§4. Support

§4.1. Support of a distribution

Let $U \subset \mathbb{R}^n$ open

Defn. Let $u \in \mathcal{D}'(U)$. Define the support of u, $\operatorname{supp} u \subset U$ follows: $x \in U$ does **not** lie in $\operatorname{supp} u$ iff $\exists$ open V, $x \in V \subset U$, such that $u|_V = 0$.

That is, $x \notin \operatorname{supp} u$ iff $\exists$ a neighborhood $V \subset U$ of x such that $(u, \varphi) = 0 \quad \forall \varphi \in C_c^\infty(V)$

Note: by definition, $\operatorname{supp} u$ is a (relatively) closed subset of U.

Examples: (1) if $u \in L^1_{loc}(U)$ then $\operatorname{supp} u$ is the complement of the set $\{x \in U \mid u = 0$ almost everywhere on a nbhd of $x\}$

In particular, if $u \in C^0(U)$ then supp u (as a distribution) = supp u (as a function), i.e. the closure in U of $\{x \in U \mid u(x) \neq 0\}$.

2 If $u = \delta_y$ for some $y \in U$ then supp $u = \{y\}$. Indeed, $(u, \varphi) = \varphi(y)$, so for any $V \subset U$ open

$$(u, \varphi) = 0 \quad \forall \varphi \in C_c^\infty(V)$$
$$\Updownarrow$$
$$y \notin V.$$

<u>Theorem</u> Let $u \in \mathcal{D}'(U)$. Then

$$u|_{U \setminus \operatorname{supp} u} = 0.$$

<u>Proof</u> Let $\varphi \in C_c^\infty(U \setminus \operatorname{supp} u)$. Then $\forall x \in \operatorname{supp} \varphi$ $\exists$ open set V_x, $x \in V_x \subset U$ and $(u, \psi) = 0$ $\forall \psi \in C_c^\infty(V_x)$. Using a partition of unity, write $\varphi = \varphi_1 + \dots + \varphi_m$, each $\varphi_j \in C_c^\infty(U)$ is supported in one of the sets V_x. Then $(u, \varphi) = (u, \varphi_1) + \dots + (u, \varphi_m) = 0$. $\square$

§4.2. Distributions with compact support

Defn Let $U \subset \mathbb{R}^n$ be open.
Denote $\mathcal{E}'(U) := \{u \in \mathcal{D}'(U) \mid \text{supp}\, u \text{ is compact}\}$

Note: $\mathcal{E}'(U)$ is a subspace since $\text{supp}\,(u+v) \subset \text{supp}\, u \cup \text{supp}\, v$.

Example: $\delta_y \in \mathcal{E}'(U)$ when $y \in U$.

An important feature of elements of $\mathcal{E}'$ is that they can be paired with functions in $C^\infty(U)$ (not just $C_c^\infty(U)$):

Let $u \in \mathcal{E}'(U)$, $\varphi \in C^\infty(U)$.
Take any $\chi \in C_c^\infty(U)$ such that

$$\text{supp}\,(1-\chi) \cap \text{supp}\, u = \emptyset$$

(i.e. $\chi = 1$ near $\text{supp}\, u$)

and define $\widetilde{(u, \varphi)} := (u, \chi\varphi)$

where $\chi\varphi \in C_c^\infty(U)$.

Note: 1 if $\varphi \in C_c^\infty(\bar{U})$, then

$(u, \chi\varphi) = (u, \varphi)$ in the sense of $\mathcal{D}'$

since $(u, (1-\chi)\varphi) = 0$ as

$\operatorname{supp}(1-\chi)\varphi \cap \operatorname{supp} u = \emptyset$.

2 $\widetilde{(u,\varphi)}$ does not depend on χ:

if χ' is another cutoff then

$(u, \chi\varphi) = (u, \chi'\varphi)$ as

$\operatorname{supp}(\chi - \chi')\varphi \cap \operatorname{supp} u = \emptyset$

We henceforth write $(u,\varphi) := \widetilde{(u,\varphi)}$.

for $u \in \mathcal{E}'$, $\varphi \in C^\infty$

Topology on $C^\infty(\bar{U})$:

We say $\varphi_k \in C^\infty(\bar{U})$ converges to φ in $C^\infty(\bar{U})$ if

$$\sup_K |\partial^\alpha(\varphi_k - \varphi)| \xrightarrow[k\to\infty]{} 0$$

for every compact $K \subset \bar{U}$

and every multiindex α

This topology is <u>metrizable</u>:
take a sequence of compact sets
$K_1 \subset K_2 \subset \cdots \subset K_N \subset \cdots$ in U
such that $U = \bigcup_N \mathring{K}_N$
and define the N-th seminorm

$$\|\varphi\|_N := \max_{|\alpha| \le N} \sup_{K_N} |\partial^\alpha \varphi|.$$

Then $\varphi_k \to \varphi$ in $C^\infty(U)$ iff
$\|\varphi_k - \varphi\|_N \to 0$ for all N.
Now define the metric $d(\cdot,\cdot)$
on $C^\infty(U)$ by

$$d(\varphi, \psi) := \sum_{N=0}^{\infty} 2^{-N} \frac{\|\varphi-\psi\|_N}{1+\|\varphi-\psi\|_N}$$

Then (C^∞, d) is a complete metric space
and $\varphi_k \to \varphi$ in $C^\infty(U) \iff$
$\iff d(\varphi_k, \varphi) \to 0$. (Details in Pset 3)

Coming back to $\mathcal{E}'$, we have

<u>Thm</u> 1 If $u \in \mathcal{E}'(U)$ then $\varphi \in C^\infty(U) \longmapsto (u,\varphi) \in \mathbb{C}$ is continuous

2 If $\tilde{u} : C^\infty(U) \to \mathbb{C}$ is linear continuous then the restriction $\tilde{u}\big|_{C_c^\infty(U)}$ lies in $\mathcal{E}'(U)$.

<u>Note</u>: L. Schwartz denoted $\mathcal{D} := C_c^\infty(U)$, $\mathcal{E} := C^\infty(U)$ which explains the notation $\mathcal{D}'$, $\mathcal{E}'$

<u>Proof</u> 1 Fix $\chi \in C_c^\infty(U)$, $\chi = 1$ near supp u, and put $K := \operatorname{supp} \chi$.

Then $|(u,\varphi)| = |(u, \chi\varphi)| \leq$ (as u is a distribution)

$\leq C \|\chi\varphi\|_{C^N}$ for some C, N

$\leq C' \max\limits_{|\alpha| \leq N} \sup\limits_K |\partial^\alpha \varphi|$.

So if $\varphi_k \to \varphi$ in $C^\infty(U)$ then $(u,\varphi_k) \to (u,\varphi)$.

And sequential continuity $\Leftrightarrow$ continuity in metric spaces

2 Let $\tilde{u} : C^\infty(U) \to \mathbb{C}$ be linear continuous. Then

$$u := \tilde{u}|_{C_c^\infty(U)} : C_c^\infty(U) \to \mathbb{C}$$

is linear and sequentially continuous: if $\varphi_k \to 0$ in $C_c^\infty(U)$ then

$$\varphi_k \to 0 \text{ in } C^\infty(U)$$

So $(u, \varphi_k) = (\tilde{u}, \varphi_k) \to 0$.

Next, supp u is compact.

Indeed, take a sequence of compact sets $K_1 \subset \dots \subset K_N \subset \dots \subset U$, $U = \bigcup_N K_N^\circ$.

If supp u is not compact then $\forall N\ \exists x_N \in \operatorname{supp} u \setminus K_N$

So $\forall N\ \exists \varphi_N \in C_c^\infty(U)$,

$\operatorname{supp} \varphi_N \cap K_N = \emptyset$, $(u, \varphi_N) \neq 0$.

Can rescale to make $(u, \varphi_N) = 1$.

But $\varphi_N \to 0$ in $C^\infty(U)$, so

$u : C^\infty(U) \to \mathbb{C}$ cannot be continuous. $\square$

§4.3. Distributions supported at one point

Here we show

Thm. Let $u \in \mathcal{D}'(U)$ and $\operatorname{supp} u \subset \{y\}$ for some $y \in U$.
Then $\exists N$: $u = \sum_{|\alpha| \le N} c_\alpha \partial^\alpha \delta_y$ for some constants $c_\alpha \in \mathbb{C}$.

Proof Let $y = 0$ for simplicity. We will show that

$\exists N$: $(u, \varphi) = 0$ for all $\varphi \in C_c^\infty(U)$ such that $\partial^\alpha \varphi(0) = 0$ for all $|\alpha| \le N$. $(*)$

This gives the theorem: indeed, take any $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi = 1$ near 0

and write the Taylor expansion

$$\varphi(x) = \chi(x) \sum_{|\alpha| \le N} \frac{x^\alpha}{\alpha!} \partial^\alpha \varphi(0) + \tilde{\varphi}(x)$$

where $\partial^\alpha \tilde{\varphi}(0) = 0$ for all $|\alpha| \le N$. Then

$$(u, \varphi) = \sum_{|\alpha| \le N} \frac{1}{\alpha!} (u, x^\alpha \chi) \partial^\alpha \varphi(0) =$$

$$= \Big(\sum_{|\alpha| \le N} c_\alpha \partial^\alpha \delta_0, \varphi \Big)$$

where $c_\alpha := \dfrac{(-1)^{|\alpha|}}{\alpha!} (u, x^\alpha \psi)$.

Now let us show $(*)$.

Take some closed ball $\overline{B(0, \varepsilon_0)} \subset U$.

Since u is in $\mathcal{D}'(U)$, there exist C, N s.t.

$$|(u, \psi)| \le C \|\psi\|_{C^N}$$ ← that's the N we take

for every $\psi \in C_c^\infty(U)$, $\operatorname{supp} \psi \subset \overline{B(0, \varepsilon_0)}$.

Assume that $\varphi \in C_c^\infty(U)$,

$$\partial^\alpha \varphi(0) = 0 \quad \forall \alpha, |\alpha| \le N$$

Fix a cutoff function

$\chi \in C_c^\infty(B(0,1))$, $\chi = 1$ on $\overline{B(0, \tfrac{1}{2})}$,

and put $\varphi_\varepsilon(x) := \varphi(x) \cdot \chi\left(\frac{x}{\varepsilon}\right)$

Then $\varphi_\varepsilon \in C_c^\infty(U)$ and

$(u, \varphi) = (u, \varphi_\varepsilon) \quad \forall \varepsilon$ since

$\operatorname{supp}(\varphi - \varphi_\varepsilon) \cap \{0\} = \emptyset$, $\operatorname{supp} u \subset \{0\}$

Now to show $(u, \varphi) = 0$ it suffices to prove that

$$(u, \varphi_\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0.$$

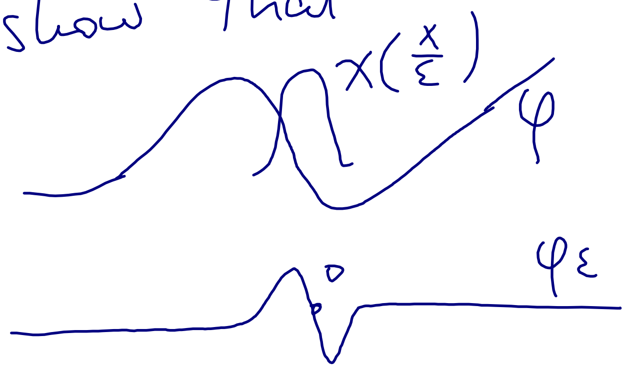
We bound for $\varepsilon < \varepsilon_0$, $\operatorname{supp} \varphi_\varepsilon \subset B(0, \varepsilon_0)$

$$|(u, \varphi_\varepsilon)| \le C \|\varphi_\varepsilon\|_{C^N}$$

where C is independent of ε.

So it is enough to show that

$$\|\varphi_\varepsilon\|_{C^N} \xrightarrow[\varepsilon \to 0]{} 0$$



Basic case: $\boxed{N = 0.}$

We have $\operatorname{supp} \varphi_\varepsilon \subset B(0, \varepsilon)$.

Now, for $|x| < \varepsilon$ we bound

$$|\varphi_\varepsilon(x)| \le \sup |x| \cdot |\varphi(x)| = O(\varepsilon)$$

Since