Math 54-1 Your name: Key
Quiz 11, August 3, 2010

Please write your name **on each sheet**. Show your work clearly and in order, including intermediate steps in the solutions and the final answer.

1. (4 pt) Find the least-squares solution to the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Normal system:

$(A^T A)\hat{x} = \hat{b} = A^T\vec{b}$. Compute $A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} =$

$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $A^T\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \to \hat{x}_1 = \hat{x}_2 = \frac{1}{3}$

2. (6 pt) Consider the space $\mathbb{P}_2$ of polynomials of degree no more than 2, equipped with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt, \quad f, g \in \mathbb{P}_2.$$

Starting from the basis $\{1, t, t^2\}$, use the Gram–Schmidt algorithm to find an orthogonal basis of $\mathbb{P}_2$ with respect to the inner product introduced above.

We find
$$\langle 1, 1\rangle = \int_{-1}^{1} dt = 2$$
$$\langle 1, t\rangle = \int_{-1}^{1} t\,dt = 0$$
$$\langle 1, t^2\rangle = \langle t, t\rangle = \int_{-1}^{1} t^2\,dt = \frac{2}{3}$$
$$\langle t, t^2\rangle = \int_{-1}^{1} t^3\,dt = 0$$

Then, our basis $\{f_1, f_2, f_3\}$ is constructed as follows:

$$f_1 = 1$$
$$f_2 = t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle} 1 = t - 0 = t$$
$$f_3 = t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle} t =$$
$$= t^2 - \frac{1}{3} - 0 \cdot t = t^2 - \frac{1}{3}.$$

So, an orthogonal basis is $\{1, t, t^2 - \frac{1}{3}\}$.