(M,g) a hyperbolic surface with one cusp

$$\mathcal{C} = [a,\infty)_r \times \mathbb{S}^1_\theta, \qquad \mathbb{S}^1 = \mathbb{R}/\ell\mathbb{Z}$$

$$g = dr^2 + e^{-2r} d\theta^2$$

If $u \in H^2_{loc}(M)$ solves $(-\Delta_g - \frac{1}{4} - \lambda^2) u = f \in L^2_{comp}(M)$

then, taking the Fourier series in $\mathcal{C}$ $u(r,\theta) = \sum_{k\in\mathbb{Z}} u_k(r) e^{ik\cdot\frac{2\pi}{\ell}\theta}$

the 0 mode solves

$$(-\partial_r^2 + \partial_r - \tfrac{1}{4} - \lambda^2) u_0 = f_0 \quad \text{so}$$

$$u_0(r) = c_+ e^{(\frac{1}{2}+i\lambda)r} + c_- e^{(\frac{1}{2}-i\lambda)r} \quad \text{for } r \gg 1.$$

- u is <u>outgoing</u> if $u_{k\neq 0} \in L^2(\mathbb{R}; e^{-r}dr)$ for $k \neq 0$ and $c_- = 0$.

<u>Scattering resolvent</u>: $R(\lambda): \begin{cases} L^2_{comp} \to H^2_{loc}, & \lambda \in \mathbb{C} \\ L^2 \to H^2, & \operatorname{Im}\lambda > 0 \end{cases}$

λ <u>not</u> a resonance $\Rightarrow$ for $f \in L^2_{comp}$, $u = R(\lambda)f$ is the unique outgoing solution to $(-\Delta_g - \frac{1}{4} - \lambda^2) u = f$

λ a resonance $\Rightarrow$ $\exists\, u \not\equiv 0$ outgoing solution to $(-\Delta_g - \frac{1}{4} - \lambda^2) = 0$.

<u>Eisenstein "series"</u>: $E(z;\lambda)$, $z \in M$, $\lambda \in \mathbb{C}$

$$\begin{cases} (-\Delta_g - \frac{1}{4} - \lambda^2) E = 0 \\ E \in L^2 \text{ on } \neq 0 \text{ modes in } \mathcal{C} \\ E_0(r) = e^{(\frac{1}{2}-i\lambda)r} + S(\lambda) e^{(\frac{1}{2}+i\lambda)r} \end{cases}$$

$S(\lambda)$ scattering coefficient. E, S <u>meromorphic</u> in λ.

λ <u>not</u> a resonance $\Rightarrow$ E, S are holomorphic at λ.

λ is a resonance $\Rightarrow$ $\exists$ <u>resonant state</u> u.

<u>2 cases:</u> $(\lambda \neq 0)$

ⓐ $u_0(r) = c\, e^{(\frac{1}{2}+i\lambda) r}$, $c \neq 0$.

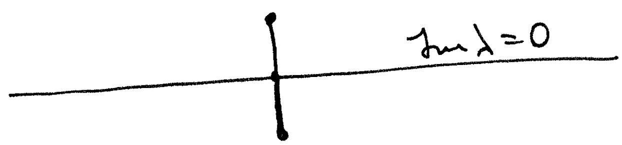
If $-\lambda$ is not a resonance, then

$$\boxed{S(-\lambda) = 0}$$

ⓑ $u_0 \equiv 0$, then u is an L^2 eigenvalue.

Note that since $-\Delta_g \geq 0$, case ⓑ can only happen when $\lambda^2 + \frac{1}{4} \geq 0$, i.e.

$\lambda \in \mathbb{R} \cup i[-\frac{1}{2}, \frac{1}{2}]$:



<u>Properties of $S(\lambda)$</u>:

① $S(\lambda)^{-1} = S(-\lambda)$ ② $S(\lambda)\overline{S(\bar{\lambda})} = 1$

In particular, $\lambda \in \mathbb{R} \Rightarrow |S(\lambda)| = 1$.

So $\lambda \in \mathbb{R} \setminus \{0\}$ a resonance $\Rightarrow$

$\Rightarrow \frac{1}{4} + \lambda^2$ is an L^2 eigenvalue (embedded eigenvalue)

(this is a weaker version of Rellich's Theorem)

<u>Note</u>: people often use another spectral parameter

$s = \frac{1}{2} - i\lambda$, so $\{\mathrm{Im}\,\lambda > 0\} \sim \{\mathrm{Re}\, s > \frac{1}{2}\}$

$\lambda \in \mathbb{R} \sim \mathrm{Re}\, s = \frac{1}{2}$

$-\Delta_g - \lambda^2 - \frac{1}{4} \sim -\Delta_g - s(1-s)$

An algebraic approach to hyperbolic scattering

Thm. Each hyperbolic surface is a quotient
$M \simeq \Gamma \backslash \mathbb{H}^2$ where $\mathbb{H}^2$ is the hyperbolic plane and Γ is a discrete group of isometries on $\mathbb{H}^2$ acting without fixed points.
Conversely each such $\Gamma\backslash\mathbb{H}^2$ is a hyperbolic surface.

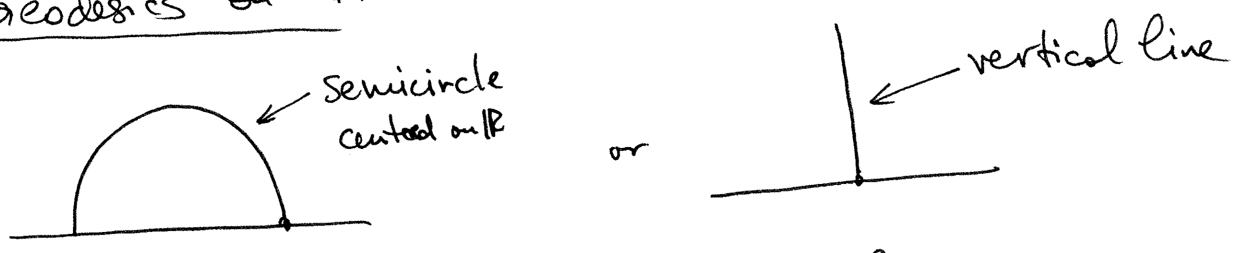
We use the upper half-plane model for $\mathbb{H}^2$:

$$\mathbb{H}^2 = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \} = \{ x+iy \mid x \in \mathbb{R},\ y>0 \}$$

Metric: $$g = \frac{|dz|^2}{|\operatorname{Im} z|^2} = \frac{dx^2+dy^2}{y^2},$$

g is hyperbolic, $(\mathbb{H}^2, g)$ is complete.
Note: $\{y=0\}$ corresponds to infinity. More precisely
$\partial\overline{\mathbb{H}^2} = \mathring{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

Geodesics on $\mathbb{H}^2$:



Orientation preserving isometries on $\mathbb{H}^2$:
Möbius transformations $\gamma : z \mapsto \dfrac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{R}$, $ad-bc=1$.

To each such γ we correspond a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d\end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\}$$

Turns out that composition of Möbius transformations corresponds to <u>matrix multiplication</u> in $SL(2,\mathbb{R})$.

Also, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \to \gamma(z) = z$.

So, the group of orientation preserving isometries of $\mathbb{H}^2$ is $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\mathbb{Z}_2$,

$\mathbb{Z}_2 = \{I, -I\} \subset SL(2,\mathbb{R})$

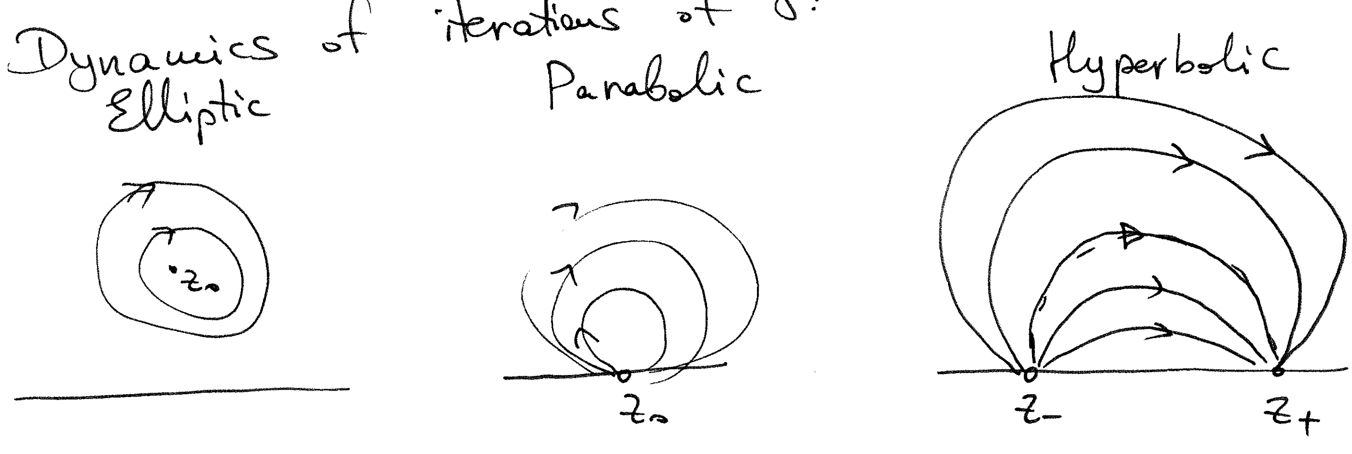
<u>Types of Möbius transformations</u>

Take $\gamma \in SL(2,\mathbb{R})$, $\gamma \neq \pm I$. Want to solve

$\gamma(z) = z$ (quadratic in z!). Cases: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

1. $|a+d| < 2 \Rightarrow \gamma$ is <u>elliptic</u>:
   $\gamma(z) = z$ has 1 solution z_0 with $\operatorname{Im} z > 0$
   1 solution with $\operatorname{Im} z < 0$
2. $|a+d| = 2 \Rightarrow \gamma$ is <u>parabolic</u>:
   $\gamma(z) = z$ has 1 solution $z_0 \in \mathring{\mathbb{R}}$
   (parabolic fixed point)
3. $|a+d| > 2 \Rightarrow \gamma$ is <u>hyperbolic</u>:
   $\gamma(z) = z$ has 2 solutions $z_-, z_+ \in \mathring{\mathbb{R}}$

Dynamics of iterations of γ:

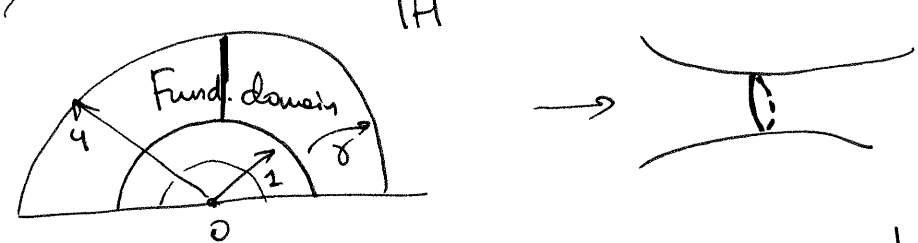


Note: elliptic transformations produce cone points in M ... strictly speaking, not smooth (we'll ignore it for now though)

Basic examples: $\Gamma = \langle \gamma \rangle = \{ \gamma^j \mid j \in \mathbb{Z} \}$ a cyclic subgroup generated by $\gamma \in SL(2,\mathbb{R})$.

(1) γ is hyperbolic, e.g. $\gamma = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

Then $\gamma.z = 4z$, the quotient $\langle\gamma\rangle \backslash \mathbb{H}^2$ is a hyperbolic cylinder:



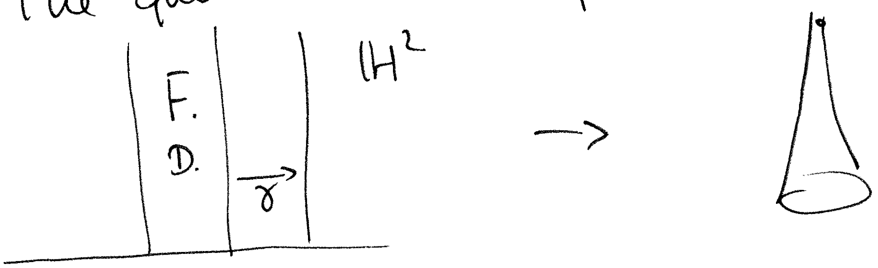
$\gamma.z = z$ has 2 solutions: $z=0$, $z=\infty$

The geodesic going $0 \to \infty$ projects to the closed geodesic.

(2) γ is parabolic, e.g. $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Then $\gamma.z = z+1$, so $\gamma.z = z$ has 1 solution $z = \infty$

The quotient is a parabolic cylinder:



Here $g = \dfrac{dx^2 + dy^2}{y^2}$
$= dr^2 + e^{-2r} d\theta^2$
where $\theta = x \bmod 1$,
$y = e^r$

Recall the basic incoming/outgoing solns $e^{(\frac{1}{2} \pm i\lambda) r}$. They become

$(\operatorname{Im} z)^{\frac{1}{2} \pm i\lambda}$ or $(\operatorname{Im} z)^s$ [incoming]
$(\operatorname{Im} z)^{1-s}$ [outgoing]

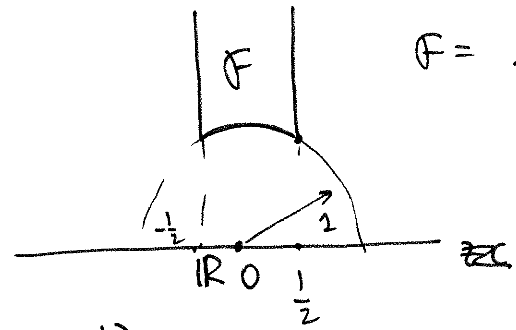
where $s = \frac{1}{2} - i\lambda$.

An interesting example: $M = \Gamma \backslash \mathbb{H}^2$ (modular curve) where

$\cancel{M = \mathbb{H}}$ $\mathbb{H}$ $\Gamma = PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ and

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) : a,b,c,d \in \mathbb{Z} \right\}$$

Fundamental domain:



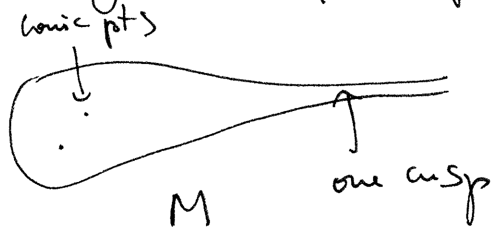
$F = \{ z : \operatorname{Im} z > 0, |z| > 1, |\operatorname{Re} z| < \frac{1}{2} \}$

$SL(2,\mathbb{Z})$ is generated by

$\gamma_0(z) = z + 1$ (parabolic)

and $\gamma_1(z) = -\frac{1}{z}$ ~~(elliptic)~~ (elliptic!) In fact, there is a cone point on M because of this, but we'll ignore this fact..

Specifically an ~~elliptic~~ conic point is $e^{\frac{i\pi}{3}}$. Another one is i.



The cusp corresponds to $z \to \infty$ along the strip.

We will compute the scattering coefficient for the modular curve.

~~Note:~~ ~~$\operatorname{Im} z$~~ We want a formula for the Eisenstein function $E(z;s)$. We can lift it to a function of $z \in \mathbb{H}^2$, $s \in \mathbb{C}$ where we require invariance:

$$\boxed{\gamma \in \Gamma \Rightarrow E(\gamma . z; s) = E(z;s)}$$

A first thing to take would be $(\operatorname{Im} z)^s$.

This solves $(-\Delta - s(1-s))(\operatorname{Im} z)^s = 0$

but it's not invariant under Γ.

Instead we could take $(\cancel{\operatorname{Im} z})(\operatorname{Im} \gamma . z)^s$

where $\gamma \in PSL(2, \mathbb{R})$, in particular

we could take $\gamma \in \Gamma = PSL(2, \mathbb{Z})$.

Note: if $\gamma_0(z) = z+1$, $\gamma_0 \in \Gamma$, then

$\operatorname{Im} \gamma_0(z) = \operatorname{Im} z \Rightarrow \forall \gamma, \operatorname{Im}(\gamma_0 \gamma . z) = \operatorname{Im}(\gamma . z)$.

So if $\tilde{\gamma} \in \Gamma_0 \gamma$, $\Gamma_0 = \langle \gamma_0 \rangle$, then

$\operatorname{Im}(\tilde{\gamma}.z) = \operatorname{Im}(\gamma.z)$

We then take the Eisenstein series

$$E(z; s) = \sum_{[\gamma] \in \underbrace{\Gamma_0 \backslash \Gamma}_{\text{space of left cosets}}} \operatorname{Im}(\gamma . z)^s.$$

For $\operatorname{Re} s \gg 1$, the series will converge (will not prove here...)

And it will actually give sth. in L^2 on $\neq 0$ modes (also won't prove here...)

Note: $E(\tilde{\gamma}.z; s) = E(z; s) \quad \forall \tilde{\gamma} \in \Gamma$.

Let's compute the 0 mode of E in the cusp.

Write $z = x + iy$. The below computation is due to Selberg, see

Titchmarsh, The theory of the Riemann zeta-function, 2nd edition, § 2.18

Parametrize cosets: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ gives $ad - bc = 1$.

In particular $\gcd(c, d) = 1$. Given (c, d), we determine a, b uniquely up to $(a, b) \mapsto (a, b) + k(c, d)$, $k \in \mathbb{Z}$.

But $\gamma_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} \ldots$

So $\Gamma_0 \backslash \Gamma$ is peramatrized by pairs (c,d) such that $\gcd(c,d)=1$.

In addition, we have the symmetry $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.

So we will sum over $(c,d) \in \mathcal{W}_0$ where $\mathcal{W}_0 = \{ (c,d) \in \mathbb{Z}^2 : \gcd(c,d)=1;$ either $c>0$ or $c=0,\ d>0\}$

We have $\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\operatorname{Im} z}{|cz+d|^2} = \cancel{\frac{y}{(cx+d)^2+d^2}}\ \frac{y}{(cx+d)^2+y^2}$.

So $$\boxed{E(\overset{x+iy}{\overset{\shortparallel}{z}}, s) = \sum_{(c,d)\in \mathcal{W}_0} \frac{y^s}{((cx+d)^2+c^2y^2)^s}}.$$

The condition $\gcd(c,d)=1$ is inconvenient.

Take $\mathcal{W}_1 = \{ (c,d) \in \mathbb{Z}^2 :$ either $c>0$ or $c=0,\ d>0\}$.

Then $$\sum_{(c,d)\in \mathcal{W}_1} \frac{y^s}{((cx+d)^2+c^2y^2)^s} = \sum_{k\in\mathbb{N}} \sum_{(c,d)\in \mathcal{W}_0} \frac{y^s}{((kcx+kd)^2+(kc)^2y^2)^s}$$

$$= \sum_{k\in\mathbb{N}} k^{-2s} E(z;s) = \zeta(2s) E(z;s)$$

where ζ is the <u>Riemann ζ-function</u>:

$$\zeta(s) := \sum_{k=1}^{\infty} k^{-s},$$ converges for $\operatorname{Re} s > 1$.

So $$\zeta(2s) E(z;s) = \sum_{(c,d)\in \mathcal{W}_1} \frac{y^s}{((cx+d)^2+c^2y^2)^s}.$$

Split into <u>2 parts</u>:

Part 1: $c = 0,\ d > 0$

Get $\displaystyle\sum_{d>0} \frac{y^s}{d^{2s}} = \zeta(2s)\, y^s.$

Part 2: $c > 0$.

Only need the 0 mode, i.e.

$$\zeta(2s)\int_0^1 E(x+iy; s)\,dx$$

$$= y^s \sum_{c>0} \sum_{d\in\mathbb{Z}} \int_0^1 \frac{dx}{((cx+d)^2+c^2y^2)^s}$$

$$= y^s \sum_{c>0} \sum_{d=0}^{c-1} \int_{-\infty}^{\infty} \frac{dx}{((cx+d)^2+c^2y^2)^s}$$

Change of variables $\frac{cx+d}{cy} = t$, $x = yt - \frac{d}{c}$

$$= y^s \sum_{c>0} c\cdot y \int_{-\infty}^{\infty} \frac{dt}{(cy)^{2s}(1+t^2)^s}$$

$$= y^{1-s}\cdot\Big(\sum_{c\in\mathbb{N}} c^{1-2s}\Big)\cdot \int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^s}.$$

$$= y^{1-s}\cdot\zeta(2s-1)\cdot\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^s}.$$

Now putting $u := \frac{1}{1+t^2}$ get

$$\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^s} = \int_0^1 u^{s-\frac{3}{2}}(1-u)^{-1/2}\,du = B\left(s-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$= \frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}.$$

So, we get

$$y^{1-s}\,\zeta(2s-1)\cdot\frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}.$$

Adding part 1 + part 2 we get

$$\zeta(2s)E_0(y,s) = \zeta(2s)\int_0^1 E(x+iy;s)\,dx$$

$$= \zeta(2s)y^s + \zeta(2s-1)\frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}y^{1-s}.$$

Therefore we get the scattering coefficient:

$$\boxed{S(s) = \frac{\zeta(2s-1)}{\zeta(2s)}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sqrt{\pi}.}$$

Since S is meromorphic, this gives <u>meromorphic continuation of the Riemann ζ-function.</u>

<u>Exercise</u>: get the identity $S(s)S(1-s) = 1$ using the functional equation for the Riemann ζ function.

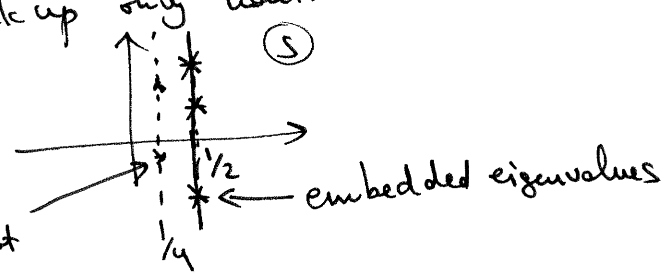
<u>What could the resonances of the modular curve be?</u>

Either $s(1-s)$ is an ~~embed~~ <u>L^2 eigenvalue</u> or

$S(1-s) = 0 \Leftrightarrow$ basically, $\zeta(2s) = 0$.

But pick up only nontrivial zeroes of ζ because of the Γ factors.

So:



s such that $\zeta(2s)=0$.

<u>Riemann hypothesis</u> tells us that most resonances are either on $\operatorname{Re} s = \frac{1}{2}$ (emb. eig.) or on $\operatorname{Re} s = \frac{1}{4}$ (nontrivial 0-s of ζ)

<u>Selberg</u>: there are embedded eigenvalues. In fact, in a ball of radius R the # of e.e. is $\sim R^2$.

<u>Philips - Sarnak '85, Colin de Verdière '82, '83</u>: a generic perturbation of $PSL(2,\mathbb{Z})\backslash\mathbb{H}^2$ destroys embedded eigenvalues (<u>they move to $\operatorname{Re} s < \frac{1}{2}$</u>)

"How not to prove the Riemann hypothesis..."

See [Dy Zw, Thm 4.22]
<u>Fermi Golden Rule</u>