1. **Fix usual pairing**

| Note: since $\langle \cdot \rangle$ |
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18.155, FALL 2021, PROBLEM SET 8

**Review / helpful information:**

- $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$. Note that $C^{-1}(1 + |\xi|) \leq \langle \xi \rangle \leq C(1 + |\xi|)$ for some global constant $C > 0$ and $\langle \xi \rangle$ is smooth in $\xi$.

- Plancherel Theorem: for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ we have $\langle \hat{\varphi}, \hat{\psi} \rangle_{L^2(\mathbb{R}^n)} = (2\pi)^n \langle \varphi, \psi \rangle_{L^2(\mathbb{R}^n)}$.

- Sobolev space $H^s(\mathbb{R}^n)$: $u \in \mathcal{S}'(\mathbb{R}^n)$ lies in $H^s(\mathbb{R}^n)$ if and only if $\langle \xi \rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)$. Define $\|u\|_{H^s} := (2\pi)^{-n/2} \langle \xi \rangle^s \hat{u}(\xi)_{L^2(\mathbb{R}^n)}$.

- Note that $H^0 = L^2$ and $H^t \subset H^s$ when $t \geq s$.

- If $s \in \mathbb{N}_0$ is a nonnegative integer, then $u \in \mathcal{S}'(\mathbb{R}^n)$ lies in $H^s(\mathbb{R}^n)$ if and only if each distributional derivative $\partial^\alpha u$, $|\alpha| \leq s$, lies in $L^2(\mathbb{R}^n)$.

- If $0 < s < 1$, then for each $u \in L^2(\mathbb{R}^n)$

$$u \in H^s(\mathbb{R}^n) \iff \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy < \infty. \quad (1)$$

- Local Sobolev spaces: if $U \subset \mathbb{R}^n$ is open, then $H^s_{\text{loc}}(U) \subset \mathcal{D}'(U)$ is defined as follows: $u \in \mathcal{D}'(U)$ lies in $H^s_{\text{loc}}(U)$ if and only if $\psi u \in H^s(\mathbb{R}^n)$ for each $\psi \in C^\infty_c(U)$. (Here $\psi u$ is in $\mathcal{E}'(U)$ which naturally embeds into $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.)

- Sobolev spaces with compact support: if $U \subset \mathbb{R}^n$ is open, then $H^s_c(U) \subset \mathcal{E}'(U)$ consists of elements of $H^s(\mathbb{R}^n)$ whose support is contained in $U$.

- Hölder space $C^\gamma(\mathbb{R}^n)$, $0 < \gamma < 1$: a function $u \in C^0(\mathbb{R}^n)$ lies in $C^\gamma(\mathbb{R}^n)$ if for each compact set $K \subset \mathbb{R}^n$ there exists a constant $C$ such that for all $x, y \in K$ we have $|u(x) - u(y)| \leq C|x - y|\gamma$. The space $C^\gamma_c(\mathbb{R}^n)$ consists of compactly supported functions in $C^\gamma(\mathbb{R}^n)$.

- Constant coefficient differential operators of order $m \in \mathbb{N}_0$ have the form $P = \sum_{|\alpha| \leq m} c_\alpha D^\alpha$ where $c_\alpha \in \mathbb{C}$ and $D := -i \partial$. The principal symbol is $p_0(\xi) = \sum_{|\alpha| = m} c_\alpha \xi^\alpha$. We say $P$ is elliptic if the equation $p_0(\xi) = 0$ has no solutions $\xi \in \mathbb{R}^n \setminus \{0\}$.

1. **Fix $s \in \mathbb{R}$. This exercise shows that $H^{-s}(\mathbb{R}^n)$ is dual to $H^s(\mathbb{R}^n)$ with respect to the usual pairing**

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x) g(x) \, dx. \quad (2)$$

(Note: since $H^s$ is a Hilbert space, Riesz representation theorem shows that $H^s$ is dual to itself, but this duality features the inner product $\langle \cdot, \cdot \rangle_{H^s}$ rather than $(2)$.)

(a) **Show that there exists a unique bilinear map**

$$u \in H^s(\mathbb{R}^n), v \in H^{-s}(\mathbb{R}^n) \mapsto (u, v) \in \mathbb{C}$$
such that (i) for all $u, v \in \mathcal{S}(\mathbb{R}^n)$, $(u, v)$ is given by (2) and (ii) there exists a constant $C$ such that for all $u, v$ we have the bound $|\langle u, v \rangle| \leq C \|u\|_{H^s} \|v\|_{H^{-s}}$. A consequence of this is that each $v \in H^{-s}(\mathbb{R}^n)$ defines a bounded linear functional on $H^s(\mathbb{R}^n)$ by the rule $u \mapsto (u, v)$.

(b) Assume that $F : H^s(\mathbb{R}^n) \to \mathbb{C}$ is a bounded linear functional. Show that there exists $v \in H^{-s}(\mathbb{R}^n)$ such that $F(u) = (u, v)$ for all $u \in H^s(\mathbb{R}^n)$.

2. This exercise studies the relation between the spaces $C^k(\mathbb{R}^n)$ of $k$ times continuously differentiable functions and the Sobolev spaces $H^s(\mathbb{R}^n)$.

(a) Show that for each $k \in \mathbb{N}_0$, the space $C^k_c(\mathbb{R}^n)$ (where ‘c’ stands for ‘compactly supported’) embeds into $H^k(\mathbb{R}^n)$: that is, $C^k_c(\mathbb{R}^n) \subset H^k(\mathbb{R}^n)$ and for each sequence $u_j \in C^k_c(\mathbb{R}^n)$ converging to 0 (in a way similar to convergence in $C^\infty_c$ but with only $k$ derivatives), we have $\|u_j\|_{H^k(\mathbb{R}^n)} \to 0$ as well.

(b) Show the following version of Sobolev embedding: if $k \in \mathbb{N}_0$ and $s > k + \frac{n}{2}$ then $H^s(\mathbb{R}^n)$ embeds into the space $\tilde{C}^k(\mathbb{R}^n)$ of functions in $C^k(\mathbb{R}^n)$ with bounded derivatives up to order $k$. (Hint: for $u \in \mathcal{S}(\mathbb{R}^n)$, use Fourier inversion formula and the Cauchy–Schwarz inequality to bound the $\tilde{C}^k$ norm of $u$ by $\|\langle \xi \rangle^k \hat{u}(\xi)\|_{L^1}$, which is bounded in terms of $\|u\|_{H^s}$. Now, each $u \in H^s(\mathbb{R}^n)$ can be approximated by Schwartz functions, and this approximating sequence will be a Cauchy sequence in $\tilde{C}^k$, which is a Banach space – this step is similar to the proof of the Continuous Linear Extension theorem.)

3. (Optional) This exercise extends the previous one by comparing Sobolev spaces with Hölder spaces. Assume that $0 < \gamma < 1$.

(a) Show that $C^\gamma(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ for each $s < \gamma$. (Hint: use (1). Note that the integral there is bounded for any $u \in L^2(\mathbb{R}^n)$ if we restrict to the region $|x - y| \geq 1$.)

(b) Show that $H^s(\mathbb{R}^n) \subset C^\gamma(\mathbb{R}^n)$ for each $s > \gamma + \frac{n}{2}$. (Hint: write each $u \in H^s(\mathbb{R}^n)$ in terms of $\hat{u}$ using the Fourier inversion formula, and use the inequality $|e^{ix\xi} - e^{iy\xi}| = |e^{i(x-y)\xi} - 1| \leq C_\gamma |x - y|^{\gamma} |\xi|^\gamma$.)

4. Let $U \subset \mathbb{R}^n$ be an open set. Assume that $P$ is an elliptic constant coefficient differential operator of order $m$. Following Step 2 of the proof of Elliptic Regularity II in §12.2 of the lecture notes, show that for each $u \in \mathcal{D}'(U)$ such that $Pu \in H^{s-m}_{\text{loc}}(U)$, we have $u \in H^s_{\text{loc}}(U)$. (You do not need to reprove the existence of elliptic parametrix.)

5. For the distributions below, find out for which $s$ they lie in $H^s(\mathbb{R}^n)$:

(a) $\delta_0$;

(b) the indicator function of the some interval $[a, b] \subset \mathbb{R}$ (here $n = 1$).

6. (Optional) This exercise forms the basis for the theorem about restricting elements of Sobolev spaces to hypersurfaces, which is important for the study of boundary
value problems. We write elements of $\mathbb{R}^n$ as $(x_1, x')$ where $x' \in \mathbb{R}^{n-1}$, and consider the restriction operator to $\{x_1 = 0\}$,

$$T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^{n-1}), \quad T\varphi(x') = \varphi(0, x').$$

Show that when $s > \frac{1}{2}$, there exists a constant $C$ such that we have the bound

$$\|T\varphi\|_{H^{s - \frac{1}{2}}(\mathbb{R}^{n-1})} \leq C\|\varphi\|_{H^s(\mathbb{R}^n)}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Thus by Continuous Linear Extension $T$ extends to a bounded operator $H^s(\mathbb{R}^n) \rightarrow H^{s - \frac{1}{2}}(\mathbb{R}^{n-1})$. (Hint: use Fourier Inversion Formula to write the Fourier transform of $T\varphi$ in terms of the integral of $\hat{\varphi}$ in the $\xi_1$ variable. Next, if $v \in L^2(\mathbb{R}^n)$, then we can use Cauchy–Schwartz to estimate $\int_{\mathbb{R}} \langle \xi \rangle^{-s} v(\xi_1, \xi') d\xi_1$ in terms of the $L^2$ norms of the functions $\xi_1 \mapsto (1 + |\xi_1|^2 + |\xi'|^2)^{-s/2}$ and $\xi_1 \mapsto v(\xi_1, \xi')$. It remains to show that the first of these norms is bounded by $C\langle \xi' \rangle^{\frac{1}{2} - s}$.)

7. This exercise establishes coordinate invariance of Sobolev spaces, which is key for defining Sobolev spaces on manifolds. Assume that $U, V \subset \mathbb{R}^n$ are open sets and $\Phi : U \rightarrow V$ is a $C^\infty$ diffeomorphism. Recall the pullback operator $\Phi^* : \mathcal{E}'(V) \rightarrow \mathcal{E}'(U)$.

We will show that $v \in H^s_c(V) \Rightarrow \Phi^* v \in H^s_c(U)$ (3) and for each compact $K \subset V$ there exists a constant $C$ such that $\|\Phi^* v\|_{H^s} \leq C\|v\|_{H^s}$ for all $v \in H^s_c(V)$ such that supp $v \subset K$. (A similar argument shows that $\Phi^*$ maps $H^s_{\text{loc}}(V)$ to $H^s_{\text{loc}}(U)$ as well.)

(a) Show (3) when $s$ is a nonnegative integer. (Hint: use the Chain Rule.)

(b) Show (3) when $0 < s < 1$. You may use the following stronger version of (1): if $A(u)$ is the square root of the right-hand side of (1) then for all $u \in L^2(\mathbb{R}^n)$

$$\|u\|_{H^s} \leq C(\|u\|_{L^2} + A(u)), \quad A(u) \leq C\|u\|_{H^s}.$$

(c) (Optional) Show (3) for all $s \in \mathbb{R}$. (Hint: show that for $s \geq 0$, a function $u \in H^s(\mathbb{R}^n)$ lies in $H^{s+1}(\mathbb{R}^n)$ if and only if $\partial_j u \in H^s(\mathbb{R}^n)$ for all $j$, and reduce to parts (a)–(b). For $s < 0$ and $v \in H^s_c(V)$, show that the functional $\varphi \in \mathcal{S}(\mathbb{R}^n) \mapsto (\Phi^* v, \varphi)$ is bounded in terms of the $H^{-s}$ norm of $\varphi$ and thus extends to a bounded functional on $H^{-s}(\mathbb{R}^n)$, and use Exercise 1.)