\,d\mu.$$

In other words, the measure $|E_j(\lambda_n)|^2\,d\operatorname{Vol}$ converges weakly to the pullback of μ under the projection $\pi : T^*M \to M$.

Basic properties of semiclassical measures can be proved using the calculus of semiclassical pseudodifferential operators. By (1.2),

$$P(h_n;\omega_n)E_j(\lambda_n) = 0, \tag{1.6}$$

where $P(h;\omega)$ is the following second order semiclassical differential operator:

$$P(h;\omega) = h^2\Delta - h^2/4 - (1 + h\omega)^2.$$

The principal symbol of P is given by

$$p(z,\zeta) = g_z^{-1}(\zeta,\zeta) - 1,\ z \in M,\ \zeta \in T_z^*M,$$

with g^{-1} the inner product on the fibers of the cotangent bundle induced by g. The critical set

$$S^*M = \{p = 0\}$$

is the unit cotangent bundle. Furthermore, if

$$G^t : T^*M \to T^*M,\ t \in \mathbb{R},$$

is the geodesic flow of g on T^*M, then the Hamiltonian flow of p is given by G^{2t}.

[1] Same methods apply with $\operatorname{Re}\lambda_n \to -\infty$, with signs in certain formulas inverted and semiclassical measures concentrated on the outgoing, rather than incoming, set $\mathcal{A}_j^+$.

Proposition 1.1. *1. For every j, each sequence λ_n satisfying* (1.4) *has a subsequence λ_{n_l} such that $E(\lambda_{n_l})$ converges to some Radon measure μ.*

2. Each semiclassical measure μ is supported on the unit cotangent bundle:

$$\mu(T^*M \setminus S^*M) = 0.$$

3. Each semiclassical measure μ is decaying exponentially with rate 2ν along the geodesic flow:

$$\mu(G^t A) = e^{-2\nu t}\mu(A),\ A \subset S^*M,\ t \in \mathbb{R}. \tag{1.7}$$

To study semiclassical measures in greater detail, we define the incoming set $\mathcal{A}_j^-$ and the outgoing set $\mathcal{A}_j^+$ for each cusp $\mathcal{C}_j$ as follows: if (r,θ) are the canonical coordinates on $\mathcal{C}_j$ and (r,θ,p_r,p_θ) is the induced system of coordinates on $T^*\mathcal{C}_j$, then

$$\begin{gathered}\mathcal{A}_j^\pm = \{\rho \in S^*M \mid \exists t > 0 : G^{\pm t}\rho \in \widehat{\mathcal{A}}_j^\pm\},\\ \widehat{\mathcal{A}}_j^\pm = \{(r,\theta,p_r,p_\theta) \in T^*\mathcal{C}_j \mid p_r = \pm 1,\ p_\theta = 0\}.\end{gathered} \tag{1.8}$$

In other words, $\mathcal{A}_j^+$ is the union of all geodesics going directly into jth cusp and $\mathcal{A}_j^-$ is the union of all geodesics emanating directly from it. Note that $\mathcal{A}_j^\pm$ need not be closed; in fact, for hyperbolic surfaces each of them is dense in S^*M.

For each $\nu > 0$, we construct the measure $\mu_{j\nu}$ on $\mathcal{A}_j^-$ as follows:

$$\mu_{j\nu}(A) = \lim_{t\to+\infty} e^{-2\nu t} \int_{\widehat{\mathcal{A}}_j^- \cap G^{-t}(A)} e^{2\nu r}\, dr d\theta,\ A \subset \mathcal{A}_j^-. \tag{1.9}$$

Since $\nu > 0$, the limit exists (in fact, the function under the limit is increasing and it is bounded when A is bounded) and yields a Radon measure satisfying parts 2 and 3 of Proposition 1.1. We can now state the main result of the paper, which can be viewed as quantum unique ergodicity for Eisenstein functions in the upper half-plane:

Theorem 1. *For every sequence λ_n satisfying* (1.4)*, the sequence $E_j(\lambda_n)$ converges to $\mu_{j\nu}$.*

As an application of Theorem 1, we derive a bound on the scattering matrix $S(\lambda)$. For each two cusps $\mathcal{C}_j$, $\mathcal{C}_{j'}$, define $S_{jj'}(\lambda)$ by

$$E_j|_{\mathcal{C}_{j'}}(\lambda; r,\theta) = \delta_{jj'} e^{(1/2-i\lambda)r} + S_{jj'}(\lambda) e^{(1/2+i\lambda)r} + \cdots,$$

where (r,θ) are canonical coordinates on $\mathcal{C}_{j'}$, δ is the Kronecker delta, and $\cdots$ denotes the terms corresponding to terms with $k \neq 0$ in the Fourier series expansion (3.1) of $E_j|_{\mathcal{C}_{j'}}$ in the θ variable.

Theorem 2. *Consider two cusps $\mathcal{C}_j, \mathcal{C}_{j'}$ and assume that $\mu_{j\nu}(\mathcal{A}_{j'}^+) = \emptyset$ (in particular, this is true for hyperbolic surfaces, as $\mathcal{A}_{j'}^+ \cap \mathcal{A}_j^-$ consists of countably many geodesics). Then for each sequence λ_n satisfying* (1.4),

$$S_{jj'}(\lambda_n) \to 0 \text{ as } n \to \infty.$$

In other words,

$$S_{jj'}(\lambda) = o(1),\ 0 < C^{-1} < \operatorname{Im}\lambda < C,\ \operatorname{Re}\lambda \to \infty.$$

This estimate is far from optimal: in the special case of the modular surface $M = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$, the scattering coefficient $S(\lambda)$ is related to the Riemann zeta function by the formula [Ti, Section 2.18]

$$S(\lambda) = \sqrt{\pi}\frac{\zeta(-2i\lambda)\Gamma(-i\lambda)}{\zeta(1-2i\lambda)\Gamma(1/2-i\lambda)}.$$

Given that both $\zeta(z)$ and $\zeta^{-1}(z)$ are bounded in every half-plane $\{\operatorname{Re} z > 1 + C^{-1}\}$ (either by Dirichlet series or by Euler product representation), the basic bound on the zeta function in the critical strip [Ti, (5.1.4)] gives

$$|S(\lambda)| = O(|\lambda|^{-\min(\operatorname{Im}\lambda,1/2)-}),\ \operatorname{Im}\lambda \geq C^{-1}. \tag{1.10}$$

The bound (1.10) is optimal for $\operatorname{Im}\lambda > 1/2$, and no optimal bounds are known for $0 < \operatorname{Im}\lambda < 1/2$. It would be interesting to see if semiclassical methods can yield an effective bound on the scattering coefficients, and compare such bound to (1.10).

Finally, assume that for some λ, the matrix $S(\lambda)$ is not invertible; that is, there exists $\alpha \in \mathbb{C}^m \setminus \{0\}$ such that for each j',

$$\sum_j \alpha_j S_{jj'}(\lambda) = 0. \tag{1.11}$$

Then $-\lambda$ is a resonance; i.e., a pole of the meromorphic continuation of the resolvent $(-\Delta - 1/4 - \lambda^2)^{-1}$ to the lower half-plane (see for example [Mü, Section 5]), and a resonant state at $-\lambda$ is given by $\sum_j \alpha_j E_j(\lambda)$. We arrive to

Theorem 3. *Assume that λ_n is a sequence satisfying* (1.4) *such that each $-\lambda_n$ is a resonance, and let $u_{(n)}$ be a sequence of corresponding resonant states converging to some measure μ. Then μ is a linear combination of the measures $\mu_{1\nu},\dots,\mu_{m\nu}$ defined by* (1.9).

The fact that semiclassical measures are exponentially decaying along the geodesic flow is parallel to [NoZw, Theorem 4]. However, the concentration statement [NoZw, (1.15)] is vacuous in our case, as the set Γ_E^- from [NoZw] (not Γ_E^+, as $\operatorname{Re}(-\lambda) < 0$) is the whole cosphere bundle. In fact, [NoZw] heavily use the fact that resonant states are outgoing, while Eisenstein series studied in the present paper need not satisfy the outgoing condition (which in our case is (1.11)).

To prove Theorem 1, we first use exponential decay of semiclassical measures along the geodesic flow together with the Borel–Cantelli lemma and the structure of the flow in the cusp to show that each semiclassical measure μ has to be supported on $\bigcup_{j'} \mathcal{A}_{j'}^-$. We then consider the Fourier series (3.1) in some cusp region $\mathcal{C}_{j'}$. The restriction of μ to $\mathcal{A}_{j'}^-$ is composed of:

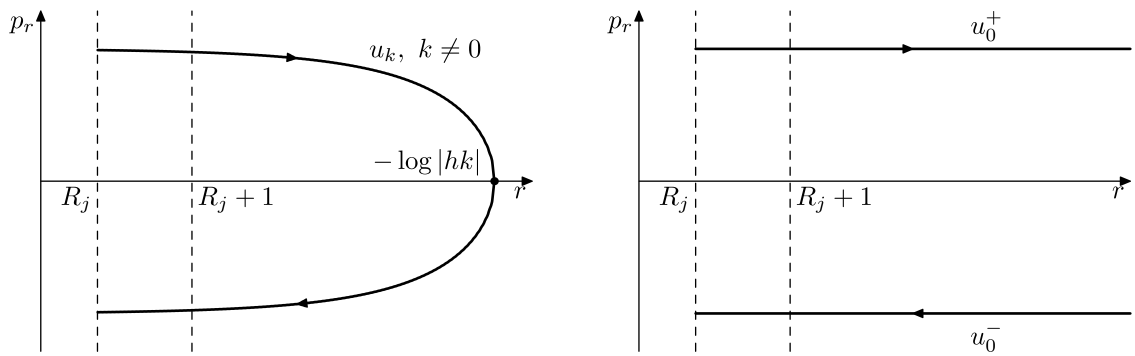

FIGURE 1. Semiclassical characteristic set for (3.3)

- the zero mode $u_0(r)$, which gives the measure $\delta_{jj'}\mu_{j\nu}$, and
- the modes $u_k(r)$, for $0 < |hk| \ll 1$.

It then remains to show that the contribution of nonzero modes is negligible. For that, we use the equation (3.3) satisfied by each u_k. Its semiclassical characteristic set, corresponding to the geodesics on $\{p = 0\} \cap \{p_\theta = hk\}$, is a single curve bending at the point $r = -\log|hk|$. If we approximate u_k using the WKB method, then the transport equation tells us that the amplitude is going to decay exponentially as we go along the flow (the precise rate of decay controlled by $\operatorname{Im}\lambda$). Therefore, the mass of u_k on the incoming (bottom) part of the characteristic set in the region $\{R_j < r < R_j + 1\}$ is $O(|hk|^\nu)$ and thus indeed negligible. Since the standard WKB approximation is no longer valid as $hk \to 0$, we use a version of the exact WKB method to justify the above observation. One could also establish the needed properties by expressing solutions to (3.3) via modified Bessel functions; however, we give an (arguably longer) semiclassical proof because it provides a more direct explanation for the phenomena encountered using the geometry of the problem.

The paper is organized as follows. In Section 2, we review some notation and facts from semiclassical analysis. In Section 3, we use standard methods of semiclassical analysis and spectral theory to show existence of Eisenstein functions and prove Proposition 1.1. In Section 4, we consider the special case of finite area hyperbolic surfaces and describe the canonical measures $\mu_{j\nu}$ from (1.9) via the action of the fundamental group of M; we also prove Theorem 1 in this case for $\operatorname{Im}\lambda > 1/2$ using the classical definition of Eisenstein functions as series. Section 5 studies solutions of a certain auxiliary ordinary differential equation, which are related to the nonzero modes in the Fourier series decomposition (3.1). Finally, in Section 6 we prove Theorems 1 and 2.

2. Semiclassical preliminaries

In this section, we briefly review the portions of semiclassical analysis used below; the reader is referred to [EvZw] for a detailed account on the subject.

We assume that $h > 0$ is a parameter, the smallness of which is implied in all statements below. Consider the algebra $\Psi^s(\mathbb{R}^d)$ of pseudodifferential operators with symbols in the class

$$S^s(\mathbb{R}^d) = \{a(x,\xi;h) \in C^\infty(\mathbb{R}^d) \mid \sup_{K\subset\mathbb{R}^d} |\partial^x_\alpha \partial^\xi_\beta a(x,\xi;h)| \leq C_{\alpha\beta K}\langle\xi\rangle^{s-|\beta|}\};$$

here K runs through compact subsets of $\mathbb{R}^d$. The only difference with the classes studied in [EvZw, Section 8.6] is that we do not require uniform bounds as $x \to \infty$. However, this does not matter in our situation, as we will mostly use *compactly supported* operators; e.g. those operators whose Schwartz kernels are compactly supported in $\mathbb{R}^d \times \mathbb{R}^d$. As in [EvZw, Appendix E], we can define the algebra $\Psi^s(M)$ for any manifold M. The compactly supported elements of $\Psi^s(M)$ act $H^t_{\hbar,\mathrm{loc}}(M) \to H^{t-s}_{\hbar,\mathrm{comp}}(M)$ with norm $O(1)$ as $h \to 0$, where $H^t_{\hbar,\mathrm{loc}}$ and $H^{t-s}_{\hbar,\mathrm{comp}}$ are semiclassical Sobolev spaces.

To avoid discussion of simultaneous behavior of symbols as $\xi \to \infty$ and $h \to 0$, we further require that the symbols of elements of Ψ^s are *classical*, in the sense that they posess an asymptotic expansion in powers of h, with the term next to h^k lying in S^{s-k} (see [Dya, Section 2.1] for details). Following [Va, Section 2], we introduce the fiber-radial compactification $\overline{T}^*M$ of the cotangent bundle. Each $A \in \Psi^s$ has an invariantly defined (semiclassical) principal symbol $\sigma(A) = a \in C^\infty(T^*M)$, and $\langle\xi\rangle^{-s}a$ extends to a smooth function on $\overline{T}^*M$. We then define the characteristic set of A as $\{\langle\xi\rangle^{-s}a \neq 0\} \subset \overline{T}^*M$ and say that A is elliptic on some $U \subset \overline{T}^*M$, if U does not intersect the characteristic set of A.

Finally, as in [Va] or [Dya], we define the semiclassical wavefront set $\mathrm{WF}_\hbar(A) \subset \overline{T}^*M$. The wavefront set of A is empty if and only if A lies in the algebra $h^\infty\Psi^{-\infty}(M)$ of smoothing operators such that each of $C^\infty(M \times M)$ seminorms of their Schwartz kernels decays faster than any power of h. For $A, B \in \Psi^s(M)$, we say that $A = B$ microlocally on some open $U \subset \overline{T}^*M$, if $\mathrm{WF}_\hbar(A - B) \cap U = \emptyset$. Also, we say that $A \in \Psi^s(M)$ is compactly microlocalized, if $\mathrm{WF}_\hbar(A)$ does not intersect the fiber infinity $\partial(\overline{T}^*M)$; in this case, $A \in \Psi^s(M)$ for all s.

We recall the following fundamental estimates:

Proposition 2.1. *Let $P \in \Psi^s(M)$ have real-valued principal symbol p. Then:*

1. (Elliptic estimate) Let $A \in \Psi^0(M)$ be compactly supported and assume that P is elliptic on $\mathrm{WF}_\hbar(A)$. Then for each $u \in L^2(M)$,

$$\|Au\|_{H^N_\hbar(M)} \leq O(1)\|Pu\|_{H^{N-s}_\hbar(M)} + O(h^\infty)\|u\|_{L^2(M)}.$$

2. (Propagation of singularities) Let $A, B \in \Psi^0(M)$ *be compactly supported and compactly microlocalized and assume that for each Hamiltonian flow line* $\gamma(t)$ *of* p *such that* $\gamma(0) \in \mathrm{WF}_\hbar(A) \cap \{p = 0\}$*, there exists* $T \geq 0$ *such that* B *is elliptic at* $\gamma(T)$*. Then for each* $u \in L^2(M)$*,*

$$\|Au\|_{L^2(M)} \leq O(1)\|Bu\|_{L^2(M)} + O(h^{-1})\|Pu\|_{L^2(M)} + O(h^\infty)\|u\|_{L^2(M)}.$$

Proof. The elliptic estimate is proved by constructing a *parametrix*; i.e. a compactly supported operator $Q \in \Psi^{-s}(M)$ such that $A = QP + h^\infty\Psi^{-\infty}$; see [HöIII, Theorem 18.1.24'] for the microlocal case and for example [Dya, Section 2.2] for the semiclassical case. Propagation of singularities can be proved either by conjugation to a model case by a Fourier integral operator or by the positive commutator method; see [EvZw, Theorem 10.21] for the former and [DaVa, end of Section 4] for the latter. □

We now prove an elliptic estimate for a Schrödinger operator. It would not follow from standard elliptic estimates on the real line because the potential we will use is exponentially increasing and thus the operator does not lie in any reasonable symbolic class.

Proposition 2.2. *Consider the Schrödinger operator*

$$P(h) = h^2 D_x^2 + V(x),$$

where the potential $V(x)$ *satisfies*

$$\operatorname{Re} V(x) \geq \varepsilon \text{ for } x \leq \varepsilon;$$
$$|\partial^j V(x)| \leq C_j \text{ for each } j \text{ and } -1 \leq x \leq 1,$$

for some constants $\varepsilon > 0$ *and* C_j*. Then*

$$v(x) \in H^1(-\infty, \varepsilon),\ P(h)v = 0 \Longrightarrow \|v\|_{L^2(-\infty,0)} = O(h^\infty)\|v\|_{L^2(0,\varepsilon)}.$$

Proof. First of all, we integrate by parts:

$$0 = \operatorname{Re}\int_{-\infty}^0 (P(h)v(x))\overline{v(x)}\,dx$$
$$= -h^2\operatorname{Re}(\bar v\partial_x v)|_{x=0} + \int_{-\infty}^0 h^2|\partial_x v(x)|^2 + (\operatorname{Re} V(x))|v(x)|^2\,dx.$$

Therefore,

$$\|v\|^2_{L^2(-\infty,0)} = O(h^2)|v(0)|\cdot|\partial_x v(0)|.$$

Now, since P is elliptic on $(-\varepsilon, \varepsilon)$, we have by Proposition 2.1(1) for each N,

$$\|v\|_{H^N_\hbar(-\varepsilon/2,\varepsilon/2)} = O(h^\infty)\|v\|_{L^2(-\varepsilon,\varepsilon)};$$

it follows by Sobolev embedding that

$$|v(0)|,\ |\partial_x v(0)| = O(h^\infty)\|v\|_{L^2(-\varepsilon,\varepsilon)}. \quad \square$$

Finally, we will need the following very special case of the theory of (local) Lagrangian distributions (see [GuiSt, Chapter 6] or [VũNg, Section 2.3] for a detailed account, and [HöIV, Section 25.1] or [GrSj, Chapter 11] for the microlocal case; however, the proposition below can be proved in a straightforward manner using the definition of a pseudodifferential operator and the method of stationary phase):

Proposition 2.3. *Assume that $I \subset \mathbb{R}$ is an interval, and let $\varphi_0(x)$ be a smooth real-valued function on I. Define the set*

$$\Lambda = \{(x,\xi) \mid \xi = \partial_x \varphi_0(x)\} \subset T^*I.$$

Let $a(x;h)$ be a smooth function in $x \in I$ and in $h \geq 0$ (which is a classical symbol in h, owing to the asymptotic Taylor expansion at $h = 0$) and define

$$u(x;h) = e^{i\varphi_0(x)/h} a(x;h). \tag{2.1}$$

We call $u(x;h)$ a Lagrangian distribution associated to Λ. Let $A \in \Psi^0(I)$ be a compactly supported operator. Then:

1. If $\mathrm{WF}_\hbar(A) \cap \Lambda = \emptyset$, then $\|Au\|_{L^2} = O(h^\infty)$.

2. If A is elliptic at some $(x_0, \xi_0) \in \Lambda$ and $a(x_0; 0) \neq 0$, then $\|Au\|_{L^2} \geq C^{-1}$ for some constant C independent of h.

3. If $\tilde{a}(x,\xi)$ is the principal symbol of A, then $Au(x;h) = \tilde{a}(x, \partial_x\varphi_0(x))u(x;h) + O_{L^2}(h)$.

3. Basic properties

In this section, we review the construction of Eisenstein functions and prove the basic properties of semiclassical measures given in Proposition 1.1.

We start by studying the equation (1.2) in some cusp $\mathcal{C}_j$. Consider the Fourier series

$$u|_{\mathcal{C}_j}(r,\theta) = \sum_{k\in\mathbb{Z}} u_k^{(j)}(r) e^{ik\theta}; \tag{3.1}$$

when it is unambiguous which cusp we are considering, we simply write $u_k^{(j)} = u_k$. Note that since the volume form induced by g in $\mathcal{C}_j$ is $e^{-r}\,dr d\theta$, we have

$$\|u\|^2_{L^2(\mathcal{C}_j \cap \{R' < r < R''\})} = \sum_{k\in\mathbb{Z}} \|e^{-r/2} u_k^{(j)}(r)\|^2_{L^2(R',R'')},\ R_j < R' < R''. \tag{3.2}$$

By (1.1), (1.2) takes the form

$$[(D_r + i/2)^2 + k^2 e^{2r} - \lambda^2] u_k^{(j)}(r) = 0,\ k \in \mathbb{Z}. \tag{3.3}$$

For $k = 0$, (3.3) is a constant-coefficient ODE and we have

$$u_0^{(j)}(r) = u_0^{+(j)} e^{(1/2+i\lambda)r} + u_0^{-(j)} e^{(1/2-i\lambda)r}, \tag{3.4}$$

for some constants $u_0^{\pm(j)}$.

Now, let $\chi_j(r) \in C^\infty((R_j, \infty))$ be supported in $[R_j + 1, \infty)$, but $\chi_j(r) = 1$ for $r \geq R_j + 2$. Then

$$(\Delta - 1/4 - \lambda^2)(\chi_j(r) e^{(1/2-i\lambda)r}) = [\Delta, \chi_j] e^{(1/2-i\lambda)r} \in C_0^\infty(M).$$

Take $\lambda \in \mathbb{C}$ such that $\operatorname{Im} \lambda > 0$, $\lambda \notin (0, i/2]$; then the resolvent

$$(\Delta - 1/4 - \lambda^2)^{-1} : L^2(M) \to L^2(M)$$

is well-defined and the only solution to (1.2) satisfying (1.3) is given by [Mü, Section 3]

$$E_j(\lambda; z) = \chi_j(r) e^{(1/2-i\lambda)r} - (\Delta - 1/4 - \lambda^2)^{-1} [\Delta, \chi_j] e^{(1/2-i\lambda)r}. \tag{3.5}$$

The second term in the right-hand side of (3.5) lies in $C^\infty(M) \cap H^1(M)$. Therefore, if we put $u = E_j(\lambda; z)$ and consider Fourier series (3.1) and the decompositions (3.4), then for each cusp $\mathcal{C}_{j'}$,

$$u_0^{-(j')} = \delta_{jj'}; \tag{3.6}$$

$$u_k^{(j')} \in H^1((R_{j'}, \infty)),\ k \neq 0. \tag{3.7}$$

Moreover, by the standard resolvent estimate for Δ away from the spectrum we get that for each compact set $K \subset M$ and each constant $\nu > 0$, there exists a constant $C(K, \nu)$ such that

$$|\operatorname{Re} \lambda| \geq 1,\ \nu/2 \leq \operatorname{Im} \lambda \leq 2\nu \Longrightarrow \|E_j(\lambda; z)\|_{L^2(K)} \leq C(K, \nu). \tag{3.8}$$

Indeed, the first term on the right-hand side of (3.5) clearly satisfies (3.8); the norm of the second term in $L^2(M)$ is estimated by the product of $\|(\Delta - 1/4 - \lambda^2)^{-1}\|_{L^2 \to L^2} = O(|\lambda|^{-1})$ and $\|[\Delta, \chi_j] e^{(1/2-i\lambda)r}\|_{L^2} = O(|\lambda|)$.

Proof of Proposition 1.1. We denote $u_{(n)} = E_j(\lambda_n)$; we also note that we can get rid of the cutoff χ in Definition 1.1 by restricting our attention to compactly supported pseudodifferential operators.

1. We follow the argument of [EvZw, Theorem 5.2]. Let

$$\{A_l(h) \in \Psi^0(M)\}_{l \in \mathbb{N}}$$

be a countable set of compactly supported and compactly microlocalized operators such that, if a_l is the principal symbol of A_l, then for each $a \in C_0^\infty(T^*M)$ there exists a subsequence a_{l_m} converging to a in the sup-norm and such that the Schwartz kernels of the operators A_{l_m} are supported inside some compact set independent of m.

By (3.8), for each l the sequence $\langle A_l(h_n)u_{(n)}, u_{(n)}\rangle_{L^2(M)}$ is bounded as $n \to \infty$. Using a diagonal argument, we can replace $u_{(n)}$ by a subsequence of it such that for each l, the limit

$$\lim_{n\to\infty} \langle A_l(h_n)u_{(n)}, u_{(n)}\rangle_{L^2(M)} = \mathcal{F}(a_l)$$

exists. We now use [EvZw, Theorem 5.1]: for each compact $K \subset M$, there exists a constant $C(K)$ such that for each $A \in \Psi^0(M)$ with Schwartz kernel supported in $K \times K$ and principal symbol a,

$$\|A(h_n)u_{(n)}\|_{L^2} \leq C(K) \sup_{T^*M} |a| + O(h_n^{1/2}),$$

where the constant in $O(\cdot)$ depends on some fixed seminorm of the full symbol of A. By a standard 3ε argument using the density of $\{a_l\}$, we see that for each compactly supported and compactly microlocalized $A \in \Psi^0(M)$ with principal symbol a, there exists the limit

$$\lim_{n\to\infty} \langle A(h_n)u_{(n)}, u_{(n)}\rangle_{L^2(M)} = \mathcal{F}(a). \tag{3.9}$$

The map $\mathcal{F} : C_0^\infty(T^*M) \to \mathbb{C}$ is a linear functional such that for each compact $K \subset M$, there exists a constant $C(K)$ such that for each $a \in C_0^\infty(T^*M)$ with the projection of $\operatorname{supp} a$ onto M contained in K, we have

$$|\mathcal{F}(a)| \leq C(K) \sup_{T^*M} |a|.$$

Also, by sharp Gårding inequality (see [EvZw, Theorem 5.3]), $\mathcal{F}(a) \geq 0$ whenever $a \geq 0$. It follows from Riesz Representation Theorem that there exists a Radon measure μ on T^*M such that for each $a \in C_0^\infty(M)$,

$$\mathcal{F}(a) = \int_{T^*M} a\, d\mu.$$

Finally, as follows from the proof of part 2 of this proposition, (3.9) is valid for any compactly supported $A \in \Psi^0(M)$ (with no compact microlocalization requirement), and in fact $\mathcal{F}(a) = 0$ whenever $\operatorname{supp} a \cap S^*M = \emptyset$.

2. It suffices to note that by (1.6) and Proposition 2.1(1), for each compactly supported $A \in \Psi^0(M)$ with $\mathrm{WF}_\hbar(A) \cap S^*M = \emptyset$, we have

$$\lim_{n\to\infty} \|A(h_n)u_{(n)}\|_{L^2} = 0.$$

3. It suffices to show that for each $a \in C_0^\infty(M)$ and each t,

$$\int_{S^*M} a \circ G^t\, d\mu = e^{2\nu t} \int_{S^*M} a\, d\mu.$$

Differentiating this equality in t, we see that it is enough to prove

$$\int_{S^*M} \{p, a\}\, d\mu = 4\nu \int_{S^*M} a\, d\mu. \tag{3.10}$$

Let $A(h) \in \Psi^0(M)$ be compactly supported and compactly microlocalized with principal symbol a; then by (1.6),

$$\begin{aligned}
0 &= h_n^{-1}\langle A(h_n)P(h_n)u_{(n)}, u_{(n)}\rangle \\
&= h_n^{-1}\langle [A(h_n), P(h_n)]u_{(n)}, u_{(n)}\rangle + h_n^{-1}\langle A(h_n)u_{(n)}, P(h_n)^*u_{(n)}\rangle \\
&= h_n^{-1}\langle [A(h_n), P(h_n)]u_{(n)}, u_{(n)}\rangle - 4i(1 + h_n \operatorname{Re}\omega_n)\operatorname{Im}\omega_n\langle A(h_n)u_{(n)}, u_{(n)}\rangle.
\end{aligned}$$

However, $h^{-1}[A(h), P(h)]$ lies in Ψ^0 and has principal symbol $i\{p, a\}$; taking the limit as $n \to \infty$, we obtain (3.10). □

4. Hyperbolic surfaces

In this section, we consider the special case $M = \Gamma\backslash\mathbb{H}$, where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a Fuchsian group of the first kind, so that M is a finite area hyperbolic surface. We denote by $\pi_\Gamma : \mathbb{H} \to M$ the projection map and we use both the half-plane and the ball models $\mathbb{H}$ and $\mathbb{B}$; they are mapped to each other by the transformation

$$\gamma_0 : \mathbb{H} \to \mathbb{B},\ \gamma_0(z) = \frac{z - i}{z + i};$$

note in particular that $\gamma_0(i) = 0$, $\gamma_0(\infty) = 1$, and $|\gamma_0'(z)| = 2/|z + i|^2$.

We first find an interpretation of (1.7) in terms of the group action; this is parallel to the representation of measures invariant under the Hamiltonian flow in Patterson–Sullivan theory (see for example [Bo, Section 14.2]). Consider the Poincaré ball model $\mathbb{B}$ and parametrize the unit cotangent bundle $S^*\mathbb{B}$ by the diffeomorphism

$$\mathcal{T} : (\partial\mathbb{B} \times \partial\mathbb{B})_\Delta \times \mathbb{R} \to S^*\mathbb{B},$$

where $(\partial\mathbb{B} \times \partial\mathbb{B})_\Delta$ is the Cartesian square of the circle $\partial\mathbb{B}$ minus the diagonal. The map $\mathcal{T}$ is defined as follows: take $(w_1, w_2) \in (\partial\mathbb{B} \times \partial\mathbb{B})_\Delta$ and let $\gamma_{w_1w_2}(t)$ be the unique unit length geodesic going from w_1 to w_2, parametrized so that $\gamma(0)$ is the point of γ closest to $0 \in \mathbb{B}$. We put

$$\mathcal{T}(w_1, w_2, t) = (\gamma_{w_1w_2}(t), g\dot\gamma_{w_1w_2}(t)).$$

Now, consider a Radon measure μ on S^*M satisfying (1.7). We can lift it to a measure μ' on $S^*\mathbb{B}$; then

$$\mathcal{T}^*\mu' = \tilde\mu \times e^{-2\nu t}\,dt, \tag{4.1}$$

where $\tilde\mu$ is some Radon measure on $(\partial\mathbb{B} \times \partial\mathbb{B})_\Delta$.

For each $\gamma \in \mathrm{PSL}(2, \mathbb{R})$, we can calculate

$$\gamma(\mathcal{T}(w_1, w_2, t)) = \mathcal{T}\bigg(\gamma(w_1), \gamma(w_2), t + \frac{1}{2}\log\bigg|\frac{\gamma'(w_1)}{\gamma'(w_2)}\bigg|\bigg),$$

where $\gamma'(w)$ is the derivative of γ considered as a transformation on $\mathbb{B}$. We see then that the measure μ' defined by (4.1) is invariant under the action of Γ on $S^*\mathbb{B}$ if and only if

$$\gamma \in \Gamma \Longrightarrow \gamma^*\tilde{\mu} = |\gamma'(w_1)|^{\nu}|\gamma'(w_2)|^{-\nu}\tilde{\mu}, \tag{4.2}$$

where

$$(\gamma^*\tilde{\mu})(A) = \tilde{\mu}((\gamma \times \gamma)(A)),\ A \subset (\partial\mathbb{B} \times \partial\mathbb{B})_\Delta.$$

In particular, if $\hat{\mu}$ is a Radon measure on $\partial\mathbb{B}$ such that

$$\gamma \in \Gamma \Longrightarrow \gamma^*\hat{\mu} = |\gamma'(w)|^{2\nu+1}\hat{\mu}, \tag{4.3}$$

then a measure $\tilde{\mu}$ satisfying (4.2) is given by

$$\tilde{\mu} = |w_1 - w_2|^{-2(\nu+1)}\hat{\mu} \times |dw_2|. \tag{4.4}$$

Now, fix a cusp region $\mathcal{C}_j$ on M and assume for simplicity that $1 \in \partial\mathbb{B}$ is a preimage of the corresponding cusp. Denote

$$\gamma(z) = \frac{az+b}{cz+d},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R});$$

then

$$q \in \partial\mathbb{H} \Longrightarrow |(\gamma_0\gamma\gamma_0^{-1})'(\gamma_0(q))| = \frac{q^2+1}{(aq+b)^2+(cq+d)^2}.$$

Let Γ_∞ be the group of all elements of Γ fixing ∞; without loss of generality, we may assume that it is generated by the shift $z \to z+1$. Then all the preimages of the cusp of $\mathcal{C}_j$ are given by

$$\{\gamma(\infty) \mid \gamma\Gamma_\infty \in \Gamma/\Gamma_\infty\};$$

i.e., they are indexed by right cosets of Γ_∞ in Γ. Note that

$$\gamma(\infty) = a/c,\ |(\gamma_0\gamma\gamma_0^{-1})'(\gamma_0(\infty))| = \frac{1}{a^2+c^2}. \tag{4.5}$$

A canonical system of coordinates on $\mathcal{C}_j$ is given by

$$(r,\theta) \in (R_j,\infty) \times \mathbb{S}^1 \to \pi_\Gamma\left(\frac{\theta + ie^r}{2\pi}\right). \tag{4.6}$$

Proposition 4.1. *The lift of the measure $\mu_{j\nu}$ defined in* (1.9) *corresponds under* (4.1) *to* $(4\pi)^{2\nu+1}\tilde{\mu}$, *with $\tilde{\mu}$ given by* (4.4), *and*

$$\gamma_0^*\hat{\mu} = \sum_{\gamma\Gamma_\infty \in \Gamma/\Gamma_\infty} \frac{\delta_{a/c}}{(a^2+c^2)^{2\nu+1}}; \tag{4.7}$$

here δ denotes a delta measure. (Note that the values a/c are distinct for different cosets, as Γ_∞ is the stabilizer of ∞.)

Proof. The measure $\hat{\mu}$ is well-defined, as the series

$$\sum_{\gamma\Gamma_\infty\in\Gamma/\Gamma_\infty}\frac{1}{(a^2+c^2)^{2\nu+1}}=\sum_{\gamma\Gamma_\infty\in\Gamma/\Gamma_\infty}(\operatorname{Im}\gamma^{-1}(i))^{2\nu+1} \tag{4.8}$$

converges, by convergence of Eisenstein series (4.9). By (4.5), the measure $\hat{\mu}$ satisfies (4.3); therefore, it produces a measure μ satisfying parts 2 and 3 of Proposition 1.1. Moreover, since $\hat{\mu}$ is supported on the set of the preimages of the cusp of $\mathcal{C}_j$, μ is supported on $\mathcal{A}_j^-$. It then suffices to study the restriction of μ to $\widehat{\mathcal{A}}_j^-$. To this end, take $A\subset(R_j,\infty)\times\mathbb{S}^1$ and consider

$$\widetilde{A}=\{(r,\theta;-1,0)\in T^*\mathcal{C}_j\mid(r,\theta)\in A\}\subset\widehat{\mathcal{A}}_j^-.$$

Since

$$\mathcal{T}(1,\gamma_0(q),t)=\gamma_0\bigg(q+i|i+q|e^{-t},-i\frac{e^t}{|i+q|}\bigg),$$

we get

$$\widetilde{A}=\pi_\Gamma\mathcal{T}(\{(1,\gamma_0(q),t)\mid(q,t)\in\check{A}\}),$$
$$\check{A}=\{(\theta/(2\pi),-r+\ln(2\pi)+\ln|i+\theta/(2\pi)|)\mid(r,\theta)\in A\}.$$

Then

$$\mu(\widetilde{A})=\int_{(q,t)\in\check{A}}|\gamma_0(q)-1|^{-2(\nu+1)}e^{-2\nu t}\,|d\gamma_0(q)|dt$$
$$=2^{-2\nu-1}\int_{(q,t)\in\check{A}}|i+q|^{2\nu}e^{-2\nu t}\,dqdt=(4\pi)^{-2\nu-1}\int_A e^{2\nu r}\,drd\theta$$

and the proof is finished by the definition of $\mu_{j\nu}$. □

In particular, for the modular surface the measure $\hat{\mu}$ is given by

$$\gamma_0^*\hat{\mu}=\sum_{\substack{m,n\in\mathbb{Z}\\ n\geq 0,\ m\perp n}}\frac{\delta_{m/n}}{(m^2+n^2)^{2\nu+1}}.$$

We now sketch a proof of Theorem 1 for hyperbolic surfaces in the case $\operatorname{Im}\lambda>1/2$. For that we use the series representation for the Eisenstein function

$$\widetilde{E}(\lambda;z)=(2\pi)^{1/2-i\lambda}\sum_{\Gamma_\infty\gamma\in\Gamma_\infty\backslash\Gamma}(\operatorname{Im}\gamma(z))^{1/2-i\lambda},\ z\in\mathbb{H}. \tag{4.9}$$

This series converges absolutely [Ku, Theorem 2.1.1]; since it is invariant under Γ and each of its terms solves (1.2) on $\mathbb{H}$, it gives rise to a solution $\widehat{E}(\lambda,z)$ of (1.2). It can also be seen that (4.9) converges on L^2 of a fundamental domain of Γ, if we take out the term with $\gamma=1$; therefore, $\widehat{E}(\lambda,z)$ satisfies (1.3) and we have

$$\widetilde{E}(\lambda;z)=E_j(\lambda;\pi(z)).$$

To prove Theorem 1 in the considered case, it now suffices to show that for each compactly supported $A(h) \in \Psi^0(\mathbb{H})$ with principal symbol a and a sequence λ_n satisfying (1.4), we have as $n \to \infty$

$$\langle A(h_n)\widetilde{E}(\lambda_n; z), \widetilde{E}(\lambda_n; z)\rangle_{L^2} \to (4\pi)^{2\nu+1} \int_{S^*\mathbb{H}} a \, d\mu',$$

where μ' is defined by (4.1) and $\tilde{\mu}$ is defined by (4.4), with $\hat{\mu}$ given by (4.7). Since the series (4.9) in particular converges in L^2_{loc}, it is enough to prove (bearing in mind that the parts of μ' corresponding to different terms in (4.7) are mapped to each other by elements of Γ)

$$\begin{aligned}\sum_{\Gamma_\infty\gamma, \Gamma_\infty\gamma_1 \in Z} &\langle A(h_n)(\operatorname{Im}\gamma(z))^{1/2-i\lambda_n}, (\operatorname{Im}\gamma_1(z))^{1/2-i\lambda_n}\rangle_{L^2} \\ \to 2^{2\nu+1} \sum_{\Gamma_\infty\gamma \in Z} &\int_{\partial\mathbb{B}\times\mathbb{R}} (a \circ \gamma^{-1})(\mathcal{T}(1, w, t))|1-w|^{-2(\nu+1)} e^{-2\nu t} \, |dw| dt\end{aligned} \tag{4.10}$$

for any finite subset Z of $\Gamma_\infty\backslash\Gamma$. However, each $(\operatorname{Im}\gamma(z))^{1/2-i\lambda_n}$ is microlocalized on $\mathcal{T}(\{\gamma^{-1}(\infty)\} \times \partial\mathbb{B} \times \mathbb{R})$; since these sets do not intersect for different $\gamma^{-1}(\infty)$, the cross terms in (4.10) are all $O(h_n^\infty)$. Recalling how the principal symbol of a pseudodifferential operator behaves under diffeomorphisms (in our case given by the action of Γ on $\mathbb{H}$), we see that it remains to prove

$$\begin{aligned}&\langle A(h_n)(\operatorname{Im} z)^{1/2-i\lambda_n}, (\operatorname{Im} z)^{1/2-i\lambda_n}\rangle_{L^2} \\ \to 2^{2\nu+1} &\int_{\partial B\times\mathbb{R}} a(\mathcal{T}(1, w, t))|1-w|^{-2(\nu+1)} e^{-2\nu t} \, |dw| dt \\ &= \int_{\mathbb{R}^2} a(q + ie^{-t}, -ie^t) e^{-2\nu t} \, dq dt;\end{aligned}$$

the latter is verified directly, as (by an application of the method of stationary phase)

$$A(h_n)(\operatorname{Im} z)^{1/2-i\lambda_n} = a(z, -i/\operatorname{Im} z)(\operatorname{Im} z)^{1/2-i\lambda_n} + O(h_n).$$

5. Auxiliary ODE analysis

In this section, we study the semiclassical Schrödinger operator

$$Q(h) = (hD_x)^2 + V(x; h), \ V(x; h) = e^{-2x} - (1 + h\omega)^2, \ x \in \mathbb{R}.$$

Here $\omega \in \mathbb{C}$ is bounded by some fixed constant.

Note that $Q(h)$ is a semiclassical differential operator, with real-valued principal symbol

$$q(x, \xi) = \xi^2 + e^{-2x} - 1, \ (x, \xi) \in T^*\mathbb{R}.$$

We will first use the exact WKB method, in the form similar to [Dya, Section 4.2], to construct two solutions $v^{\pm}(x;h),\ x\geq R$, to the equation

$$Q(h)v=0. \tag{5.1}$$

Here $R>0$ is a large constant independent of h, chosen below. As noted in the introduction, we use the exact WKB method to obtain asymptotics on $v_{\pm}$ in powers of h that is uniform as $x\to+\infty$; this will be the key to proving Theorem 1. We start by constructing the phase function:

Proposition 5.1. *There exists a complex-valued function $\varphi(x;h),\ x\geq 1$, solving the eikonal equation*

$$(\partial_x\varphi(x;h))^2+V(x;h)=0 \tag{5.2}$$

and such that

$$\varphi(x;h)=(1+h\omega)x+\psi(e^{-2x};h), \tag{5.3}$$

with $\psi(z;h)$ holomorphic in $\{|z|\leq e^{-2}\}$, smooth in $h\geq 0$, and satisfying $\psi(0;h)=0$. Moreover,

$$\partial_x\varphi(x;h)=\sqrt{1-e^{-2x}}+O(h), \tag{5.4}$$

$$\operatorname{Im}\varphi(x;h)=hx\operatorname{Im}\omega+O(h), \tag{5.5}$$

uniformly in $x\geq 1$.

Proof. We look for φ of the form (5.3) solving

$$\partial_x\varphi(x;h)=\sqrt{(1+h\omega)^2-e^{-2x}},\ x\geq 1. \tag{5.6}$$

Here the square root is defined on $\mathbb{C}$ minus the negative real half-line so that the square root of a positive real number is positive. Then ψ has to satisfy

$$\partial_z\psi(z;h)=\frac{1+h\omega-\sqrt{(1+h\omega)^2-z}}{2z}.$$

This equation has unique solution such that $\psi(0;h)=1$, since its right-hand side is holomorphic in $\{|z|\leq e^{-2}\}$.

Next, (5.4) follows immediately from (5.6). To show (5.5), we derive from (5.6) that

$$\partial_x\varphi(x;h)=h\omega+\sqrt{1-e^{-2x}}+O(he^{-2x});$$

therefore,

$$\operatorname{Im}\varphi(x;h)=\operatorname{Im}\varphi(1;h)+\int_1^x h\operatorname{Im}\omega+O(he^{-2x'})\,dx'=hx\operatorname{Im}\omega+O(h).\quad\square$$

Now, we construct the solutions $v_{\pm}$ themselves:

Proposition 5.2. *There exists a constant R independent of h and unique functions $a^\pm(z;h)$ holomorphic in $\{|z| \leq e^{-2R}\}$ and smooth in $h \geq 0$ such that for*

$$v^\pm(x;h) = e^{\pm i\varphi(x)/h} a^\pm(e^{-2x};h),\ x \geq R, \tag{5.7}$$

we have

$$Q(h)v^\pm(x;h) = 0, \tag{5.8}$$

$$a^\pm(0;h) = 1. \tag{5.9}$$

Proof. We consider the Taylor series of $a^\pm$ in z:

$$a^\pm(z;h) = \sum_{m\geq 0} a_m^\pm(h) z^m. \tag{5.10}$$

We first treat (5.10) as a formal series and solve (5.8) to find the coefficients $a_m^\pm(h)$. By (5.6), we have

$$\partial_x \varphi(x;h) = f(e^{-2x}),\ f(z) = \sqrt{(1+h\omega)^2 - z}.$$

Then

$$\begin{aligned} &Q(h)(e^{\pm i\varphi(x;h)/h} e^{-2mx}) = \pm 2ihe^{\pm i\varphi(x;h)/h} e^{-2mx} \\ &\cdot [2mf(e^{-2x};h) \pm 2ihm^2 + e^{-2x}(\partial_z f)(e^{-2x};h)]. \end{aligned} \tag{5.11}$$

Consider the Taylor series

$$f(z;h) = 1 + h\omega + \sum_{l\geq 1} f_l(h) z^l;$$

combining (5.7), (5.10), and (5.11), we see that the coefficients $a_m^\pm(h)$ have to satisfy the recursive relations

$$a_m^\pm(h) = \frac{1}{2m(1+h\omega \pm imh)} \sum_{0<l\leq m} (l-2m) f_l(h) a_{m-l}^\pm(h),\ m > 0. \tag{5.12}$$

The equations (5.12) have a unique solution $a_m^\pm(h)$ if we impose the condition (5.9), which takes the form

$$a_0^\pm(h) = 1. \tag{5.13}$$

It remains to estimate $a_m^\pm(h)$ uniformly in m and in h. Since $f(z;h)$ is holomorphic in $\{|z| \leq e^{-2}\}$, we have

$$|f_l(h)| \leq Ce^{2l},$$

for some constant C independent of l and h. Then, since $|1 + h\omega \pm imh| \geq 1/2$ for all m and h small enough, (5.12) implies

$$|a_m^\pm(h)| \leq 2C \sum_{0<l\leq m} e^{2l} |a_{m-l}^\pm(h)|. \tag{5.14}$$

Take R large enough so that $e^{2R-4} > 2C + 1$; then

$$1 > 2C \sum_{l>0} e^{(4-2R)l},$$

By (5.9) and (5.14), this implies that

$$|a_m^\pm(h)| \leq e^{2(R-1)m}.$$

Similar reasoning applied to the result of differentiating (5.12) in h gives that for each j, there exists a constant C_j such that for all m and h small enough,

$$|\partial_h^j a_m^\pm(h)| \leq C_j e^{2(R-1)m}. \tag{5.15}$$

It follows that the series (5.10) converges uniformly in $\{|z| \leq e^{-2R}\}$ to a function holomorphic in w and smooth in $h \geq 0$; by (5.12), the functions $v^\pm$ defined by (5.7) solve (5.8). □

Note that $v^\pm(x; h)$ are Lagrangian distributions on (R, ∞), in the sense of Proposition 2.3, associated to

$$\Gamma_\pm = \{\xi = \pm\sqrt{1 - e^{-2x}},\ x \geq R\},$$

the top and bottom branch, respectively, of the characteristic set $\{q = 0\} \cap \{x \geq R\}$. Indeed, we can write by (5.4)

$$\varphi(x; h) = \varphi_0(x) + h\varphi_1(x; h),$$

where φ_0 is real-valued, $\partial_x\varphi_0(x) = \sqrt{1 - e^{-2x}}$, and φ_1 is smooth in both x and $h \geq 0$ (but φ_1 is not bounded as $x \to \infty$). Then by (5.7),

$$v^\pm(x; h) = e^{i\varphi_0(x)/h}[e^{i\varphi_1(x;h)} a(x; h)];$$

the expression in square brackets is a symbol, which brings $v^\pm$ in the form (2.1). As $x \to +\infty$, $v^\pm$ is still a Lagrangian distribution modulo a certain exponentially growing or decaying factor:

Proposition 5.3. *Take $X \geq R$. Then the function*

$$\tilde{v}^\pm(x) = e^{\pm X \operatorname{Im}\omega} v^\pm(x + X),\ 0 < x < 1,$$

is a Lagrangian distribution associated to

$$\{(x, \xi) \mid 0 < x < 1,\ (x + X, \xi) \in \Gamma_\pm\},$$

uniformly in X.

Proof. We can write by (5.7)

$$\tilde{v}^\pm(x) = e^{\pm i\tilde{\varphi}(x;h)/h} \tilde{a}^\pm(x; h),$$

where

$$\tilde{\varphi}(x; h) = \varphi(x + X; h) - ihX \operatorname{Im}\omega$$

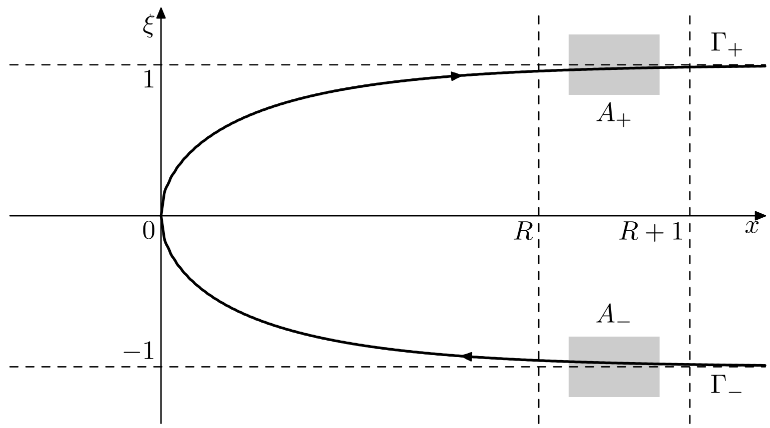

FIGURE 2. The characteristic set of $Q(h)$; compare with Figure 1

satisfies $\partial_x\tilde\varphi = \sqrt{1-e^{-2(x+X)}} + O(h)$ and $\operatorname{Im}\tilde\varphi = O(h)$ by (5.5), and $\tilde a^\pm(x;h) = a^\pm(e^{-2X}e^{-2x};h)$ is smooth in x and $h\geq 0$ uniformly in X. □

Now, let v be a nonzero solution to

$$Q(h)v = 0,\ \ v\in H^1(-\infty,0). \tag{5.16}$$

It follows from (5.7) that the Wronskian of v^+ and v^- is nonzero for h small enough; therefore, v is a linear combination of these two solutions:

$$v(x) = c_+v^+(x;h) + c_-v^-(x;h),\ \ x\geq R. \tag{5.17}$$

Here $c_\pm$ are constants depending on h and the choice of v. We can now use the facts from semiclassical analysis gathered in Section 2 to prove

Proposition 5.4. *There exists a constant C such that for h small enough, any solution v to* (5.16) *and $c_\pm$ defined by* (5.17), $|c_\pm/c_\mp|\leq C$.

Proof. Take two compactly supported and compactly microlocalized operators $A_\pm\in\Psi^0((R,R+1))$ such that $\mathrm{WF}_\hbar(A_\pm)\cap\Gamma_\mp=\emptyset$, and $A_\pm$ is elliptic at $\Gamma_\pm\cap\{x=R+1/2\}$. Then for some constant C (whose value will change with every new line),

$$\|A_\pm v\|_{L^2}\leq C|c_\pm| + O(h^\infty)|c_\mp|; \tag{5.18}$$

$$|c_\pm|\leq C\|A_\pm v\|_{L^2} + O(h^\infty)|c_\mp|. \tag{5.19}$$

Indeed, (5.18) follows from Proposition 2.3(1), while (5.19) follows from part 2 of the same proposition. Also, by (5.7) and (5.17),

$$\|v\|_{L^2(R,R+2)}\leq C(|c_+|+|c_-|). \tag{5.20}$$

Now, each forward Hamiltonian flow line of $\pm q$ starting at $\{q = 0\} \cap \{x \leq R\}$ reaches the elliptic set of $A_\pm$, while staying inside $\{0 \leq x \leq R + 1\}$; therefore, by Proposition 2.1,

$$\|v\|_{L^2(-2,R)} \leq \|A_\pm v\|_{L^2} + O(h^\infty)\|v\|_{L^2(-3,R+2)}. \tag{5.21}$$

Similarly

$$\|A_\pm v\|_{L^2} \leq C\|A_\mp v\|_{L^2} + O(h^\infty)\|v\|_{L^2(-3,R+2)};$$

using (5.18) and (5.19), we get

$$|c_\pm| \leq C|c_\mp| + O(h^\infty)\|v\|_{L^2(-3,R+2)}. \tag{5.22}$$

Finally, by Proposition 2.2,

$$\|v\|_{L^2(-\infty,-1)} = O(h^\infty)\|v\|_{L^2(-2,0)}. \tag{5.23}$$

Adding up (5.20), (5.21), (5.23), and using (5.18), we get

$$\|v\|_{L^2(-\infty,R+2)} \leq C(|c_+| + |c_-|); \tag{5.24}$$

it remains to substitute this into (5.22). □

6. Proofs of Theorems 1 and 2

First of all, we show that semiclassical measures are supported on the incoming set. The proof is based on the following property of the geodesic flow on manifolds with cusps: as $t \to -\infty$, each geodesic $\gamma(t)$ either goes straight into some cusp or it passes through a fixed compact set at arbitrarily large negative times.

Proposition 6.1. *Let $\mathcal{A}^-_{j'}$ be the sets defined in* (1.8) *and denote*

$$\mathcal{A}^- = \bigcup_{j'} \mathcal{A}^-_{j'}.$$

Then for any Radon measure μ satisfying parts 2 and 3 of Proposition 1.1, $\mu(S^*M \setminus \mathcal{A}^-) = 0$.

Proof. Take the compact set

$$K = M \setminus \bigcup_{j'} \mathcal{C}'_{j'},\ \mathcal{C}'_{j'} = \{(r,\theta) \in \mathcal{C}_{j'} \mid r > R_{j'} + 1\}. \tag{6.1}$$

Consider the sets

$$A_l = G^l(\{(x,\xi) \in S^*M \mid x \in K\}),\ l \in \mathbb{N}.$$

Then by (1.7),

$$\sum_{l\in\mathbb{N}} \mu(A_l) = \sum_{l\in\mathbb{N}} e^{-2\nu l}\mu(K) < \infty.$$

We are then done by Borel–Cantelli lemma, if we prove that

$$\rho \notin \mathcal{A}^-,\ l_0 \in \mathbb{N} \Longrightarrow \exists l \geq l_0 : \rho \in A_l. \tag{6.2}$$

To show (6.2), we first replace ρ by $G^{l_0}(\rho)$; we can then assume that $l_0 = 0$. If $\rho \in K$, then $\rho \in A_0$; therefore, we may assume that $\rho \in \mathcal{C}'_{j'}$ for some j'. Let $(r_0, \theta_0, p_{r0}, p_{\theta 0})$ be the canonical coordinates of ρ in $\mathcal{C}_{j'}$; since $\rho \notin \mathcal{A}^-_{j'}$, we have $p_{r0} \neq -1$; also, $r_0 > R_{j'}+1$. The equations of the backward geodesic $(r(t), \theta(t), p_r(t), p_\theta(t)) = G^{-t}(\rho)$ are

$$\dot r = -p_r,\ \dot p_r = e^{2r} p_\theta^2,\ \dot\theta = -e^{2r} p_\theta,\ \dot p_\theta = 0.$$

Consider a portion of the geodesic lying entirely in $\mathcal{C}_{j'}$. For $p_{\theta 0} \neq 0$, $\ddot r(t) = -e^{2r(t)} p_{\theta 0}^2 \leq -e^{2R_{j'}} p_{\theta 0}^2$ and thus $r(t)$ is a strictly concave function. For $p_{\theta 0} = 0$, since $p_{r0} \neq -1$ and $p(r_0, \theta_0, p_{r0}, p_{\theta 0}) = 0$, we have $\dot r = -p_{r0} = -1$. In either case, there exists $t > 0$ such that $r(t) = R_{j'} + 1$.

Choose minimal $t_0 > 0$ such that $r(t_0) = R_{j'} + 1$. Then $\dot r(t_0) = -p_r(t_0) \leq 0$; since $\dot p_r \geq 0$, we have $p_r(t) \geq 0$ for $t \geq t_0$ as long as we stay in $\mathcal{C}_{j'}$. However, as $p(r, \theta, p_r, p_\theta) = 0$, we have $|p_r(t)| \leq 1$. It follows that $G^{-t}\rho \in \mathcal{C}_{j'} \setminus \mathcal{C}'_{j'} \subset K$ for $t \in [t_0, t_0 + 1]$; we have proved (6.2) since the interval $[t_0, t_0 + 1]$ contains an integer point. □

We now use the analysis of Section 5 to obtain more detailed information on the behavior of the functions $u_{(n)}$ in the cusps:

Proposition 6.2. *Fix a cusp $\mathcal{C}_{j'}$ and define $\mathcal{C}'_{j'}$ by (6.1). Assume that $A \in \Psi^0((R_{j'}, R_{j'}+1))$ is a compactly supported operator such that $\mathrm{WF}_\hbar(A) \subset \{p_r < 0\}$, with principal symbol $\tilde a(r, p_r)$. Let $\chi \in C_0^\infty(-1, 1)$ be nonnegative and satisfy $\chi(0) = 1$. For each $\delta > 0$, define the operator*

$$A_\delta = \chi(hD_\theta/\delta) A \in \Psi^0(\mathcal{C}_{j'} \setminus \mathcal{C}'_{j'}).$$

Let $\lambda \in \mathbb{C}$ satisfy

$$\nu/2 < \operatorname{Im}\lambda < \nu,\ \operatorname{Re}\lambda > 1;$$

define h, ω by (1.5) and put $u(z) = E_j(\lambda; z)$ for some j. Then

$$A_\delta u(r, \theta) = \delta_{jj'} \tilde a(r, -1) e^{-ir/h} e^{(1/2 - i\omega) r} + O_{L^2}(\delta^\nu + h).$$

Proof. Consider the Fourier series (3.1) in the cusp $\mathcal{C}_{j'}$; then

$$A_\delta u(r, \theta) = \sum_{k,\ |hk| \leq \delta} \chi(hk/\delta) A u_k(r) e^{ik\theta}.$$

Using (3.6), we write (3.4) as

$$u_0(r) = u_0^+ e^{ir/h} e^{(1/2 + i\omega) r} + \delta_{jj'} e^{-ir/h} e^{(1/2 - i\omega) r}.$$

Since ω is bounded, we can treat $e^{(1/2\pm i\omega)r}$ as the amplitude; then $u_0(r)$ is a linear combination of two Lagrangian distributions associated to $\{p_r = \pm 1\}$. By Proposition 2.3,

$$Ae^{-ir/h}e^{(1/2-i\omega)r} = \tilde{a}(r,-1)e^{-ir/h}e^{(1/2-i\omega)r} + O_{L^2}(h),$$
$$Ae^{ir/h}e^{(1/2+i\omega)r} = O_{L^2}(h^\infty);$$

therefore, as $u_0^+ = O(1)$ by (3.2) and (3.8), we have

$$Au_0(r) = \delta_{jj'}\tilde{a}(r,-1)e^{-ir/h}e^{(1/2-i\omega)r} + O_{L^2}(h).$$

It remains to prove that for δ small enough,

$$k \neq 0,\ |hk| \leq \delta \Longrightarrow \|Au_k(r)\|_{L^2} = O(\delta^\nu + h^\infty)\|u_k(r)\|_{L^2(R_{j'},R_{j'}+1)}. \tag{6.3}$$

For that, put

$$v_k(x) = |hk|^{1/2}e^{x/2}u_k(-\log|hk| - x),\ k \neq 0. \tag{6.4}$$

It follows from (3.3) and (3.7) that v_k solves (5.16) on $(-\infty, -\log|hk| - R_{j'})$.

Let $v^\pm(x;h)$ be the solutions to (5.1) defined in Proposition 5.2 and let R be the constant from this proposition. Assume that δ is small enough so that $\log\delta + R_{j'} + 1 + R < 0$. By (5.17)

$$v_k(x) = c_k^+ v^+(x;h) + c_k^- v^-(x;h),\ -\log|hk| - R_{j'} - 1 < r < -\log|hk| - R_{j'},$$

for some constants $c_k^\pm$. Then by Proposition 5.3 (with $X = -\log|hk| - R_{j'} - 1$), we have

$$u_k(r) = \tilde{c}_k^+\tilde{u}_k^+(r;h) + \tilde{c}_k^-\tilde{u}_k^-(r;h),\ \tilde{c}_k^\pm = |hk|^{\mp \operatorname{Im}\omega}c_k^\pm,$$

where

$$\tilde{u}_k^\pm(r;h) = e^{r/2}e^{\mp(R_{j'}+1)\operatorname{Im}\omega}\tilde{v}^\mp(R_{j'}+1-r;h),\ R_{j'} < r < R_{j'}+1,$$

are Lagrangian distributions microlocalized in particular inside $\{\pm p_r > 0\}$. The sign reversal here happens because of the change of variables in r. By Proposition 2.3(1), we now have

$$A\tilde{u}_k^+(r;h) = O_{L^2}(h^\infty),\ A\tilde{u}_k^-(r;h) = O_{L^2}(1).$$

However, by Proposition 5.4, $|\tilde{c}_k^-| = O(\delta^\nu)|\tilde{c}_k^+|$; since

$$C^{-1} \leq \|\tilde{u}_k^\pm\|_{L^2(R_{j'},R_{j'}+1)} \leq C,$$

we have $|\tilde{c}_k^+| \leq C\|u_k\|_{L^2(R_{j'},R_{j'}+1)}$ and (6.3) follows. □

We are now ready to prove Theorems 1 and 2:

Proof of Theorem 1. By Proposition 1.1(1), it suffices to take a semiclassical measure μ associated to $\mathcal{C}_j$ and ν and show that $\mu = \mu_{j\nu}$. Fix a cusp $\mathcal{C}_{j'}$ and consider A_δ as in Proposition 6.2; also, take $b \in C^\infty(\mathbb{S}^1)$. By Proposition 6.2,

$$A_\delta E_j(\lambda_n) = w_n + O_{L^2}(\delta^\nu + h),\ w_n = \delta_{jj'}\tilde{a}(r,-1)e^{-ir/h}e^{(1/2-i\omega_n)r};$$

therefore,

$$\begin{aligned}&\langle A_\delta^* b(\theta) A_\delta E_j(\lambda_n), E_j(\lambda_n)\rangle_{L^2}\\ &= \langle b(\theta) A_\delta E_j(\lambda_n), A_\delta E_j(\lambda_n)\rangle_{L^2} = \langle b(\theta) w_n, w_n\rangle_{L^2} + O(\delta^\nu + h)\\ &= \delta_{jj'} \int_{\mathbb{S}^1} b(\theta)\,d\theta \cdot \int_{R_j}^{R_j+1} |\tilde a(r,-1)|^2 e^{2r \operatorname{Im}\lambda_n}\,dr + O(\delta^\nu + h).\end{aligned}$$

The principal symbol of $A_\delta^* b(\theta) A_\delta$ is $b(\theta)|\tilde a(r,p_r)|^2\chi(p_\theta/\delta)^2$; passing to the limit $n \to \infty$, we get

$$\int_{S^*\mathcal{C}_{j'}} b(\theta)|\tilde a(r,p_r)|^2 \chi(p_\theta/\delta)^2\,d\mu = \delta_{jj'} \int_{S^*M \cap \widehat{\mathcal{A}}_{j'}^-} b(\theta)|\tilde a(r,p_r)|^2 e^{2\nu r}\,dr d\theta + O(\delta^\nu).$$

We now let $\delta \to 0$. The left-hand side converges to the integral of $b(\theta)|\tilde a(r,p_r)|^2$ over the restriction of μ to $\widehat{\mathcal{A}}_{j'}^-$; since $\tilde a$ and b were chosen arbitrarily, we get

$$\mu|_{\widehat{\mathcal{A}}_{j'}^- \cap T^*(\mathcal{C}_{j'} \setminus \mathcal{C}'_{j'})} = \delta_{jj'} e^{2\nu r}\,dr d\theta.$$

Using (1.7), we have

$$\mu|_{\mathcal{A}_{j'}^-} = \delta_{jj'} \mu_{j\nu};$$

we are now done by Proposition 6.1. □

Proof of Theorem 2. Fix a cusp $\mathcal{C}_{j'}$ and take compactly supported and compactly microlocalized $A \in \Psi^0((R_{j'}, R_{j'}+1))$ such that $\operatorname{WF}_\hbar(A) \subset \{p_r > 0\}$, and the principal symbol $a(r,p_r)$ of A satisfies $a(R_{j'}+1/2, 1) \neq 0$. Let $\chi \in C_0^\infty(\mathbb{R})$ have $\chi(0) = 1$. Denote $u_{(n)} = E_j(\lambda_n)$ and recall that $S_{jj'}(\lambda_n) = u_{(n)0}^{+(j')}$ is defined by (3.4). We then have for each $\delta > 0$,

$$|u_{(n)0}^{+(j')}| = O(1)\|A u_{(n)0}^{(j')}\|_{L^2} + O(h_n^\infty) = O(1)\|A_\delta u_{(n)}\|_{L^2(M)} + O(h_n^\infty), \tag{6.5}$$

uniformly in δ, where $A_\delta = \chi(hD_\theta/\delta)A$. However, by Theorem 1 we have as $n \to \infty$

$$\|A_\delta u_{(n)}\|_{L^2(M)}^2 = \langle A_\delta^* A_\delta u_{(n)}, u_{(n)}\rangle_{L^2(M)} \to \int_{S^*\mathcal{C}_{j'}} |\chi(hp_\theta/\delta) a(r,p_r)|^2\,d\mu_{j\nu}.$$

By our assumption, $\mu_{j\nu}(\widehat{\mathcal{A}}_{j'}^+) = 0$; therefore,

$$\lim_{\delta\to 0}\lim_{n\to\infty} \|A_\delta u_{(n)}\|_{L^2(M)} = 0$$

and we are done by (6.5). □

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E-mail address: dyatlov@math.berkeley.edu

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA