Resonance expansion

Wave equation: $P_V = -\Delta + V$, $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$, n odd

$$\begin{cases} (\partial_t^2 + P_V)\, w(t,x) = 0, & t \geq 0,\ x \in \mathbb{R}^n \\ w(0,x) = w_0(x) \in H^1_{comp}(\mathbb{R}^n) \\ \partial_t w(0,x) = w_1(x) \in L^2_{comp}(\mathbb{R}^n) \end{cases}$$

Assume $w_j = \chi w_j$, $\chi \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}\chi \subset B(0,K)$

and we are given $A > 0$.

Then $\exists\ T_0 = T(K,V)$, $C = C(K,V,A)$

such that

$$w(t,x) = \sum_{\substack{\lambda_j \text{ resonance} \\ \operatorname{Im}\lambda_j \geq -A}} \sum_{\ell=0}^{L_j - 1} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x) + E_A(t),$$

where $\|\chi E_A(t)\|_{H^2} \leq C e^{-tA} (\|w_0\|_{H^1} + \|w_1\|_{L^2})$, $t \geq T$.

Proof. We will show a somewhat weaker statement (with worse remainder bounds) under some simplifying assumptions.

* $w_0 \equiv 0$, $w_1 \in C_c^\infty(\mathbb{R}^n)$
* P_V has no resonances in $\operatorname{Im}\lambda > 0$ (e.g. $V \geq 0$)
* P_V has no resonances on the line $\{\operatorname{Im}\lambda = -A\}$.

1) <u>Write the solution via the scattering resolvent:</u>

First of all, using the spectral theory of P_V we write

$w(t) = \dfrac{\sin(t\sqrt{P_V})}{\sqrt{P_V}} w_1$. That is, applying the entire fn. $\lambda \mapsto \dfrac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}$ to P_V which is <u>self-adjoint.</u>

No resonances in $\{\operatorname{Im}\lambda > 0\}$

Spectrum $(P_V) \subset [0,\infty) \Rightarrow \sup_{\lambda \in \operatorname{Spec}} \left| \dfrac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} \right| \leq Ct$.

So $\|w(t)\|_{L^2} \leq Ct\|w_1\|_{L^2}$. And as $w_1 \in C_c^\infty$

$\|P_V w(t)\|_{L^2} = \left\| \sqrt{P_V}\dfrac{\sin(t\sqrt{P_V})}{\sqrt{P_V}} P_V w_1 \right\|_{L^2} \leq Ct\|P_V w_1\|_{L^2} \leq Ct$

Together these statements give

$$\|w(t)\|_{H^2} \leq Ct. \quad \leftarrow \text{a priori bound}$$

Take $\hat{w}(\lambda) := \int_0^\infty e^{it\lambda} w(t)\,dt$, Im $\lambda > 0$

The integral converges, giving $\hat{w}(\lambda) \in H^2$, Im $\lambda > 0$

[if P_V had ~~res~~ negative eigenvalues, could do this for Im $\lambda \gg 1$]

Fourier transform the wave equation, taking care of the boundary terms:

$$\widehat{\partial_t^2 w}(\lambda) = \int_0^\infty e^{it\lambda} w''(t)\,dt = -w_1 - \int_0^\infty \lambda^2 e^{it\lambda} w(t)\,dt.$$

So $-\widehat{P_V w}(\lambda)$. Thus

$$(P_V - \lambda^2)\hat{w}(\lambda) = w_1, \quad \text{Im}\,\lambda > 0$$

It follows that $\hat{w}(\lambda) = R_V(\lambda) w_1$, Im $\lambda > 0$

where $R_V(\lambda)$ is the scattering resolvent.

Apply Fourier inversion formula to

$$\tilde{w}(t) = \begin{cases} e^{-t} w(t), & t > 0 \\ 0, & t < 0 \end{cases}.$$

$\hat{\tilde{w}}(\lambda) = \hat{w}(\lambda + i)$, $\lambda \in \mathbb{R}$. So,

$$\tilde{w}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda t} R_V(\lambda + i)\, w_1\, d\lambda$$

We get the formula we want:

$$\boxed{w(t) = \frac{1}{2\pi} \int_{\text{Im}\,\lambda = 1} e^{-i\lambda t} R_V(\lambda) w_1\, d\lambda.}$$

2 The contour deformation argument:

From the high frequency resolvent bound we had last time we get in particular:

for $-A \le \operatorname{Im}\lambda \le 1$, $|\operatorname{Re}\lambda| \gg 1$,

λ is not a resonance and

$$\|\chi R_V(\lambda) w_1\|_{H^s} \le C |\lambda|^{s-1} \cdot \|w_1\|_{L^2}, \quad s=0,1,2$$

Want integrable decay in λ however.

For that we write

$$R_V(\lambda)(P_V-\lambda^2)w_1 = w_1$$

(note: w_1 compactly supported can use analytic continuation.)

Thus $\lambda^2 R_V(\lambda) w_1 = R_V(\lambda) P_V w_1 - w_1$. So,

$$\|\chi R_V(\lambda) w_1\|_{H^1} \le \frac{\|\chi R_V(\lambda) P_V w_1\|_{H^1} + \|w_1\|_{H^1}}{|\lambda|^2}$$

$$\le \frac{C|\lambda|^{-1}\|P_V w_1\|_{L^2} + \|w_1\|_{H^1}}{|\lambda|^2}$$

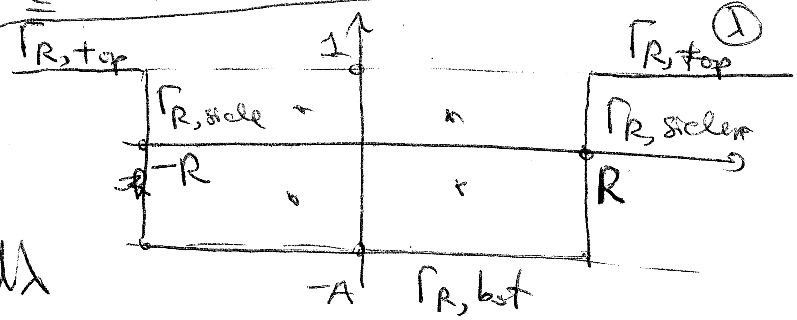
(lost some derivatives here...)

$$\le C|\lambda|^{-2}\|w_1\|_{H^2}.$$

Again: $-A \le \operatorname{Im}\lambda \le 1$, $|\operatorname{Re}\lambda| \gg 1$

$\Downarrow$

$$\|\chi R_V(\lambda) w_1\|_{H^1} \le C|\lambda|^{-2}\|w_1\|_{H^2}.$$



Deform the contour:

$$w(t) = \frac{1}{2\pi}\int_{\Gamma_{R,top}+\Gamma_{R,side}+\Gamma_{R,bot}} e^{-i\lambda t}\chi R_V(\lambda) w_1 \, d\lambda + \boxed{\text{residues}}$$

Let $R \to \infty$. Using the $|\lambda|^{-2}$ bound, we see that $\int_{\Gamma_{R,top} + \Gamma_{R,side}} \to 0$,

$\int_{\Gamma_{R,bot}} \to \int_{\operatorname{Im}\lambda = -A} (\ldots)$. So, we get

<u>another useful formula:</u>

$$\chi w(t) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = -A} e^{-it\lambda} \chi R_V(\lambda) w_1 \, d\lambda + \boxed{\text{residues}}$$

③ <u>Endgame</u>: the $\int_{\operatorname{Im}\lambda = -A} \cdots =: \chi E_A(t)$

satisfies $\|\chi E_A(t)\|_{H^1} \le C e^{-At} \|w_1\|_{H^2}$

(how to get a better bound? see the book, need to work a bit more...)

What about $\boxed{\text{residues}}$?

Each of them comes from a resonance...

At λ_j: $\operatorname{Res}_{\lambda=\lambda_j} e^{-i\lambda t} \chi R_V(\lambda) w_1$, write

$$R_V(\lambda) = (\text{Holomorphic at } \lambda_j) + \sum_{\ell=1}^{L_j} \frac{A_{j\ell}}{(\lambda - \lambda_j)^\ell}, \quad A_{j\ell} : L^2_{comp} \to H^2_{loc} \text{ finite rank}$$

$$\operatorname{Res}_{\lambda=\lambda_j} e^{-i\lambda t} \chi R_V(\lambda) w_1 = \sum_{\ell=1}^{L_j} \frac{1}{(\ell-1)!} \left(\partial_\lambda^{\ell-1} e^{-i\lambda t}\right)\Big|_{\lambda=\lambda_j} \cdot \chi A_{j\ell} w_1$$

$$= \sum_{\ell=0}^{L_j-1} t^\ell e^{-i\lambda_j t} \underbrace{\frac{(-i)^\ell}{\ell!} \chi A_{j\ell} w_1}_{f_{j,\ell}} . \quad \square$$

Outgoing asymptotics

Recall: $u \in H^2_{loc}$ is outgoing at $\lambda \in \mathbb{C}\setminus\{0\}$ if $u = R_0(\lambda) f$ for some $f \in L^2_{comp}$.

When λ is real, we can get an asymptotic expansion of $u(x)$ as $|x| \to \infty$:

Thm (see Thm. 3.5 in the book) Assume $f \in L^2_{comp}$.

Then for $\lambda \in \mathbb{R}\setminus\{0\}$, $r>0$, $\theta \in \mathbb{S}^{n-1}$

$$R_0(\lambda) f(r\theta) = e^{i\lambda r} r^{-\frac{n-1}{2}} g(r,\theta), \quad r>0, \quad \theta \in \mathbb{S}^{n-1};$$

$$g(r,\theta) \sim \sum_{j=0}^{\infty} r^{-j} g_j(\theta) \quad \text{as} \quad r \to \infty,$$

$$g_0(\theta) = \frac{1}{4\pi}\left(\frac{\lambda}{2\pi i}\right)^{\frac{n-3}{2}} \hat{f}(\lambda \cdot \theta).$$

Proof Will do the case $n=3$, using the formula

$$R_0(\lambda) f(x) = \int_{\mathbb{R}^3} \frac{e^{i\lambda|x-y|} \cdot f(y)}{4\pi|x-y|} dy.$$

Here y is bounded (as supp f is cpct) & $|x| \to \infty$.

$$R_0(\lambda) f(r\theta) = \frac{e^{i\lambda r}}{r} \int_{\mathbb{R}^3} \frac{e^{i\lambda r(|\theta - \frac{y}{r}| - 1)}}{4\pi|\theta - \frac{y}{r}|} f(y)\, dy.$$

It remains to take the Taylor expansion of $\dfrac{e^{i\lambda r(|\theta - \frac{y}{r}|-1)}}{4\pi|\theta - \frac{y}{r}|}$ in $\frac{1}{r} \to 0$.

The first term? We have

$$|\theta - \tfrac{y}{r}| = \sqrt{1 - \frac{2\langle\theta,y\rangle}{r} + O\left(\frac{1}{r^2}\right)} = 1 - \frac{\langle\theta,y\rangle}{r} + O\left(\frac{1}{r^2}\right), \text{ so}$$

$$\frac{e^{i\lambda r(|\theta - \frac{y}{r}|-1)}}{4\pi|\theta - \frac{y}{r}|} = \frac{e^{-i\lambda\langle\theta,y\rangle}}{4\pi} \cdots \qquad \Box$$

Note: u outgoing at $\lambda \in \mathbb{R}\setminus\{0\}$ $\to$ have a similar expansion for $\partial_r u$

u satisfies the Sommerfeld Radiation Condition:

$$\text{(SRC)} \quad (\partial_r - i\lambda) u(r;\theta) = o\left(r^{-\frac{n-1}{2}}\right), \quad r \to \infty, \ \theta \in \mathbb{S}^{n-1}.$$

Rellich's Uniqueness Theorem (Thm 3.30)

Assume $\lambda \in \mathbb{R} \setminus \{0\}$ & u is an outgoing solution to $(P_V - \lambda^2)u = 0$.

Then $u \equiv 0$.

Remarks 1) Implies there are no resonances on $\mathbb{R}\setminus\{0\}$.

In particular, it gives the ~~following~~ limiting absorbtion principle which is ~~often~~ true under weaker assumptions on V (in particular in some cases with no merom. cont. of $R_V(\lambda)$):

λ: $\exists \lim_{\varepsilon \to 0+} \chi(P_V - (\lambda + i\varepsilon)^2)^{-1}\chi = \chi R_V(\lambda)\chi$ in norm $L^2 \to H^2$ $\forall \chi \in C_c^\infty$

2) One can show a stronger statement (Thm 3.32): $(P_V - \lambda^2)u = 0$, u satisfies (SRC) $\Rightarrow u \equiv 0$. $\lambda \in \mathbb{R}\setminus\{0\}$,

3) The proof (unlike most of the stuff discussed below) uses strongly that V is real-valued.

4) Given the resonance expansion, we see that we get exponential local energy decay of ~~waves~~ solutions to $(\partial_t^2 + P_V)w = 0$ assuming (a) P_V has no eigenvalues < 0

(b) P_V has no resonance at 0.

Proof Write $u = R_0(\lambda) f$ for some $f \in L^2_{comp}$.

We have $u(r\theta) = e^{i\lambda r} r^{-\frac{n-1}{2}} a(\theta) + O(r^{-\frac{n+1}{2}})$, $r \to \infty$

for $a(\theta) = c_\lambda \hat{f}(\lambda\theta)$, $c_\lambda \neq 0$.

3 steps of the proof:

1. $a \equiv 0$
2. u is compactly supported
3. $u \equiv 0$.

Step 1 Take large $R>0$

& integrate by parts on the ball $B(0,R)$:

$$0 = \int_{B(0,R)} \bar{u}\cdot(P_V-\lambda^2)u - u\cdot(P_V-\lambda^2)\bar{u}\,dx$$

$$= \int_{B(0,R)} u\cdot\Delta\bar{u} - \bar{u}\cdot\Delta u\,dx = \int_{\partial B(0,R)} u\cdot\partial_r\bar{u} - \bar{u}\cdot\partial_r u.$$

Using the asymptotics

$$u(r\theta) = e^{i\lambda r} r^{-\frac{n-1}{2}} a(\theta) + O(r^{-\frac{n+1}{2}})$$

$$\partial_r u(r\theta) = i\lambda e^{i\lambda r} r^{-\frac{n-1}{2}} a(\theta) + O(r^{-\frac{n+1}{2}})$$

this is equal to:

$$0 = -2i\lambda\int_{S^{n-1}} |c_\lambda|^2\cdot|a(\theta)|^2 dS(\theta) + O(R^{-1})$$

Taking $R\to\infty$, we get $a\equiv 0$.

Step 2 To show u is compactly supported, we use Paley-Wiener Thm for Fourier transform. It's convenient to state it for the class $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions, dual to $\mathcal{S}(\mathbb{R}^n)$, Schwartz functions

$\mathcal{E}'(\mathbb{R}^n)\subset\mathcal{S}'(\mathbb{R}^n)$ compactly supported distributions

Paley-Wiener Thm (PWT)

1. If $f\in\mathcal{E}'(\mathbb{R}^n)$ then $\hat{f}\in C^\infty$ extends to an entire fn. on $\mathbb{C}^n$ and (*) $\exists N,C$: $|\hat{f}(\xi)|\le C(1+|\xi|)^N e^{C|\operatorname{Im}\xi|}$ for all $\xi\in\mathbb{C}^n$.
2. If $f\in\mathcal{S}'(\mathbb{R}^n)$, $\hat{f}\in C^\infty$ extends to an entire fn. on $\mathbb{C}^n$, and (*) holds, then $f\in\mathcal{E}'(\mathbb{R}^n)$.

This is a beautiful Thm – if you don't know the proof, ask me and I'll explain it! Hörmander, Vol. I, Thm 7.3.1

Coming back to Step 2:

we know $a(\theta) \equiv 0$. But $a(\theta) = c_\lambda \hat{f}(\lambda\theta)$.

Therefore, if $\mathcal{H} = \{\xi \in \mathbb{C}^n \mid \underbrace{\langle \xi, \xi \rangle_{\mathbb{C}^n}}_{\sum_1^n \xi_j^2} = \lambda^2\} \subset \mathbb{C}^n$,

then $\hat{f}|_{\mathcal{H} \cap \mathbb{R}^n} = 0$.

By PWT, $f \in L^2_{comp} \subset \mathcal{E}' \Rightarrow \hat{f}$ is entire (on $\mathbb{C}^n$)

and $|\hat{f}(\xi)| \leq C(1+|\xi|)^N e^{C|\operatorname{Im}\xi|}$, $\xi \in \mathbb{C}^n$.

2 things from complex analysis (need more details in principle):

① $\hat{f}|_{\mathcal{H} \cap \mathbb{R}^n} = 0 \Rightarrow \hat{f}|_{\mathcal{H}} = 0$

② $\hat{f}|_{\mathcal{H}} = 0 \Rightarrow \hat{f}(\xi) = (\langle \xi, \xi \rangle - \lambda^2) F(\xi)$, F entire

(how to show these? do it locally, ~~taking~~ a change of var's to replace $\langle \xi, \xi \rangle - \lambda^2$ by ξ_1)

Now, for $\varepsilon > 0$ $R_0(\lambda + i\varepsilon) f \in H^2 \subset \mathcal{S}'(\mathbb{R}^n)$ and ~~for~~ $\xi \in \mathbb{R}^n$,

$$\widehat{R_0(\lambda+i\varepsilon)f}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 - (\lambda+i\varepsilon)^2} \xrightarrow{\varepsilon \to 0} F(\xi) \text{ in } \mathcal{S}'.$$

Thus $u = R_0(\lambda) f = \lim_{\varepsilon \to 0+} R_0(\lambda + i\varepsilon) f$ in $\mathcal{D}'$

actually lies in $\mathcal{S}'$ and $\hat{u}(\xi) = F(\xi)$. ($\xi \in \mathbb{R}^n$)

Now $|F(\xi)| \leq C(1+|\xi|)^N e^{C|\operatorname{Im}\xi|}$ $\xi \in \mathbb{C}^n$.

(in principle needs a bit of checking — ~~easiest~~ to use Cauchy estimates / Taylor series)

Thus by PWT $u \in \mathcal{E}'$ as needed, i.e.

u is compactly supported.