MATH 279 HOMEWORK 3

1. Let $U \subset \mathbb{R}^m$ be an open set and consider the oscillatory integral (see September 11 lecture or Remark (ii) following [Zw, Theorem 3.18])

$$u(z) = \int_{\mathbb{R}^k} e^{i\varphi(w,z)} a(w,z)\, dw := \lim_{\epsilon \to 0+} \int_{\mathbb{R}^k} e^{i\varphi(w,z)} a(w,z)\chi(\epsilon w)\, dw, \quad z \in U$$

where $\chi \in C_c^\infty(\mathbb{R}^k)$ satisfies $\chi = 1$ near 0. Here:

- $\varphi(w,z) = \frac{1}{2}\langle A(z)w, w\rangle + \langle b(z), w\rangle + c(z)$ where $A(z)$ is an invertible symmetric matrix, $b(z) \in \mathbb{R}^k$, and $c(z) \in \mathbb{R}$, all depending smoothly on z;
- $a \in C^\infty(\mathbb{R}^k \times U)$ and there exists M such that for each multiindex α and each compact set $K \subset U$ we have the derivative bounds

$$|\partial^\alpha_{(w,z)} a(w,z)| \le C_{\alpha,K} \langle w\rangle^M, \quad z \in K.$$

(a) Show that $u \in C^\infty(U)$.

(b) In the special case when φ is z-independent prove the estimate

$$|\partial_z^\beta u(z)| \le C \sup_{|\alpha| \le |\beta| + M + k + 1} \sup_w |\langle w\rangle^{-M} \partial^\alpha_{(w,z)} a(w,z)|, \quad z \in U.$$

2. Assume that $a(\xi) \in C^\infty(\mathbb{R}^n)$ satisfies the derivative bounds for some M and all α

$$|\partial_\xi^\alpha a(\xi)| \le C_\alpha \langle \xi\rangle^{M-|\alpha|}. \tag{0.1}$$

Show that the Fourier transform $\hat{a}(x)$ is in C^∞ when restricted to $\mathbb{R}^n \setminus \{0\}$. (Hint: a proper way to do this would be to write $\hat{a}(x) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} a(\xi)\, d\xi$ in the sense of oscillatory integrals and use the strategy of the previous exercise. However there is a much shorter solution.)

Remark: distributions of the form $\hat{a}(x)$ with a satisfying (0.1) are called *conormal* to $0 \in \mathbb{R}^n$. This class includes the delta function and its derivatives (when a is polynomial in ξ) but is much more rich – for instance it includes the cutoff fundamental solution to the Laplace's operator (e.g. $\frac{\chi(x)}{4\pi|x|}$ in $\mathbb{R}^3$) where $\chi \in C_c^\infty(\mathbb{R}^3)$ where $a(\xi) \sim |\xi|^{-2}$ as $\xi \to \infty$.

References

[Zw] Maciej Zworski, *Semiclassical analysis,* Graduate Studies in Mathematics **138**, AMS, 2012.