$P_V = -\Delta + V : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, n odd.

Our goal is to prove

Thm 1 The resolvent

$$R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n),$$
$$\operatorname{Im}\lambda > 0$$

has a meromorphic continuation to (with poles of finite rank – see after the proof)

$$R_V(\lambda) : L^2_{comp}(\mathbb{R}^n) \to H^2_{loc}(\mathbb{R}^n), \quad \lambda \in \mathbb{C}$$

Here $L^2_{comp} = \{ u \in L^2(\mathbb{R}^n) \mid \operatorname{supp} u \Subset \mathbb{R}^n \}$

$H^2_{loc} = \{ u \in \mathcal{D}'(\mathbb{R}^n) \mid \forall \chi \in C_c^\infty(\mathbb{R}^n), \chi u \in H^2 \}$

The free resolvent

$$R_0(\lambda) = (-\Delta - \lambda^2)^{-1} : L^2 \to H^2, \quad \operatorname{Im}\lambda > 0$$

Thm 2 (from last lecture)

$R_0(\lambda)$ has a meromorphic continuation to

$$R_0(\lambda) : L^2_{comp} \to H^2_{loc}, \quad \lambda \in \mathbb{C}$$

& has no poles unless $n=1$, $\lambda = 0$.

There are explicit formulas for $R_0(\lambda)$, e.g.

$n=1$: $R_0(\lambda) f(x) = \frac{i}{2\lambda} \int_{\mathbb{R}} e^{i\lambda|x-y|} f(y)\, dy$

$n=3$: $R_0(\lambda) f(x) = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{e^{i\lambda|x-y|}}{|x-y|} f(y)\, dy$

(see the book, Thm 3.3 for more)

n even $\to$ $R_0(\lambda)$ continues instead to the log-cover of $\mathbb{C}$...

Also, recall for $\operatorname{Im}\lambda > 0$,

$$\widehat{R_0(\lambda) f}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 - \lambda^2}.$$

So for $s>0$, $\|R_0(is)\|_{L^2 \to L^2} = \frac{1}{s^2}$.

<u>Proof of Thm 1</u>

(1) Let's find another formula for $\operatorname{Im}\lambda > 0$.

Will ~~take~~ $\lambda = is$, $s \gg 1$.

The idea is to use $R_0(\lambda)$ as an <u>approximate inverse</u>:

$$(P_V - \lambda^2) R_0(\lambda) = (-\Delta - \lambda^2 + V) R_0(\lambda) = I + VR_0(\lambda)$$

If $\lambda = is$, $s \gg 1$, then $\|VR_0(\lambda)\|_{L^2 \circlearrowleft} \le \frac{\|V\|_{L^\infty}}{s^2} < \frac{1}{2}$.

Then $(I + VR_0(\lambda))^{-1} = \sum_{j \ge 0}^{\infty} (-VR_0(\lambda))^j : L^2 \circlearrowleft$ exists.

We get $R_V(\lambda) = (P_V - \lambda^2)^{-1} = R_0(\lambda)(I + VR_0(\lambda))^{-1}$.

(2) We want to put a cutoff on the other side of $VR_0(\lambda)$ as well. Fix $\rho \in C_c^\infty(\mathbb{R}^n)$: $\rho = 1$ on $\operatorname{supp} V$, $\rho V = V$

~~Then $I + VR_0(\lambda) = (I + \rho VR_0(\lambda))(I + (1-\rho)VR_0(\lambda))$~~

~~$= (I + VR_0(\lambda)\rho)(I + VR_0(\lambda)(1-\rho))$~~

Then $I + VR_0(\lambda) = (I + VR_0(\lambda)(1-\rho))(I + VR_0(\lambda)\rho)$

And $(I + VR_0(\lambda)(1-\rho))^{-1} = I - VR_0(\lambda)(1-\rho)$

(take Neumann series again; $(VR_0(\lambda)(1-\rho))^2 = 0$).

Thus $(I + VR_0(\lambda))^{-1} = (I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1-\rho))$

Thus we write for $\lambda = is$, $s \gg 1$,

$$(*)\ R_V(\lambda) = R_0(\lambda)\left(I + VR_0(\lambda)\rho\right)^{-1}\left(I - VR_0(\lambda)(1-\rho)\right).$$

3 Now take any $\lambda \in \mathbb{C}$ (except $\lambda = 0, n=1$, more work needed since $R_0(\lambda)$ has a pole).

Note: $R_0(\lambda): L^2_{comp} \to H^2_{loc}$

$I - VR_0(\lambda)(1-\rho): L^2_{comp} \to L^2_{comp}$

$VR_0(\lambda)\rho: L^2 \to H^2$, compactly supported fns in range $\Rightarrow$

$\Rightarrow$ by Rellich's Thm $VR_0(\lambda)\rho: L^2 \to L^2$ compact.

So $I + VR_0(\lambda)\rho : L^2 \to L^2$ Fredholm, all λ!

Will use

Thm 3 [Analytic Fredholm Theory]

Assume $\Omega \subset \mathbb{C}$ open, connected, $\mathcal{H}$ Hilbert space, $A(\lambda): \mathcal{H} \to \mathcal{H}$, $\lambda \in \Omega$, holomorphic family of Fredholm operators

& $\exists \lambda_0 \in \Omega$: $A(\lambda_0)$ invertible.

Then $A(\lambda)^{-1}: \mathcal{H} \to \mathcal{H}$ is a meromorphic family of operators w/ poles of finite rank

Using Thm 3 (note $I + VR_0(\lambda)\rho$ invertible for $\lambda = iS$, $S \gg 1$),

set $(I + VR_0(\lambda)\rho)^{-1} : L^2 \to L^2$ ~~mero~~ meromorphic.

Note: $(I + VR_0(\lambda)\rho)^{-1} = I \mp VR_0(\lambda)\rho\,(I + VR_0(\lambda)\rho)^{-1}$,

thus it maps $L^2_{comp} \to L^2_{comp}$.

Using $(*)$, see that it gives $R_0(\lambda) : L^2_{comp} \to H^2_{loc}$ which is the needed meromorphic continuation. $\square$

To prove Thm 3, first need to give

<u>Definition</u>. Let $\mathcal{H}_1 \to \mathcal{H}_2$ be Hilbert spaces, $\Omega \subset \mathbb{C}$ open. A continuous family $A(\lambda) : \mathcal{H}_1 \to \mathcal{H}_2$, $\lambda \in \Omega$ of bdd operators is called: ~~holomor~~

- holomorphic, if $\forall f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$,

  $\lambda \mapsto \langle A(\lambda) f, g \rangle$ is holomorphic

  (note: this implies $A(\lambda) = \oint_\lambda \frac{A(t)}{t-\lambda}\,dt$ & thus holomorphy in operator norm)

- meromorphic w/ poles of finite rank, if it is defined & continuous except a finite set of poles and at each pole λ_0, we have

$$A(\lambda) = A_0(\lambda) + \sum_{j=1}^{J} \frac{A_j}{(\lambda - \lambda_0)^j},$$

A_0 holomorphic near λ,

$A_1, \dots, A_J : \mathcal{H}_1 \to \mathcal{H}_2$ finite rank operators.

Proof of Thm 3

1. If $A(\lambda_1)$ is invertible,
   then $A(\lambda)^{-1}$ exists & is holomorphic
   for λ near λ_1. In fact,
   $$\partial_\lambda(A(\lambda)^{-1}) = -A(\lambda)^{-1}\cdot\partial_\lambda A(\lambda)\cdot A(\lambda)^{-1}$$

2. So now let's assume that
   $A(\lambda_1)$ is <u>not</u> invertible & see what happens
   to $A(\lambda)^{-1}$ near λ_1.
   By assumption, $A(\lambda_1)$ is Fredholm,
   & it has index 0 since Ω is connected,
   $A(\lambda_0)$ invertible $\Rightarrow$ has index 0
   & index is a continuous fn. on Fredholm operators with operator norm.

3. Let $n = \dim \operatorname{Ker} A(\lambda_1) = \operatorname{codim} \operatorname{Range} A(\lambda_1)$.
   We construct $A_-: \mathbb{C}^n \to \mathcal{H}$, $A_+: \mathcal{H} \to \mathbb{C}^n$
   such that the Grushin operator
   $$\tilde{A}(\lambda) := \begin{pmatrix} A(\lambda) & A_- \\ A_+ & 0 \end{pmatrix} : \mathcal{H}\oplus\mathbb{C}^n \circlearrowleft$$
   is invertible at $\lambda=\lambda_1$.
   Note: $\tilde{A}(\lambda_1)$ is Fredholm of idx 0 so
   it's enough to take A_-, A_+ such that
   for $u\in\mathcal{H}$, $u_-\in\mathbb{C}^n$, if
   $$\begin{aligned} A(\lambda_1)u + A_-u_- &= 0 \\ A_+u &= 0 \end{aligned}, \text{ then } u=0,\ u_-=0.$$

One way to fix A_-, A_+ is as follows:

let $e_1, \ldots, e_n$ be a basis of $\operatorname{Ker} A(\lambda_1)$

$f_1, \ldots, f_n$ be a basis of $\operatorname{Ker} A(\lambda_1)^*$

& put $A_+(u) = \begin{pmatrix} \langle u, e_1 \rangle \\ \vdots \\ \langle u, e_n \rangle \end{pmatrix}$,

$A_-\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_1 f_1 + \cdots + v_n f_n.$

$$\boxed{\begin{aligned} &\langle A(\lambda_1)u, f_j \rangle \\ &= \langle u, A(\lambda_1)^* f_j \rangle \\ &= 0 \end{aligned}}$$

Then $A(\lambda_1)u + A_-u_- = 0$

$\Rightarrow 0 = \langle A(\lambda_1)u + A_-u_-, f_j \rangle$

$= \langle A_-u_-, f_j \rangle \quad \forall j \quad \Rightarrow u_- = 0.$

Thus $A(\lambda_1)u = 0 \Rightarrow u \in \operatorname{span}(e_1, \ldots, e_n)$

But $\langle u, e_j \rangle = 0 \quad \forall j \Rightarrow u = 0.$

(4) Now $\tilde{A}(\lambda): \mathcal{H} \oplus \mathbb{C}^n \circlearrowleft$ is invertible

& $\tilde{A}(\lambda)^{-1}$ holomorphic ~~for~~ λ near λ_1.

Write $\tilde{A}(\lambda)^{-1} = \begin{pmatrix} B(\lambda) & B_+(\lambda) \\ B_-(\lambda) & B_{-+}(\lambda) \end{pmatrix}.$

<u>Schur's Complement Formula</u>:

$A(\lambda)$ invertible $\Leftrightarrow B_{-+}(\lambda): \mathbb{C}^n \to \mathbb{C}^n$ invertible & if so,

$$A(\lambda)^{-1} = B(\lambda) - B_+(\lambda)\, B_{-+}(\lambda)^{-1} B_-(\lambda).$$

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5. Now, $B_{-+}(\lambda)$ is a holomorphic family of square matrices.

In particular, $\det B_{-+}(\lambda)$ is holomorphic.

2 cases:

(a) $\det B_{-+}(\lambda)$ ~~is not this~~ $\equiv 0$, λ near λ_1

(b) $\det B_{-+}(\lambda) \not\equiv 0 \Rightarrow \dfrac{1}{\det B_{-+}(\lambda)}$ is meromorphic

by Kramer's Rule, $B_{-+}(\lambda)^{-1}$ is meromorphic near $\lambda = \lambda_1$
(of course with poles of finite rank).
and thus $A(\lambda)^{-1}$ is meromorphic for λ near λ_1

6. We have proved: $\forall \lambda_1 \in \Omega$,
$\exists$ nbhd $U \ni \lambda_1$ s.t.
either (a) $A(\lambda)$ not invertible $\forall \lambda \in U$
or (b) $A(\lambda)^{-1}$ meromorphic in U.

Let Σ = closure in Ω of the set of $\lambda \in \Omega$ s.t. $A(\lambda)$ invertible.

Then
- Σ is closed
- Σ is open: imagine $\lambda_1 \in \Sigma$.
  Then cannot have case (a) at λ_1
  $\Rightarrow$ got case (b) $\Rightarrow$ a nbhd U of λ_1 is in Σ.
- $\Sigma \neq \emptyset$: $\lambda_0 \in \Sigma$.

Ω connected $\Rightarrow \Sigma = \Omega \Rightarrow$ always case (b)
$\Rightarrow A(\lambda)^{-1}$ meromorphic w/ poles of finite rank. $\square$