Review / helpful information:

• If $\mathcal{X}, \mathcal{Y}$ are Banach spaces, then a bounded operator $P : \mathcal{X} \to \mathcal{Y}$ is Fredholm if the kernel $\ker P$ is finite dimensional, the range $\text{Ran} P$ is a closed subspace of $\mathcal{Y}$, and $\text{Ran} P$ has finite codimension. The index of $P$ is
  $$\text{ind} P := \dim \ker P - \text{codim} \text{Ran} P.$$

  If $Q : \mathcal{X} \to \mathcal{Y}$ is a compact operator, then $P + Q$ is Fredholm and $\text{ind}(P + Q) = \text{ind} P$.

• If $M$ is a compact manifold and $P \in \text{Diff}^m(M)$ is an elliptic differential operator, then $P_s = P : H^s(M) \to H^{s-m}(M)$ is a Fredholm operator for each $s \in \mathbb{R}$. The kernel of $P_s$ is independent of $s$ because elements of it are in $C^\infty(M)$ by Elliptic Regularity III; denote this by $\ker P$. We have
  $$\text{Ran} P_s = \{ w \in H^{s-m}(M) \mid \forall v \in \ker P^* : \langle w, v \rangle_{L^2} = 0 \}$$
  where $P^* \in \text{Diff}^m(M)$ is the (formal) adjoint of $P$.

1. Show that the following elliptic estimate for the Laplacian $\Delta$ on $\mathbb{R}^2$,
  $$\| \psi u \|_{H^2(\mathbb{R}^2)} \leq C \| \chi \Delta u \|_{L^2(\mathbb{R}^2)} + C \| \chi u \|_{L^2(\mathbb{R}^2)}$$
  does not hold when $\psi = \chi$. (You may choose $\chi \in C^\infty_c(\mathbb{R}^2)$ as you want. Hint: try to construct a sequence of solutions to $\Delta u = 0$ of the form $f(x_1)g(x_2)$.)

2. Assume that $(M, g)$ is a compact connected Riemannian manifold and denote by $\Delta_g$ the Laplace–Beltrami operator. Using the material from lecture notes §16 (but not from later sections), show that for any $s \in \mathbb{R}$, the equation
  $$\Delta_g u = f, \quad f \in H^{s-2}(M) \text{ given},$$
  has a solution $u \in H^s(M)$ if and only if $\int_M f \, d\text{vol}_g = 0$.

3. Show that if $M$ is a compact manifold and $P \in \text{Diff}^m(M)$ is an elliptic differential operator, then $P : H^s(M) \to H^{s-m}(M)$ has index 0. (However, differential operators on sections of vector bundles, as well as scalar pseudodifferential operators, can have nonzero index. Hint: first show that $\text{ind}(P) = -\text{ind}(P^t)$ where $P^t$ is the adjoint of $P$, which has principal symbol $(-1)^m \sigma(P)$. Next show that if two operators in $\text{Diff}^m(M)$ have the same principal symbol, then their index is the same.)
4. (Optional) This exercise gives a basic example of a $0^{th}$ order pseudodifferential operator on the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ which has nonzero index. Consider the operators $\Pi^\pm$ on $L^2(S^1)$ defined using Fourier series as follows:

$$\Pi^\pm \left( \sum_{k \in \mathbb{Z}} c_k e^{ikx} \right) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}$$

for any sequence $(c_k) \in \ell^2(\mathbb{Z})$. Let $\ell \in \mathbb{Z}$ and define the operator $P$ on $L^2(S^1)$ by

$$Pf(x) = e^{i\ell x} \Pi^+ f(x) + \Pi^- f(x), \quad f \in L^2(S^1).$$

Show that $P$ is a Fredholm operator of index $-\ell$. (With more knowledge of microlocal analysis, one could actually show that this is true with $e^{i\ell x}$ replaced by any nonvanishing function $a \in C^\infty(S^1)$, and the index of $P$ is minus the winding number of the curve $a : S^1 \to \mathbb{C}$ about the origin – this is a ‘baby index theorem’.)