Meromorphic continuation in 1D

$P_V = -\partial_x^2 + V(x) : C^\infty(\mathbb{R}) \circlearrowleft$,

$V \in C_c^\infty(\mathbb{R})$ real-valued.

For $f \in C_c^\infty(\mathbb{R})$, looking for solutions u to the problem

$$(*)\begin{cases} (P_V - \lambda^2)u = f \\ u \text{ outgoing, i.e. } u(x) = c_\pm e^{\pm i\lambda x} \text{ for } \pm x \gg 1. \end{cases}$$

Thm. There exists a meromorphic family of operators $R_V(\lambda) : C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ s.t.

1. If λ is not a pole of R_V, then $\forall f \in C_c^\infty$, $u := R_V(\lambda)f$ is the unique solution to $(*)$
2. λ is a pole of R_V (called resonance) iff $\exists\, u \not\equiv 0$ solving $(*)$ with $f=0$ (resonant state).

Proof $\boxed{1}$. There exist unique solutions $e_\pm(x;\lambda)$ to

$$\begin{cases} (P_V - \lambda^2)e_\pm = 0 \\ e_\pm(x) = e^{\pm i\lambda x} \text{ for } \pm x \gg 1. \end{cases}$$

Define the Wronskian

$W(e_+, e_-) = \det\begin{pmatrix} e_+ & e_+' \\ e_- & e_-' \end{pmatrix}$. Then

$W(\lambda) = W(e_+, e_-)$ is independent of x.

[2] Assume $W(\lambda) \neq 0$. Then for each $f \in C_c^\infty$, the unique solution u to (*) is given by

$$u(x) = R_V(\lambda) f(x) := \cancel{e_+(x) e_-(y)} \Big\{$$

$$= \frac{1}{W(\lambda)} \left[\int_{-\infty}^{x} e_+(x) e_-(y) f(y)\, dy + \int_x^\infty e_-(x) e_+(y) f(y)\, dy \right].$$

$$= \int_{\mathbb{R}} R_V(x, y; \lambda) f(y)\, dy \quad \text{where}$$

$$R_V(x, y; \lambda) = \frac{e_+(x) e_-(y) [x > y] + e_-(x) e_+(y) [x < y]}{W(\lambda)}.$$

Why? See Exercise 4. One thing that's ~~ver~~ easy to see is that $u(x) = \frac{1}{W(\lambda)} \int_{\mathbb{R}} e_-(y) f(y)\, dy \cdot e_+(x)$, $x \gg 1$

$$= (\cdots) \, e_-(x), \; -x \gg 1$$

Now $W(\lambda)^{-1}$ is meromorphic (strictly speaking, need $W \not\equiv 0$; will see it later today)

So the R_V defined above is a meromorphic family of operators.

[3] Assume $W(\lambda) = 0$. Then $e_+(\lambda) \parallel e_-(\lambda)$ $\rightarrow$ either gives a solution to (*) with $f = 0$. $\square$

Example: $V \equiv 0$. $e_+(x) = e^{i\lambda x}$, $e_-(x) = e^{-i\lambda x}$

$W(\lambda) = -2i\lambda$.

$$R_V(x, y; \lambda) = \frac{i}{2\lambda} e^{i\lambda |x-y|}$$

Resonance at $\lambda = 0$ with resonant state $u \equiv 1$.

Where can resonances be?

Let's start with the upper half-plane $\operatorname{Im}\lambda > 0$.

Prop. $\lambda \in \{\operatorname{Im}\lambda > 0\}$ is a resonance $\Leftrightarrow$ λ^2 is an L^2 eigenvalue of P_V, i.e. $\exists u$ $(u \neq 0)$ $u \in C^\infty \cap L^2(\mathbb{R})$ s.t. $(P_V - \lambda^2)u = 0$.

Each such λ lies on the imaginary axis.

Proof λ is a resonance $\Leftrightarrow$ $\exists\, u \neq 0$ solving $(P_V - \lambda^2)u = 0$ & $u(x) = c_\pm e^{\pm i\lambda x}$, $\pm x \gg 1$.

But then $u \in L^2$. Similarly every L^2 solution to $(P_V - \lambda^2)u = 0$ is outgoing.

It remains to show $\lambda \in i\mathbb{R}$, i.e. $\lambda^2 \in \mathbb{R}$.

Integrate by parts (can do since u decays rapidly as $|x| \to \infty$)

$$0 = \operatorname{Im} \int_{\mathbb{R}} (P_V - \lambda^2)u(x)\,\overline{u(x)}\,dx = \operatorname{Im} \int_{\mathbb{R}} |u'(x)|^2 + V|u(x)|^2 - \lambda^2 |u(x)|^2\,dx$$

$$= -(\operatorname{Im}\lambda^2) \int_{\mathbb{R}} |u|^2\,dx \Rightarrow \operatorname{Im}\lambda^2 = 0. \quad \square$$

What about the real line?

Thm [1D version of Rellich's Thm] There are no resonances $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof Assume $\lambda \in \mathbb{R} \setminus \{0\}$ is a resonance.

So $\exists\, u$: $(P_V - \lambda^2)u = 0$, u outgoing.

Since V is real-valued, also have $(P_V - \lambda^2)\bar{u} = 0$.

Thus the Wronskian $W(u, \bar{u})$ is constant in x.

However, $u(x) = c_\pm e^{\pm i\lambda x}, \quad \pm x \gg 1$

$\overline{u(x)} = \bar{c}_\pm e^{\mp i\lambda x}, \quad \pm x \gg 1.$

So for $\pm x \gg 1$, get

$$W(u, \bar{u}) = |c_\pm|^2 \cdot (\mp 2i\lambda).$$

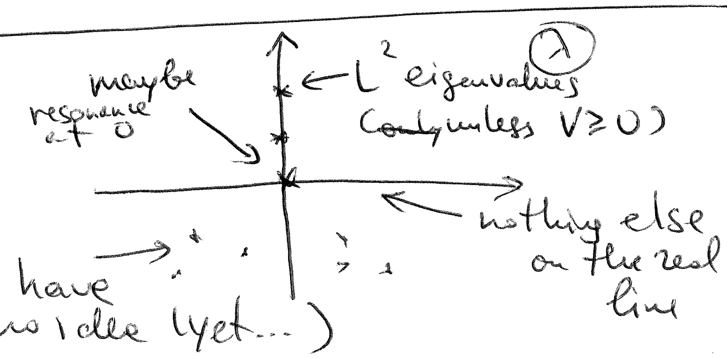
Since $W(u,\bar{u})$ is constant, this gives

$$|c_+|^2 \cdot (-2i\lambda) = |c_-|^2 \cdot 2i\lambda \Rightarrow |c_+|^2 + |c_-|^2 = 0.$$

Thus $c_\pm = 0 \Rightarrow u \equiv 0$ since u solves an ODE;

a contradiction. $\square$

So the picture is:



<u>Plane waves and scattering matrix</u>

How to describe <u>all</u> solutions to $(P_V - \lambda^2)u = 0$?

Each u has the form $u(x) = u_{in}(x) + u_{out}(x)$, $|x| \gg 1$

where $u_{in}(x) = b_\pm \cdot e^{\mp i\lambda x}, \quad \pm x \gg 1$

$u_{out}(x) = a_\pm \cdot e^{\pm i\lambda x}, \quad \pm x \gg 1$

<u>Thm</u>. There exists a 2×2 matrix $S(\lambda)$ meromorphic in $\lambda \in \mathbb{C}$, poles of $S \subset$ resonances, such that when λ is not a resonance, the equation $(P_V - \lambda^2)u = 0$ has a unique solution for each choice of b_+, b_-, and

$$S(\lambda) : \begin{pmatrix} b_- \\ b_+ \end{pmatrix} \longmapsto \begin{pmatrix} a_+ \\ a_- \end{pmatrix}.$$

Interpretation of the scattering matrix via the wave equation:

Take $\lambda \in \mathbb{R} \setminus \{0\}$ (not a resonance, as we saw above)

Fix $u(x)$ as before & put

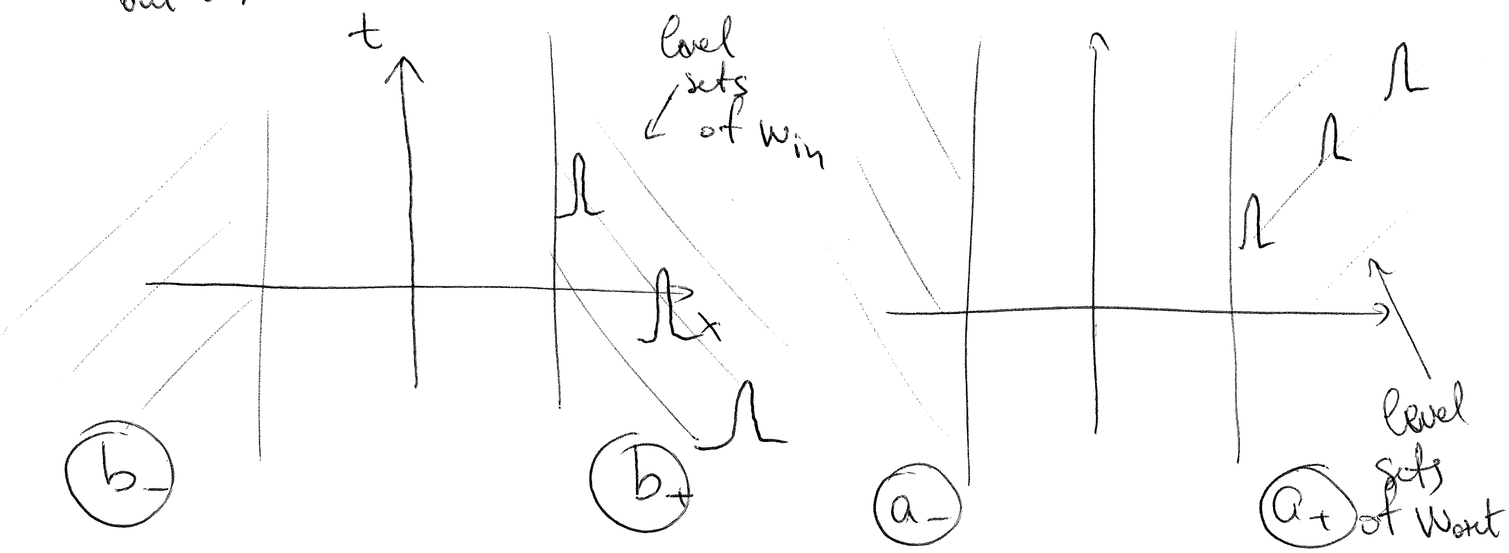
$$w(t,x) = e^{-it\lambda} u(x). \text{ Then}$$

$$\cancel{\partial_t^2} \quad (\partial_t^2 - \partial_x^2 + V)w = 0.$$

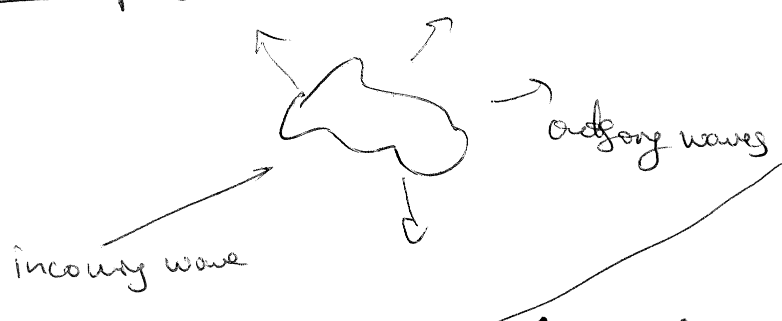
For $|x| \gg 1$, we decompose $w = w_{in} + w_{out}$

$$w_{in}(t,x) = b_\pm \cdot e^{-i\lambda(t+|x|)} = \text{function of } t+|x|$$

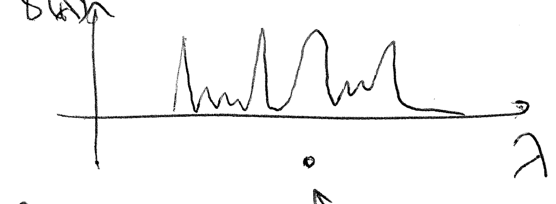
$$w_{out}(t,x) = a_\pm \cdot e^{-i\lambda(t-|x|)} = \text{function of } t-|x|$$

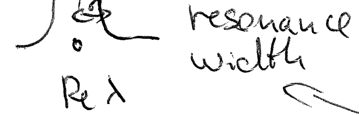


General physical interpretation:



Measure scattering cross section as function of λ:





give good approximation in practice

Breit–Wigner approximation

Example: $V \equiv 0$.

Then $u(x) =$ linear combination of $e^{i\lambda x}, e^{-i\lambda x}$ globally.

So, $b_- = a_+$, $b_+ = a_-$ $\Rightarrow S(\lambda) = I$.

Note: 0 is a resonance, but not a pole of $S(\lambda)$.

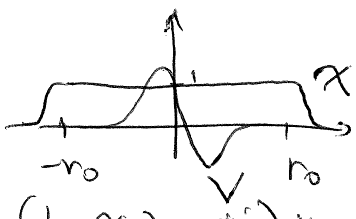
How to construct $S(\lambda)$?

Assume λ is not a resonance.

Then there exist unique solutions $u^\pm$ to $(P_V - \lambda^2) u^\pm = 0$ such that $u^\pm(x) = e^{\pm i\lambda x} + v^\pm(x)$, $v^\pm$ outgoing.

$u^\pm$ are called plane waves.

To construct $u^\pm$, take r_0: $\operatorname{supp} V \subset [-r_0, r_0]$ & fix $\chi \in C_c^\infty(\mathbb{R})$, $\chi = 1$ on $[-r_0, r_0]$.



Put

$$u^\pm := (1-\chi) e^{\pm i\lambda x} + R_V(\lambda)(P_V - \lambda^2)(1-\chi) e^{\pm i\lambda x}$$

This is possible since $P_V = -\partial_x^2$ on $\operatorname{supp}(1-\chi)$, and $(-\partial_x^2 - \lambda^2) e^{\pm i\lambda x} = 0$, so

$$(P_V - \lambda^2)(1-\chi) e^{\pm i\lambda x} = -\underset{\substack{\cap \\ C_c^\infty(\mathbb{R})}}{[P_V, \chi]} e^{\pm i\lambda x} + \underbrace{(1-\chi)(P_V - \partial_x^2 - \lambda^2) e^{\pm i\lambda x}}_{0}$$

So $(P_V - \lambda^2)(1-\chi) e^{\pm i\lambda x} \in C_c^\infty(\mathbb{R})$.

The resulting $u^\pm$ solves $(P_V-\lambda^2)u^\pm=0$
and has the form $u^\pm(x) = e^{\pm i\lambda x} + v^\pm(x)$
where $v^\pm = -\chi e^{\pm i\lambda x} + R_V(\lambda)(P_V-\lambda^2)(1-\chi)e^{\pm i\lambda x}$
is outgoing.

<u>Uniqueness</u> of $u^\pm$ with these properties is immediate because λ is not a resonance.

Plane waves $\to$ Scattering matrix

Write
$$v^\pm(x) = \begin{cases} v^\pm_+ \cdot e^{i\lambda x}, & x \gg 1 \\ v^\pm_- \cdot e^{-i\lambda x}, & -x \gg 1. \end{cases}$$

Take any solution u to $(P_V-\lambda^2)u=0$. Then:

$-r_0$ $\qquad$ r_0 $\qquad$ x

$u = b_- \cdot e^{i\lambda x} + a_- \cdot e^{-i\lambda x}$	$u = b_+ \cdot e^{-i\lambda x} + a_+ \cdot e^{i\lambda x}$
$u^+ = e^{i\lambda x} + v^+_- \cdot e^{-i\lambda x}$	$u^+ = (1+v^+_+)e^{i\lambda x}$
$u^- = (1+v^-_-)\cdot e^{-i\lambda x}$	$u^- = e^{-i\lambda x} + v^-_+ e^{i\lambda x}$

So, $\cancel{u - b_- \cdot e^{i\lambda x} - b}$
$\tilde u := u - b_- \cdot u^+ - b_+ \cdot u^-$ is an outgoing solution
to $(P_V-\lambda^2)\tilde u = 0 \Rightarrow \tilde u \equiv 0$.

So $u = b_- \cdot u^+ + b_+ \cdot u^-$.

And $a_+ = b_-(1+v^+_+) + b_+ v^-_+$
$\quad a_- = b_- v^+_- + b_+(1+v^-_-)$

$$\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = S(\lambda)\begin{pmatrix} b_- \\ b_+ \end{pmatrix} \text{ where } S(\lambda) = \begin{pmatrix} 1+v^+_+(\lambda) & v^-_+(\lambda) \\ v^+_-(\lambda) & 1+v^-_-(\lambda) \end{pmatrix}$$

Properties of the scattering matrix

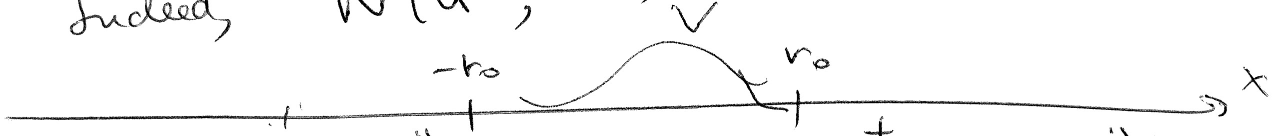
Write $S(\lambda) = \begin{pmatrix} T_+(\lambda) & R_+(\lambda) \\ R_-(\lambda) & T_-(\lambda) \end{pmatrix}$

$T_\pm$ – transmission coefficients

$R_\pm$ – reflection coefficients

1. $T_+ = T_-$. (denoted $T = T_+ = T_-$)

Indeed, $W(u^+, u^-) = \text{const}$ but



$u^+ = e^{i\lambda x} + R_- e^{-i\lambda x}$	$u^+_* = T_+ \cdot e^{i\lambda x}$
$u^- = T_- e^{-i\lambda x}$	$u^- = e^{-i\lambda x} + R_+ e^{i\lambda x}$
$W(u^+, u^-) = -2i\lambda \cdot T_-$	$W(u^+, u^-) = -2i\lambda \cdot T_+$

So $T_-(\lambda) = T_+(\lambda)$ for $\lambda \neq 0$ and thus by continuity also at $\lambda = 0$.

2. For $\lambda \in \mathbb{R}$, $S(\lambda)$ is unitary: $S(\lambda)^* = S(\lambda)$.

Enough to take a soln u to $(P_V - \lambda^2)u = 0$ & show that $|a_+|^2 + |a_-|^2 = |b_+|^2 + |b_-|^2$. $(\star)$

$\lambda \in \mathbb{R} \Rightarrow (P_V - \lambda^2)\bar{u} = 0$ as well, $W(u, \bar{u}) = \text{const}$.

$u = b_- e^{i\lambda x} + a_- e^{-i\lambda x}$	$u = b_+ e^{-i\lambda x} + a_+ e^{i\lambda x}$
$\bar{u} = \bar{b}_- e^{-i\lambda x} + \bar{a}_- e^{i\lambda x}$	$\bar{u} = \bar{b}_+ e^{i\lambda x} + \bar{a}_+ e^{-i\lambda x}$

$2i\lambda(|a_-|^2 - |b_-|^2) = W(u, \bar{u}) = W(u, \bar{u}) = 2i\lambda(|b_+|^2 - |a_+|^2)$

which implies $(\star)$.

For general λ: $S(\lambda)^* = S(\bar{\lambda})$.