An introduction to microlocal analysis

Will skip a lot of technical details and proofs.
For details, see the book

[Zw] Maciej Zworski, "Semiclassical Analysis", AMS GSM 138.

Semiclassical quantization on $\mathbb{R}^n$

Let $a(x,\xi)$ be some smooth function on $\mathbb{R}^{2n}=\mathbb{R}^n_x\times\mathbb{R}^n_\xi$.

Want to define $\mathrm{Op}_h(a)$: (fns on $\mathbb{R}^n$) $\to$ (fns on $\mathbb{R}^n$)

$\boxed{\text{READ: ZWORSKI, §4.1–4.4}}$

by the formula

$$(*)\quad \mathrm{Op}_h(a)u(x)=(2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle}a(x,\xi)\,u(y)\,dy\,d\xi.$$

(also denote $\mathrm{Op}_h(a)=a(x,hD_x)$)

Rmk

- a is called symbol, $\mathrm{Op}_h(a)$ is a pseudodifferential operator; $\mathrm{Op}_h(a)$ is the quantization of a.
- Other quantizations possible, e.g. [Zw] typically uses Weyl quantization

$$\mathrm{Op}_h^w(a)u(x)=(2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle}a\left(\tfrac{x+y}{2},\xi\right)u(y)\,dy\,d\xi$$

For the purposes of basic theory developed here, either quantization can be used.
- Our goal is: introduce a reasonable class of symbols a and study algebraic properties of $\mathrm{Op}_h(a)$ & mapping properties of $\mathrm{Op}_h(a)$ on L^2-based spaces.

Basic case: $a \in C_c^\infty(\mathbb{R}^{2n})$.

Then the integral in (*) converges for all $f \in L^1$, giving $Op_h(a): L^1 \to L^\infty$. Will often just do proofs in this case.

We can write an alternative to (*):

$$(**) \quad Op_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi)\, \hat{u}(\xi/h)\, d\xi.$$

Note: $a \in C_c^\infty(\mathbb{R}^{2n}) \Rightarrow Op_h(a): \underset{\uparrow}{S'}(\mathbb{R}^n) \to \underset{\uparrow}{S}(\mathbb{R}^n)$

tempered distributions — Schwartz fns

However, want a class that involves <u>differential operators</u>...

<u>Standard (Kohn-Nirenberg) symbol classes</u>. [Zw, §9.3]

Say $k \in \mathbb{R}$. Define $S^k_{1,0} \subset C^\infty(\mathbb{R}^{2n})$ as follows: $a(x,\xi)$ lies in $S^k_{1,0}$ iff $\forall$ multiindices α, β, $\exists\, C_{\alpha\beta} < \infty$ s.t. $\forall x, \xi$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{k-|\beta|}$$

Here $\langle \xi \rangle = \sqrt{1+|\xi|^2}$, basically like $1+|\xi|$ but smooth at $\xi=0$.

The best constants $C_{\alpha\beta}$ are seminorms on $S^k_{1,0}$ & they make it into a Fréchet space.

<u>Examples & properties</u>

- $a \in C_c^\infty \Rightarrow a \in S^k_{1,0}$ $\forall k$
- $a = \sum_{|\alpha| \le k} a_\alpha(x) \xi^\alpha$, $k \in \mathbb{N}$, a_α bdd with all derivatives $\Rightarrow a \in S^k_{1,0}$.
- $a \in S^k_{1,0}$, $b \in S^\ell_{1,0} \Rightarrow ab \in S^{k+\ell}_{1,0}$

Classical symbols (=those with good asymptotic expansions) $\boxed{\text{READ: Dy-Zw, §E.1.2.}}$

<u>Def.</u> $a \in C^\infty(\mathbb{R}^{2n})$ is <u>positively homogeneous</u> of order $k \in \mathbb{R}$ (could even take $k \in \mathbb{C}$), if

$$a(x, \tau\xi) = \tau^k a(x,\xi) \quad \text{when } |\xi| \geq 1, \ \tau \geq 1.$$

We also assume that $\partial_x^\alpha \partial_\xi^\beta a(x,\xi)$ is bdd in x when $|\xi| \leq 1$.

<u>Note</u>: then for $|\xi| \geq 1$, we have $a(x,\xi) = |\xi|^k a(x, \frac{\xi}{|\xi|})$

From here we see that $a \in S^k_{1,0}$.

We denote the class of positively homogeneous a by $\mathrm{Hom}^k \subset S^k_{1,0}$.

<u>Def.</u> Assume that $a_j \in S^{k-j}_{1,0}$ for $j = 0, 1, \dots$

We say that $a \in S^k_{1,0}$ is the asymptotic sum of a_j, and write $a \sim \sum_{j=0}^\infty a_j$, if

$$\forall J, \quad a - \sum_{j=0}^{J-1} a_j \in S^{k-J}_{1,0}.$$

<u>Rmk.</u>

1. For each $\{a_j \in S^{k-j}_{1,0}\}$, $\exists a \in S^k_{1,0}$ s.t. $a \sim \sum_j a_j$. This is a version of Borel's Lemma, see [Zw, Thm 4.15]

2. If $a \sim \sum_j a_j$, $b \sim \sum_j a_j$, then
$$a - b \in S^{-\infty} := \bigcap_{k \in \mathbb{R}} S^k_{1,0};$$
rapidly decaying (in ξ) symbols:
$$a \in S^{-\infty} \iff \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{-\infty}).$$

The following class of symbols will be ~~mostly used~~ used most of the time:

<u>Def.</u> We say $a \in S^k$, if $a \sim \sum_{j=0}^{\infty} a_j$ for some $\{a_j \in \cancel{S}\,\mathrm{Hom}^{k-j}\}$. We call a <u>classical symbol</u> of order k.

<u>Informally</u>, $a(x, r\cdot\omega) \sim r^k a_0(x,\omega) + r^{k-1} a_1(x,\omega) + \dots$ as $r \to \infty$, $|\omega| = 1$

<u>Note</u>: $S^k \subset S^{k+1}$, $S^k \cdot S^\ell \subset S^{k+\ell}$.

<u>Basic mapping properties</u>:

- Using $(**)$, we see that $a \in S^k \Rightarrow \mathrm{Op}_h(a): \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.
  Indeed, if $u, v \in \mathcal{S}(\mathbb{R}^n)$, then
  $$\langle \mathrm{Op}_h(a)u, v\rangle = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi)\, \hat{u}(\tfrac{\xi}{h}) v(x)\, dx\, d\xi$$
  is well-defined.
  However, we actually have
  $$\mathrm{Op}_h(a): \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$
  To see the first one, note that
  $u \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \mathrm{Op}_h(a)u$ is C^∞, bdd with all derivatives.
  Next, $$x_j \mathrm{Op}_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} \left(\frac{h}{i}\partial_{\xi_j} e^{\frac{i}{h}\langle x,\xi\rangle}\right) \cdot a(x,\xi)\, \hat{u}(\xi/h)\, d\xi$$
  $$\overset{\text{IBP in } \xi}{=} ih\, (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} \left(\partial_{\xi_j} a(x,\xi)\, \hat{u}(\xi/h) + h\, \overbrace{\hat{u}_{\xi_j}(\xi/h)}^{a(x,\xi)}\right) d\xi$$
  $$= \mathrm{Op}_h(a)\, x_j u(x) + ih\, \mathrm{Op}_h(\partial_{\xi_j} a)u(x) \dots$$

$\boxed{\text{READ: [Zw, Thm 4.16]}}$

- If $a(x,\xi) = \sum_{|\alpha| \le k} a_\alpha(x)\, \xi^\alpha$, then

$$\mathrm{Op}_h(a) = \sum_{|\alpha|\le k} a_\alpha(x)\,(h D_x)^\alpha \quad \text{as in pset 5;} \quad D = \tfrac{1}{i}\partial.$$

- If $a(x,\xi) = a(\xi)$, then

$\mathrm{Op}_h(a) = a(hD_x)$ is a Fourier multiplier:

$$\widehat{\mathrm{Op}_h(a)\, u}\,(\xi) = a(h\xi)\,\hat{u}(\xi).$$

To do algebra, need also h-dependent symbol classes.

__Def.__ Say $a(x,\xi;\underline{h}) \in S^k_{1,0,h}$ if $\exists\, C_{\alpha\beta}$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \le C_{\alpha\beta} \langle \xi \rangle^{k-|\beta|} \quad \text{for all } x,\xi,\ \underline{h \in (0,1]}.$$

__Def.__ Say $a \in S^k_{1,0,h}$ is an asymptotic sum,

$$a(x,\xi;h) \sim \sum_\ell h^\ell a_\ell(x,\xi), \quad a_\ell \in S^{k-\ell}_{1,0,h}, \text{ if } \forall L,$$

$$a - \sum_{\ell=0}^{L-1} h^\ell a_\ell \in h^L S^{k-L}_{1,0,h}.$$

Given $\{a_\ell\}$, such a exists & it is unique up to the class $h^\infty S^{-\infty} = \bigcap_k h^k S^{-k}_{1,0,h}$;

$a \in h^\infty S^{-\infty} \Leftrightarrow \forall \alpha,\beta,\ \partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) = O(h^\infty \langle\xi\rangle^{-\infty})$,

i.e. $O(h^N \langle \xi\rangle^{-N})\ \forall N$.

__Def.__ Say $a(x,\xi;h)$ is a semiclassical classical symbol of order k, if

$a \sim \sum h^\ell a_\ell(x,\xi)$ for some $a_\ell \in S^{k-\ell}$ ← classical symbols

Write: $\boxed{a \in S^k_h.}$ ← WILL USE THIS NOTATION A LOT

THE CALCULUS

(~~will do for S^k_h, could use the class $S^k_{1,0,h}$ instead~~)

Product Rule: Assume $a \in S^k_{1,0,h}$, $b \in S^\ell_{1,0,h}$.

Then $Op_h(a)\, Op_h(b) = Op_h(c)$ (defined as op. $\mathcal{S} \to \mathcal{S}$, $\mathcal{S}' \to \mathcal{S}'$)

for some $c \in S^{k+\ell}_{1,0,h}$ and we have

the asymptotic expansion

$$c(x,\xi;h) \sim \sum_{j=0}^{\infty} (-ih)^j \underbrace{\sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi;h) \cdot \partial_x^\alpha b(x,\xi;h)}_{\text{This is in } S^{k+\ell-j}_{1,0,h} \text{ (!)}}$$

$\boxed{\text{READ! [Zw, Theorem 4.11 + 4.17]}}$

Important Corollaries: denote $c := a \# b$, i.e. $Op_h(a)\, Op_h(b) = Op_h(a\#b)$.

① $a\#b = ab + h S^{k+\ell-1}_{1,0,h}$.

② $a\#b - b\#a = -ih\{a,b\} + h^2 S^{k+\ell-2}_{1,0,h}$

where $\{a,b\} = \sum_j \partial_{\xi_j} a \cdot \partial_{x_j} b - \partial_{x_j} a \cdot \partial_{\xi_j} b$.

Adjoint Rule: Assume $a \in S^k_{1,0,h}$. Then

$Op_h(a)^* = Op_h(a^*)$ where $a^* \in S^k_{1,0,h}$ and

$$a^*(x,\xi;h) \sim \sum_{j=0}^{\infty} (-ih)^j \underbrace{\sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha a(x,\xi;h)}_{\text{This is in } S^{k-j}_{1,0,h} \text{ (!)}}.$$

Important Corollary:

$a^* = a + h S^{k-1}_{1,0,h}$.

Another important corollary of the Product Rule:

Pseudolocality

Assume $a \in S^k_{1,0,h}$; $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^n)$ and $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$.

Then $\chi_1 \operatorname{Op}_h(a)\chi_2 = \mathcal{O}(h^\infty)_{\mathcal{D}' \to C_c^\infty}$,

namely $\chi_1 \operatorname{Op}_h(a)\chi_2$ has the integral form

$$\chi_1 \operatorname{Op}_h(a)\chi_2 u(x) = \int_{\mathbb{R}^n} K(x,y;h)\,u(y)\,dy \quad \text{and}$$

$$K \in C_c^\infty(\mathbb{R}^{2n}), \quad \|K\|_{C^N(\mathbb{R}^{2n})} \le C_N h^N \quad \forall N.$$

Indeed, since $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$, all terms in the asymptotic expansion for $c = \chi_1 \# a \# \chi_2$ are zero.

Thus $c \in h^\infty S^{-\infty}$, and $\chi_1 \operatorname{Op}_h(a)\chi_2 = \operatorname{Op}_h(c)$.

It has the integral form with

$$K(x,y;h) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} c(x,\xi)\,d\xi.$$

Since $c \in h^\infty S^{-\infty}$, we directly see that

$K \in C^\infty(\mathbb{R}^{2n})$ and $\forall \alpha,\beta,N \ \sup |\partial_x^\alpha \partial_y^\beta K(x,y;h)| \le C_{\alpha\beta N} h^N$.

Also, K is compactly supported since $\operatorname{supp} K \subset \operatorname{supp}\chi_1 \times \operatorname{supp}\chi_2$.

Note: pseudolocality tells us that as long as we do not care for $\mathcal{O}(h^\infty)_{\mathcal{D}' \to C_c^\infty}$ remainders, the interesting part of the integral kernel of $\operatorname{Op}_h(a)$ lives near the diagonal $\{x=y\}$.

Of course differential operators have locality, i.e.

$A \in \operatorname{Diff}^k_h$, $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset \Rightarrow \chi_1 A \chi_2 = 0$.

An important tool used in the proofs of Product Rule and Adjoint Rule is

$\boxed{\text{READ [Zw, Theorem 4.19]}}$

Oscillatory testing:

① Assume $a \in S^k_{h,1,0}$. Denote for $\eta \in \mathbb{R}^n$ the fn. $e_\eta(x) = e^{\frac{i}{h}\langle x, \eta\rangle}$. Then

$$\boxed{Op_h(a)\, e_\eta(x) = e^{\frac{i}{h}\langle x,\eta\rangle} \cdot a(x,\eta;h).}$$

This formula makes sense since $e_\eta \in S'$.

Proof We do the case $a \in C_c^\infty(\mathbb{R}^{2n})$. Then

$$Op_h(a) e_\eta(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi)\, \hat{e}_\eta\left(\tfrac{\xi}{h}\right) d\xi.$$

But $\hat{e}_\eta(\xi) = (2\pi)^n \delta(\xi - \frac{\eta}{h})$, so

$\hat{e}_\eta\left(\frac{\xi}{h}\right) = (2\pi h)^n \delta(\xi - \eta)$, finishing the proof

(Here we did use Fourier transform of tempered distributions.)

② Assume $A: S' \to S'$ and $a \in S^k_{h,1,0}$ satisfy $\forall \eta$, $\boxed{A e_\eta(x) = e^{\frac{i}{h}\langle x,\eta\rangle} \cdot a(x,\eta;h).}$

Then $A = Op_h(a)$.

Proof. Take any $u \in \mathcal{S}(\mathbb{R}^n)$ & write $u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\eta\rangle} \hat{u}(\eta)\, d\eta$

$= (2\pi h)^{-n} \int_{\mathbb{R}^n} e_\eta(x)\, \hat{u}(\eta/h)\, d\eta$. Write next $Au(x) =$

$= (2\pi h)^{-n} \int_{\mathbb{R}^n} \hat{u}(\eta/h) \cdot A e_\eta(x)\, d\eta = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\eta\rangle} a(x,\eta;h)\, \hat{u}(\eta/h)\, d\eta$

$= Op_h(a)\, u(x)$