Math 54, Section 214 Your name: Key
Quiz 5, March 5, 2010

Please write your name on each sheet. Show your work clearly and in order, including intermediate steps in the solutions and the final answer.

1. (7 pt) Consider the subspace

$$V = \left\{ \begin{bmatrix} a+b \\ a+c \\ b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \subset \mathbb{R}^3.$$

(a) Find a matrix A such that $V = \mathrm{Col}\, A$.

(b) Find a basis for V and the dimension of V.

(c) [Extra credit, no points] Find a matrix B such that $V = \mathrm{Nul}\, B$.

(a) We have $V = \left\{ a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\} =$

$= \mathrm{Col}\, A$, when $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(b) Row reduce: $A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$;

Basis for $V = \mathrm{Col}\, A$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$; $\dim V = 2$

(c) If $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in V$, then $x_1 - x_2 = x_3$; therefore,

V is contained in Nul B, where $B = [\underset{3}{1} \;\; -1 \;\; -1]$.

B has rank $1 \Rightarrow \dim \mathrm{Nul}\, B = 3 - 1 = 2 = \dim V$;

therefore, $V = \mathrm{Nul}\, B$.

2. (6 pt) Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 10 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

Explain.

Row reduce: $A \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

Basis for the row space: $\{[1, 2, 0], [0\ 3\ 1]\}$.

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3. (7 pt) Let $\mathbb{P}_n$ be the space of polynomials in the variable t of degree no more than n. Find the rank of the linear transformation $T : \mathbb{P}_1 \to \mathbb{P}_2$ that maps each polynomial $P(t) \in \mathbb{P}_1$ to the polynomial $(1+t)P(t)$. Explain.

Solution 1: $\dim \mathbb{P}_1 = 2$, $\dim \mathbb{P}_2 = 3$.

T is 1-to-1 because if $(1+t)P(t) \equiv 0$, then $P(t) \equiv 0$. So, $\dim \mathrm{Nul}\, T = 0$ and $\mathrm{rank}\, T = 2 - \dim \mathrm{Nul}\, T = 2$.

Solution 2: Consider the bases $\{1, t\}$ for $\mathbb{P}_1$ and $\{1, t, t^2\}$ for $\mathbb{P}_2$ and let A be the matrix of T wrt these bases.

We know: $T(1) = 1+t$,

$T(t) = (1+t)t = t + t^2$.

Therefore, $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$. Row reduce A:

$\to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$, rank $A = 2$.