Semiclassical wavefront set of a distribution [Dy2w, §E.2.3]

<u>Def.</u> A family of distributions $u = u(h) \in \mathcal{D}'(U)$, ($U \subset \mathbb{R}^n$ open), is called h-tempered, if $\forall \chi \in C_c^\infty(U)$ $\exists N, C$: $\|\chi u\|_{H_h^{-N}} \le C h^{-N}$ for all $0 < h \le 1$.

<u>Def.</u> Let $u(h) \in \mathcal{D}'(U)$ be an h-tempered family. We say that $(x_0, \xi_0) \in \overline{T^*U} = \{(x,\xi) \in \overline{T^*\mathbb{R}^n} : x \in U\}$ does <u>not</u> lie in $WF_h(u)$, if $\exists$ a ~~tab~~ nbhd $V(x_0,\xi_0) \subset \overline{T^*U}$ such that $\forall A \in \Psi_h^0(\mathbb{R}^n)$ compactly supported in U, $WF_h(A) \subset V$, we have $\|Au\|_{H_h^N} \le C_N h^N$ $\forall N$, i.e. $\underline{Au = O(h^\infty)}_{C^\infty}$.

This defines a closed set $WF_h(u) \subset \overline{T^*U}$.

<u>Remarks</u> (0) Same as the Fourier tr. definition we had before – see pset 7

(1) This definition is inspired by the following definition of support supp u for $u \in \mathcal{D}'(U)$: $x_0 \notin \operatorname{supp} u \iff \exists$ nbhd $V(x_0)$ s.t. $\forall \chi \in C_c^\infty(U)$, $\operatorname{supp} \chi \subset V$, we have $\chi u = 0$.

(2) h-temperedness is useful because $A \in \Psi_h^0$ cpctly supported in U, u h-tempered, ~~$\Rightarrow A$~~ $WF_h(A) = \emptyset \Rightarrow \|Au\|_{H_h^N} \le C_N h^N$ $\forall N$;

indeed, $A = O(h^\infty)_{\Psi^{-\infty}} \ldots$

(3) For $u \in \mathcal{D}'(U)$ (h-tempered), $B \in \Psi_h^k$ compactly supported in U, we have $WF_h(\cancel{A}Bu) \subset WF_h(B) \cap WF_h(u)$.

<u>Indeed</u>, • $(x_0,\xi_0) \notin WF_h(u) \Rightarrow$ take V from <u>Def</u> above, then $\forall A \in \Psi_h^0$, $WF_h(AB) \subset WF_h(A)$. So, $WF_h(A) \subset V \Rightarrow A(Bu) = O(h^\infty)_{C^\infty} \ldots$

• $(x_0,\xi_0) \notin WF_h(B) \Rightarrow$ Just take

$V := T^*U \setminus WF_h(B)$ in Def.

Then $WF_h(A) \subset V \Rightarrow WF_h(AB) = \emptyset$, since

$$\boxed{WF_h(AB) \subset WF_h(A) \cap WF_h(B).}$$

So since u is h-tempered, get $ABu = O(h^\infty)_{C^\infty}$...

④ <u>Elliptic estimate</u> gives the following:

if $P \in \Psi_h^k$ is differential, $u \in \mathcal{D}'(U)$ h-tempered,

then $WF_h(u) \subset WF_h(Pu) \cup (\overline{T^*U} \setminus \mathrm{Ell}_h(P))$.

<u>Indeed</u>, <u>put</u> $f := Pu$. Assume that

$(x_0,\xi_0) \in \overline{T^*U}$ satisfies

$(x_0,\xi_0) \notin WF_h(f)$, $(x_0,\xi_0) \in \mathrm{Ell}_h(P)$.

We need to show that $(x_0,\xi_0) \notin WF_h(u)$.

Since $(x_0,\xi_0) \notin WF_h(f)$, can choose $V \subset \overline{T^*U}$ a nbhd of (x_0,ξ_0)

such that $\forall B \in \Psi_h^0$ comp. supp. in U, $WF_h(B) \subset V$,

$Bf = O(h^\infty)_{C^\infty}$. ~~We have~~ Fix B like that & satisfying $(x_0,\xi_0) \in \mathrm{ell}_h(B)$ [by quantizing: $B = \tilde{\chi}\,\mathrm{Op}_h(b)\tilde{\chi}$, $\mathrm{supp}\, b \subset V$, $b(x_0,\xi_0) = 1$]

Then $(x_0,\xi_0) \in \mathrm{Ell}_h(BP)$, since $\sigma_h(BP) = \sigma_h(B)\sigma_h(P)$.

Apply the <u>elliptic estimate</u> to the operator BP –

– can still do it with same <u>proof</u> (<u>note</u>: BP compactly supp. inside U). $\boxed{\text{Put } W := \mathrm{Ell}_h(BP) \subset \overline{T^*U}.}$

Get: ~~$\exists \chi \in C_c^\infty(U)$ s.t.~~ $\forall A \in \Psi_h^0$, comp. supp. in U,

$WF_h(A) \subset \mathrm{Ell}_h(BP) =: W$, ~~we have~~ $\exists \chi \in C_c^\infty(U)$ s.t.

$$\|Au\|_{H_h^N} \le C\|\chi BPu\|_{H_h^{N-k}} + C_N h^N \|\chi u\|_{H_h^{-N}} = O(h^\infty).$$

$O(h^\infty)$ as $Bf = O(h^\infty)_{C^\infty}$ — finite for large N & poly. bdd. in h

So we constructed $W \subset \overline{T^*U}$ a nbhd of (x_0, ξ_0) s.t. $\forall A \in \Psi_h^0(\mathbb{R}^n)$ comp. supp. in U, $\mathrm{WF}_h(A) \subset W$, we have $Au = O(h^\infty)_{C^\infty}$. Thus $(x_0, \xi_0) \notin \mathrm{WF}_h(u)$ as needed. $\square$

<u>Recall</u>: in Feb 28 lecture, we had the elliptic WF set statement which follows from (4):

$$Pu = 0,\ u \text{ } h\text{-tempered} \Rightarrow \mathrm{WF}_h(u) \subset \overline{T^*U} \setminus \underset{\{(x,\xi)\,|\,\langle\xi\rangle^{-k}\sigma_h(P)(x,\xi)\neq 0\}}{\underset{\|}{\mathrm{Ell}_h(P)}}$$

in the special case of $P = -h^2\partial_x^2 + V$ on $\mathbb{R}$... We finally get a proof of that one.

Let us introduce one last fundamental tool which will be used in the proof of propagation of singularities:

<u>Sharp Gårding Inequality</u> $\boxed{\text{READ [Zw, Thm 4.32 + Thm 9.11]}}$

Assume $A \in \Psi_h^k(\mathbb{R}^n)$ and $\boxed{\mathrm{Re}\,\sigma_h(A) \geq 0.}$ Then $\exists C$ $\forall u \in C_c^\infty(\mathbb{R}^n)$, we have

$$\mathrm{Re}\,\langle Au, u\rangle_{L^2} \geq -Ch\|u\|_{H_h^{\frac{m-1}{2}}}.$$

<u>"Proof"</u> Will do the easy case when $\mathrm{Re}\,\sigma_h(A) = |b|^2$ for some $b \in S_h^{k/2}$. See [Zw] for the harder general case.

$$\mathrm{Re}\,\langle Au, u\rangle = \tfrac{1}{2}\left(\langle Au, u\rangle + \langle A^*u, u\rangle\right) = \tfrac{1}{2}\langle (A + A^*)u, u\rangle.$$

Replacing A with $\frac{A+A^*}{2}$, may assume that $A^* = A$ and $\sigma_h(A) = |b|^2$. Now, put $B := \mathrm{Op}_h(b)$. Then $A = B^*B + h\Psi_h^{k-1}$, i.e. $A = B^*B + hR$, $R \in \Psi_h^{k-1}$.

So then $\langle Au,u\rangle = \langle B^*Bu,u\rangle + h\langle Ru,u\rangle$.

Now, $\langle B^*Bu,u\rangle = \|Bu\|^2_{L^2} \geq 0$.

And $|h\langle Ru,u\rangle| \leq h \cdot \|Ru\|_{H_h^{\frac{1-m}{2}}} \cdot \|u\|_{H_h^{\frac{m-1}{2}}}$

$\leq Ch\|u\|^2_{H_h^{\frac{m-1}{2}}}$ since $\|R\|_{H_h^{\frac{m-1}{2}} \to H_h^{\frac{1-m}{2}}} \leq C$. $\square$

Actually, we will use a slightly stronger version:

<u>Thm[Upgraded sharp Gårding inequality]</u> [DyZw, Proposition E.35]

Assume that $A \in \Psi_h^k(\mathbb{R}^n)$, $B, B_1 \in \Psi_h^0(\mathbb{R}^n)$ are compactly supported inside $U \subset \mathbb{R}^n$ and

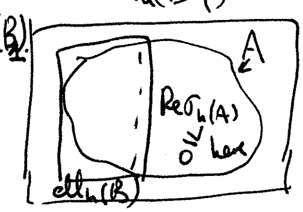
& $\operatorname{Re}\sigma_h(A) \geq 0$ in a nbhd of $T^*U \setminus \operatorname{ell}_h(B)$

& $\operatorname{WF}_h(A) \subset \operatorname{ell}_h(B_1)$.

Then $\exists C$, $\exists \chi \in C_c^\infty(U)$ s.t. $\forall N$, $\forall u \in C^\infty(U)$,

$$\operatorname{Re}\langle Au,u\rangle \geq -C\|Bu\|^2_{H_h^{k/2}} - Ch\|B_1u\|^2_{H_h^{\frac{k-1}{2}}} - O(h^\infty)\|\underbrace{\chi}_{\operatorname{ell}_h(B_1)} u\|^2_{H_h^{-N}}$$

<u>Proof</u>. For simplicity assume $k=0$, $\operatorname{WF}_h(B) \subset \operatorname{ell}_h(B_1)$.



(1) Reduce to the case $B=0$, $\operatorname{Re}\sigma_h(A) \geq 0$ everywhere:

we can find a large constant $C_0 > 0$ such that

$\operatorname{Re}\sigma_h(A) + C_0|\sigma_h(B)|^2 \geq 0$ everywhere.

So we may replace A with $\tilde{A} := A + C_0B^*B$, so

$\operatorname{Re}\sigma_h(\tilde{A}) \geq 0$ everywhere, $\operatorname{WF}_h(\tilde{A}) \in \operatorname{ell}_h(B_1)$,

$\langle \tilde{A}u,u\rangle = \langle Au,u\rangle + C_0\|Bu\|^2_{L^2}$.

2. Now it remains to handle the case $B=0$.

We have: $WF_h(A) \subset \mathrm{ell}_h(B_1)$. So there exists $X \in \Psi_h^0$, X comp. supp. in U (since A is...)

and $WF_h(A) \cap WF_h(I-X) = \emptyset$,

$WF_h(X) \subset \mathrm{ell}_h(B_1)$.

(indeed, take $X = \chi \mathrm{Op}_h(\tilde{\chi})\chi$ for some cutoff $\chi \in C_c^\infty(U)$ and $\tilde{\chi}$ s.t. $\tilde{\chi} = 1$ near $WF_h(A)$, $\mathrm{supp}\,\tilde{\chi} \subset \mathrm{ell}_h(B_1)$).

Now, write for some $\chi \in C_c^\infty(U)$

$$\mathrm{Re}\langle Au, u\rangle = \mathrm{Re}\langle AXu, Xu\rangle + O(h^\infty)\|\chi u\|^2_{H_h^{-N}}$$

because $X^*AX - A = (X^* - I)AX + A(X-I)$
$= O(h^\infty)_{\Psi^{-\infty}}$ comp. supp. in U.

Note: $Xu \in \mathcal{E}'(U) \subset \mathcal{E}'(\mathbb{R}^n)$

Finally, by the original sharp Gårding inequality

$$\mathrm{Re}\langle AXu, Xu\rangle \geq -Ch\|Xu\|_{H_h^{-1/2}}.$$

Here we applied it to X^*AX, $\mathrm{Re}\,\sigma_h(X^*AX) = \mathrm{Re}\,\sigma_h(A)\cdot|\sigma_h(X)|^2 \ldots$

Now $WF_h(X) \subset \mathrm{ell}_h(B_1)$ so by the elliptic estimate

$$\|Xu\|_{H_h^{-1/2}} \leq C\|B_1 u\|_{H_h^{-1/2}} + O(h^\infty)\|\chi u\|^2_{H_h^{-N}} \ldots \quad \square$$

Hamiltonian flow

Assume $p \in S^k(T^*\mathbb{R}^n)$ and p is real valued.

Define the Hamiltonian vector field

H_p on $T^*\mathbb{R}^n$ by

$$H_p = \sum_j (\partial_{\xi_j} p) \cdot \partial_{x_j} - \sum_j (\partial_{x_j} p) \cdot \partial_{\xi_j}.$$

Note: for $a \in C^\infty(T^*\mathbb{R}^n)$, $H_p a = \{p, a\}$.

Can consider the flow $\exp(tH_p) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$

(might not be defined for all t ...)

How to extend to $\overline{T}^*\mathbb{R}^n$? Want to get sth.

homogeneous of degree 0, i.e. $H_p a \in \mathrm{Hom}^0$ for $a \in \mathrm{Hom}^0$.

Easy to compute: $a \in \mathrm{Hom}^k$, $b \in \mathrm{Hom}^l \Rightarrow \{a, b\} \in \mathrm{Hom}^{k+l-1}$.

So, $p \in S^1 \Rightarrow \exp(tH_p)$ could just be extended as is.

In general: consider the vector field

$\langle \xi \rangle^{1-k} H_p$ on $T^*\mathbb{R}^n$. It extends to a

smooth vector field on $\overline{T}^*\mathbb{R}^n$ which is tangent

to the fiber infinity $\partial \overline{T}^*\mathbb{R}^n$.

Note: all this works for more general manifolds, M,

with $\{\cdot, \cdot\}$ and H_p defined since T^*M has

natural symplectic form $\omega = d\alpha$, $\alpha = \xi dx$ canonical 1-form.