

Properties of matrix operations

The operations are as follows:

- Addition: if A and B are matrices of the same size $m \times n$, then $A + B$, their sum, is a matrix of size $m \times n$.
- Multiplication by scalars: if A is a matrix of size $m \times n$ and c is a scalar, then cA is a matrix of size $m \times n$.
- Matrix multiplication: if A is a matrix of size $m \times n$ and B is a matrix of size $n \times p$, then the product AB is a matrix of size $m \times p$.
- Vectors: a vector of length n can be treated as a matrix of size $n \times 1$, and the operations of vector addition, multiplication by scalars, and multiplying a matrix by a vector agree with the corresponding matrix operations.
- Transpose: if A is a matrix of size $m \times n$, then its transpose A^T is a matrix of size $n \times m$.
- Identity matrix: I_n is the $n \times n$ identity matrix; its diagonal elements are equal to 1 and its offdiagonal elements are equal to 0.
- Zero matrix: we denote by 0 the matrix of all zeroes (of relevant size).
- Inverse: if A is a **square** matrix, then its inverse A^{-1} is a matrix of the same size. Not every square matrix has an inverse! (The matrices that have inverses are called **invertible**.)

The properties of these operations are (assuming that r, s are scalars and the sizes of the matrices A, B, C are chosen so that each operation is well defined):

$$A + B = B + A, \tag{1}$$

$$(A + B) + C = A + (B + C), \tag{2}$$

$$A + 0 = A, \tag{3}$$

$$r(A + B) = rA + rB, \tag{4}$$

$$(r + s)A = rA + sA, \tag{5}$$

$$r(sA) = (rs)A; \tag{6}$$

$$A(BC) = (AB)C, \tag{7}$$

$$A(B + C) = AB + AC, \tag{8}$$

$$(B + C)A = BA + CA, \tag{9}$$

$$r(AB) = (rA)B = A(rB), \quad (10)$$

$$I_m A = A = A I_n; \quad (11)$$

$$(A^T)^T = A, \quad (12)$$

$$(A + B)^T = A^T + B^T, \quad (13)$$

$$(rA)^T = rA^T, \quad (14)$$

$$(AB)^T = B^T A^T, \quad (15)$$

$$(I_n)^T = I_n; \quad (16)$$

$$AA^{-1} = A^{-1}A = I_n, \quad (17)$$

$$(rA)^{-1} = r^{-1}A^{-1}, \quad r \neq 0, \quad (18)$$

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (19)$$

$$(I_n)^{-1} = I_n, \quad (20)$$

$$(A^T)^{-1} = (A^{-1})^T, \quad (21)$$

$$(A^{-1})^{-1} = A. \quad (22)$$

We see that in many cases, we can treat addition and multiplication of matrices as addition and multiplication of numbers. However, here are some differences between operations with matrices and operations with numbers:

- Note the reverse order of multiplication in (15) and (19).
- (19) can only be applied if we know that both A and B are invertible.
- In general, $AB \neq BA$, even if A and B are both square. If $AB = BA$, then we say that A and B **commute**.
- For a general matrix A , we cannot say that $AB = AC$ yields $B = C$. (However, if we know that A is invertible, then we can multiply both sides of the equation $AB = AC$ to the left by A^{-1} and get $B = C$.)
- The equation $AB = 0$ does not necessarily yield $A = 0$ or $B = 0$. For example, take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$