We're studying

$P = -\Delta_g + V$, (M, g) Riemannian mfld with Euclidean ends, $V \in C_c^\infty(M; \mathbb{R})$

~~$R(\lambda) = (-\Delta_g + V$~~ Assume $\underline{\partial M = \emptyset}$.

$$R(\lambda) = (P - \lambda^2)^{-1} : \begin{cases} L^2(M) \to H^2(M), & \operatorname{Im}\lambda > 0 \\ L^2_{comp}(M) \to H^2_{loc}(M), & \lambda \in \mathbb{C} \end{cases}$$

<u>Def.</u> P has a gap of size $\beta > 0$, if $\exists C_0, N$: for $|\operatorname{Re}\lambda| \geq C_0$, $-\beta \leq \operatorname{Im}\lambda \leq 1$, we have $\forall \chi \in C_c^\infty(M)$

$$\|\chi R(\lambda) \chi\|_{L^2 \to L^2} \leq C(\chi) \cdot |\lambda|^{-N}$$

To understand $P - \lambda^2$ for $\operatorname{Re}\lambda \gg 1$, use <u>semiclassical rescaling</u>: (here $h \approx (\operatorname{Re}\lambda)^{-1} \ll 1$)

$$h^2(P - \lambda^2) = P_h - \omega^2, \quad P_h = h^2 P = -h^2\Delta_g + h^2 V$$
$$\omega = h\lambda, \quad \boxed{\omega = 1 + O(h)}$$

We have $P_h \in \Psi_h^2(M)$ and $\sigma_h(P_h) = p$ where

$$p(x, \xi) = |\xi|^2_{g^{-1}(x)} = \sum_{j,k} g^{jk}(x)\xi_j \xi_k$$

Define $\varphi_t = \exp(tH_p) : T^*M \to T^*M$

The trapped set

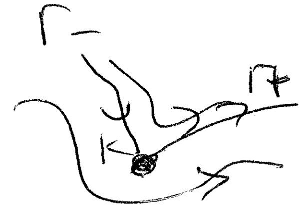
Define Γ_+ (outgoing tail), Γ_- (incoming tail), K (trapped set) as the following subsets of $T^*M \setminus 0 = \{(x,\xi) \in T^*M \mid \xi \neq 0\}$

$$\Gamma_\pm := \{(x,\xi) \in T^*M \setminus 0 \mid \varphi_t(x,\xi) \text{ stays bdd as } t \to \mp\infty\}$$

$$K := \Gamma_+ \cap \Gamma_-.$$

Basic properties: $\boxed{\text{READ [DyZw, §6.1.1]}}$

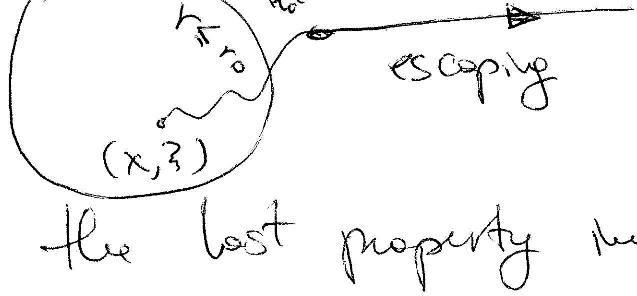
- $\Gamma_\pm$ are closed in $T^*M \setminus 0$, homogeneous: $(x,\xi)\in\Gamma_\pm \Leftrightarrow (x,\tau\xi)\in\Gamma_\pm$ $(\tau>0)$
- if r is a radius fn on M & M is Euclidean on $\{r \geq r_0\}$, then $K \subset \{r<r_0\}$.



- It follows that $K \cap S^*M$ is compact.
- $(x,\xi)\in\Gamma_\pm \Rightarrow \varphi_t(x,\xi)\to K$ as $t\to\mp\infty$

The proofs are not long but I'll skip them.
Basic idea is as follows:
imagine that $(x,\xi)\in T^*M\setminus 0$, $r(x)\leq r_0$.
Then there are 2 cases for $\varphi_t(x,\xi)$ as $t\to\infty$:

1. $\varphi_t(x,\xi)\in\{r\leq r_0\}$ for all $t>0$.
   Then $(x,\xi)\in\Gamma_-$
2. $\exists\, t_0>0$: $\varphi_{t_0}(x,\xi)\in\{r>r_0\}$.
   Then $\varphi_t(x,\xi)$ escapes to infinity as $t\to\infty$:



Note that the last property implies that
$K=\emptyset \Rightarrow \Gamma_\pm=\emptyset$ as well.

The nontrapping case

Thm. Assume that $K = \emptyset$. Then P has a gap of size β for all $\beta > 0$, and one can take $N = \frac{d}{} -1$ in the resolvent estimate.

Proof 1 It suffices to show the following:

if $|\mathrm{Im}\,\lambda| \le \beta \leftarrow$ some fixed cst

& $|\mathrm{Re}\,\lambda| > C_0 \gg 1$, then $\forall f \in L^2(M)$, $\mathrm{supp}\, f \subset \{r \le r_0\}$

and $\forall u \in H^2_{loc}(M)$, $\begin{cases} (P-\lambda^2)u = f, \\ u \text{ outgoing} \end{cases}$,

$\forall \chi \in C_c^\infty(M)$

we have $\quad \|\chi u\|_{L^2} \le C_\chi |\lambda|^{-1} \cdot \|f\|_{L^2}$

(strictly speaking, need more work to show that C_0 does not depend on χ - will skip this step)

Enough to take $\mathrm{Re}\,\lambda > 0$...

In semiclassical rescaling, get:

h small enough, ~~$\omega = 1$~~ $\mathrm{Re}\,\omega = 1$, $|\mathrm{Im}\,\omega| \le \beta h$

u outgoing at ω/h, $(P_h - \omega^2)u = f$

$$\Rightarrow \|\chi u\|_{L^2} \le C h^{-1} \|f\|_{L^2}.$$

2 For simplicity assume M has only 1 infinite end.

Fix $\chi \in C_c^\infty(M)$, $\chi \equiv 1$ near $\{r \le r_0\}$.

We can express u ~~using~~ via χu.

(or rather, the behavior of u near supp χ...)

Indeed, $(1-\psi)u$ is supported in the infinite end $\{r > r_0\}$ and it's outgoing. We have

$$(-h^2\Delta_g - \omega^2)(1-\psi)u = (P_h - \omega^2)(1-\psi)u$$

$= [h^2\Delta, \psi]u$. Take the semiclassical free resolvent

$\cancel{(-h^2\Delta-\omega^2)^{-1}}$ $R_{0,h}(\omega) = (-h^2\Delta - \omega^2)^{-1} = h^{-2} R_0(\omega/h)$.

Then $(1-\psi)u = R_{0,h}(\omega)\underbrace{[h^2\Delta, \psi]u}_{\in L^2_{comp}}$.

Here we used that u is outgoing...

So $\boxed{u = \psi u + R_{0,h}(\omega)[h^2\Delta, \psi]u}$ (1)

Recall: $\forall \chi \in C_c^\infty$ $\|\chi R_{0,h}(\omega)\chi\|_{L^2 \to} \le C_\chi h^{-1}$

(from the estimates on $R_0(\lambda)$, $|\mathrm{Im}\,\lambda| \le C$ ...)

Therefore, if $\psi_1 \in C_c^\infty(M)$, $\psi_1 = 1$ near $\mathrm{supp}\,\psi$, then $\forall \chi \in C_c^\infty(M)$,

$\boxed{\|\chi u\|_{L^2} \le C_\chi \|\psi_1 u\|_{H^1_h}}$. (2)

So if we estimate $\psi_1 u$, we estimate χu for any χ...

3 It remains to estimate $\Psi_1 u$.

We'll use ~~...~~ semiclassical estimates now

Assume that $A \in \Psi_h^0(M)$ is compactly supported (e.g. $A = \Psi_1$)

Can we estimate $\|Au\|_{H_h^2}$?

Case 1: $WF_h(A) \cap \{p=1\} = \emptyset$

where $\{p=1\} = S^*M = \{(x,\xi) : |\xi|_g = 1\}$.

We have $(P_h - \omega^2)u = f$ and $\omega^2 = 1 + O(h)$.

So $\sigma_h(P_h - \omega^2) = p - 1$ and

$WF_h(A) \subset \mathrm{ell}_h(P_h - \omega^2)$.

By the elliptic estimate, we get ($\exists \chi \in C_c^\infty$)

$$\|Au\|_{H_h^2} \le C\|f\|_{L^2} + O(h^\infty)\|\chi u\|_{L^2}.$$

4) Next we use propagation of singularities:

Case 2: $WF_h(A) \subset T^*M \setminus 0$ (is compact).

Note that since ~~A~~ $WF_h(A)$ does not intersect the fiber infinity

$$\|Au\|_{H_h^2} \le C\|Au\|_{H_h^{-N}} + O(h^\infty)\|\chi u\|_{L^2} \quad \forall N$$

Propagation of singularities tells us that

$$\|Au\|_{H_h^2} \leq C\|Bu\|_{H_h^2} + Ch^{-1}\|f\|_{L^2} + O(h^\infty)\|\chi u\|_{L^2}^2$$

as long as $\exists\ T > 0$ such that

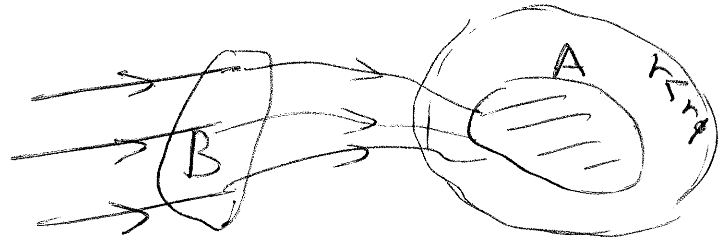
$$\varphi_{-T}(WF_h(A)) \subset \mathrm{ell}_h(B)$$

Here we use that $P_h\ |\mathrm{Im}\ \lambda\omega| \leq Ch\ldots$

Does not seem to be useful because <u>how do we estimate Bu??</u>

Here's where we use that we are nontrapping: $K = \emptyset \Rightarrow \Gamma_+ = \emptyset$

$\Rightarrow \forall (x,\xi) \in WF_h(A)$, $\varphi_t(x,\xi)$ escapes as $t \to -\infty$.



So if T is large enough, then

$\forall (x,\xi) \in WF_h(A)$, $\varphi_t(x,\xi)$ is on a directly incoming trajectory in the infinite end, i.e.

$$r(x) > r_0, \quad \langle x, \xi\rangle < 0.$$

Thus $\forall A$ we can find T, B s.t. ~~wl~~ $\varphi_{-T}(WF_h(A)) \subset \mathrm{ell}_h(B)$

and $WF_h(B) \subset \{\cancel{r(x)}\ |x| > r_0,\ \langle x,\xi\rangle < 0\}$.

5 Time to use the outgoing condition, more precisely its consequence (1):

<u>Case 3</u>: $B \in \Psi_h^0$ comp. supported in $\{r > r_2\}$, large!
$$WF_h(B) \subset \{\langle x, \xi\rangle < 0\}.$$
By taking large r_1, can assume that
$B\psi_1 = 0$, so $B\psi = 0$ & recall $\psi_1 = 1$ ~~near~~ $\operatorname{supp}\psi$
$$Bu = B(1-\psi)u = BR_{0,h}(\omega)\psi_1 v \text{ where}$$
$$v = [h^2\Delta, \psi]u.$$
For simplicity we'll assume we're in dimension <u>$n=3$</u>

We use the formula for the free resolvent:
$$R_0(\lambda)v(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i\lambda|x-y|}v(y)}{|x-y|}dy.$$
So $$R_{0,h}(\omega)v(x) = \frac{1}{4\pi h^2}\int_{\mathbb{R}^3} e^{\frac{i}{h}\Phi(x,y)}a(x,y)v(y)dy$$
where $\Phi(x,y) = |x-y|$, $a(x,y) = \dfrac{e^{-\operatorname{Im}\lambda\cdot|x-y|}}{|x-y|}$.

We have $\Phi \in C^\infty(\{x \neq y\})$ and so is a, uniformly in λ (since <u>$|\operatorname{Im}\lambda| \le C$</u>).

$B\psi = 0 \to$ we'll take
$$\psi_2 \in C_c^\infty(M), \quad B = B\psi_2, \quad \operatorname{supp}\psi_1 \cap \operatorname{supp}\psi_2 = \emptyset.$$
Write $B = \operatorname{Op}_h(b)$. Then

$$Bu(x) = B\psi_2 R_{0,h}(\omega)\psi_1 v(x)$$

$$= Op_h(b)\psi_2 R_{0,h}(\omega)\psi_1 v(x)$$

$$= \frac{1}{(2\pi h)^3 \cdot 4\pi h^2}\int_{\mathbb{R}^9} e^{\frac{i}{h}(\langle x-z,\xi\rangle + \Phi(z,y))}\psi_2(z)\psi_1(y) b(x,\xi)a(z,y)v(y)\,dy\,dz\,d\xi.$$

$$= \frac{1}{(2\pi h)^3 4\pi h^2}\int_{\mathbb{R}^3} K_B(x,y)v(y)\,dy \quad \text{where}$$

$$K_B(x,y) = \int_{\mathbb{R}^6} e^{\frac{i}{h}(\langle x-z,\xi\rangle + \Phi(z,y))}\psi_2(z)\psi_1(y) b(x,\xi)a(z,y)\,dz\,d\xi.$$

Now let's hit it with <u>method of nonstationary phase</u>:

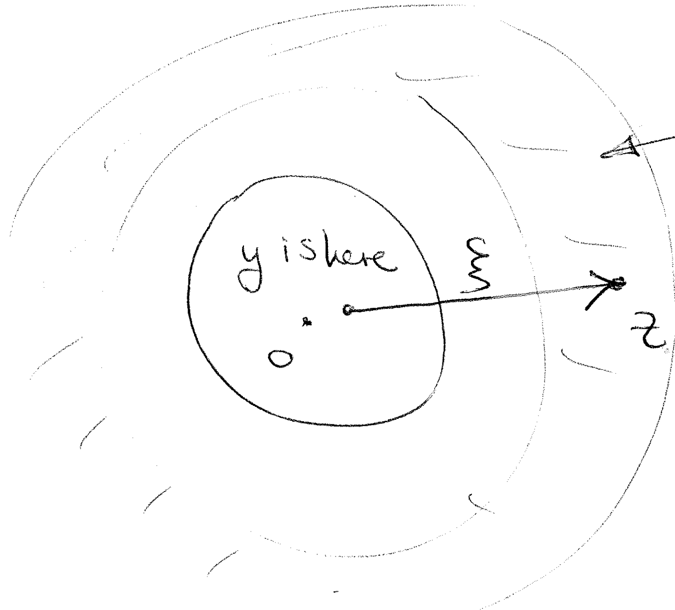
- Can make $b \in C_c^\infty(T^*\mathbb{R}^n \setminus 0) \to$ set $x \in$ cpct set, $\xi \neq 0$, $\xi \in$ cpct set
- $z \in \operatorname{supp}\psi_2$, $y \in \operatorname{supp}\psi_1 \to$ in particular $z \neq y$.
- Stationary points of the phase? $\Phi(z,y) = |z-y|$

$$\partial_\xi(\langle x-z,\xi\rangle + |z-y|) = 0 \Rightarrow x = z$$

~~(does not matter here)~~

$$\partial_z(\langle x-z,\xi\rangle + |z-y|) = 0 \Rightarrow \xi = \frac{z-y}{|z-y|}.$$

Now we arrange so that $\operatorname{supp}\psi_1 \subset \{|x| < r_2\}$
$\operatorname{supp}\psi_2 \subset \{|x| > r_2\}$ for some r_2



z is in this annulus (projection of $WF_h(B)$, we can make it quite far away by propagating more in Case 2)

So then ξ is "outgoing": $\langle z, z-y\rangle = |z|^2 - \langle y, z\rangle > 0$.

Thus on stationary pts we have $\langle x, \xi\rangle > 0$.
(of $\langle x-z, \xi\rangle + \Phi(z,y)$)

But $WF_h(B) \subset \{\langle x,\xi\rangle < 0\}$ by assumption

So can make $\operatorname{supp} b \subset \{\langle x,\xi\rangle < 0\}$

$\Rightarrow$ no stationary pts on the support of the amplitude.

Thus $K_B = O(h^\infty)_{C_c^\infty}$ $\Rightarrow$

$$\Rightarrow \|Bu\|_{L^2} = O(h^\infty)\|v\|_{L^2} = O(h^\infty)\|\chi u\|_{\cancel{H_h} L^2}$$

(6) Combine <u>Cases 1 – 3</u>: start with ψ_1,

write $\psi_1 = A' + A''$, $A', A'' \in \Psi_h^0$ comp. supp.,

$\cancel{A'}$ $WF_h(A') \cap \{p=1\} = \emptyset$, $WF_h(A'') \Subset T^*M \setminus 0$

Estimate $A'u$ by Case 1, $A''u$ by Case 2 + 3
(use Case 3 to estimate Bu)

<u>Get</u>:

$$\|\psi_1 u\|_{H_h^2} \le Ch^{-1}\|f\|_{L^2} + O(h^\infty)\|\chi u\|_{\cancel{H_h} L^2}.$$

But combining with (2) (2) we get

$$\|\psi_1 u\|_{H_h^2} \le Ch^{-1}\|f\|_{L^2} + O(h^\infty)\|\psi_1 u\|_{H_h^1}.$$

Taking h small enough, remove the remainder

& get (using (2) again)

$$\|\chi u\|_{L^2} \le \cancel{Ch} \; C\|\psi_1 u\|_{H_h^1}$$

$$\le Ch^{-1}\|f\|_{L^2}, \text{ as needed. } \square$$

Moral of the story is:

- the elliptic part was easy
- on S^*M, we used propagation of singularities to reduce to estimating Bu where B lives on incoming trajectories for ~~in~~ the infinite end
- To estimate Bu we used that u is outgoing + explicit formula for $R_0(\lambda)$ ~~(this)~~ (the "semiclassically outgoing property" of $R_0(\lambda)$)