§16. More on Sobolev spaces

§16.1. Action by pseudodifferential operators

Here we show

Thm Assume $a \in \mathcal{E}^{s}_e(U \times \mathbb{R}^n)$. Then $O_p(a) : C^\omega_c(U) \to C^\omega(U)$ extends to a continuous operator $H^s_c(U) \to H^{s-e}_c(U) \forall s \in \mathbb{R}$.

The proof will use

Lemme [Schur's bound]

Assume $B(\xi, \eta) \in C^0(\mathbb{R}^{2n})$ and define $A_f(\xi) := \int_{\mathbb{R}^n} B(\xi, \eta) f(\eta) d\eta$, $A : C^\omega_c(\mathbb{R}^n) \to C^\omega(\mathbb{R}^n)$

$C_1 := \sup_{\xi} \int_{\mathbb{R}^n} |B(\xi, \eta)| d\eta$ and

$C_2 := \sup_{\eta} \int_{\mathbb{R}^n} |B(\xi, \eta)| d\xi$ are finite.

Then $A$ extends to a bounded operator on $L^2(\mathbb{R}^n)$ and $\|A\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq \sqrt{C_1 C_2}$.
Proof. Enough to show that 

\[ \forall f \in C^0_c(\mathbb{R}^n), \]

\[ \|A f\|^2_{L^2} \leq C_1 C_2 \|f\|^2_{L^2}. \]

We estimate \( \forall \xi \in \mathbb{R}^n \)

\[ |A f(\xi)|^2 = \left| \int_{\mathbb{R}^n} s B(\xi, \eta) \phi(\eta) \, d\eta \right|^2 \leq (\text{Cauchy-Schwarz}) \]

\[ \leq \int_{\mathbb{R}^n} |s B(\xi, \eta)| \, d\eta \cdot \int_{\mathbb{R}^n} |s B(\xi, \eta)|^2 \, d\eta \int_{\mathbb{R}^n} \phi(\eta)^2 \, d\eta \]

\[ \leq C_1 \int_{\mathbb{R}^n} |s B(\xi, \eta)| \, d\eta \cdot \int_{\mathbb{R}^n} \phi(\eta)^2 \, d\eta \]

Integrating, we get

\[ \int_{\mathbb{R}^n} |A f(\xi)|^2 \, d\xi \leq C_1 \int_{\mathbb{R}^n} |s B(\xi, \eta)| \cdot |\phi(\eta)|^2 \, d\eta \, d\xi \]

\[ = C_1 \int_{\mathbb{R}^n} |\phi(\eta)|^2 \cdot \int_{\mathbb{R}^n} |s B(\xi, \eta)| \, d\xi \, d\eta \]

\[ \leq C_1 C_2 \int_{\mathbb{R}^n} |\phi(\eta)|^2. \]
We can now give

Proof of Thm

1. It suffices to show that \( \forall x \in C^a_c(U) \),
\[
\mathcal{X} \mathcal{O}_p(a) \mathcal{X} : H^s(\mathbb{R}^n) \to H^{s-\ell}(\mathbb{R}^n)
\]
For that it's enough to show:
\[
\forall x \in C \ \forall \varphi \in S(\mathbb{R}^n)
\]
\[
\| \mathcal{X} \mathcal{O}_p(a) \mathcal{X} \varphi \|_{H^{s-\ell}} \leq C \| \varphi \|_{H^s}.
\]
We have \( \mathcal{O}_p(a) \mathcal{X} \varphi(x) = (2\pi)^{-n} \int e^{ix \cdot \eta} a(x, \eta) \hat{\varphi}(\eta) d\eta \).

Now, \( \| \mathcal{X} \varphi \|_{H^s} \leq C \| \varphi \|_{H^s} \), so we can write
\[
\hat{\mathcal{X} \varphi}(\eta) = \langle \eta \rangle^{-s} \hat{v}(\eta) \text{ where } v \in S(\mathbb{R}^n) \text{ and } \| v \|_{L^2} \leq C \| \varphi \|_{H^s}.
\]

Now compute
\[
\mathcal{X} \mathcal{O}_p(a) \mathcal{X} \varphi(\xi) = (2\pi)^{-n} \int e^{i \xi \cdot \eta} \mathcal{X}(x) a(x, \eta) \langle \eta \rangle^{-s} \hat{v}(\eta) d\eta dx.
\]

So \( \langle \xi \rangle^{s-\ell} \mathcal{X} \mathcal{O}_p(a) \mathcal{X} \varphi(\xi) = \int B(\xi, \eta) v(\eta) d\eta \), where
\[ B(\xi, \eta) = (2\pi)^{-n} \langle \xi \rangle^{-\ell} \langle \eta \rangle^{-\ell} \int_{\mathbb{R}^n} e^{-ix \cdot (\eta - \xi)} X(x) a(x, \eta) \, dx \]

\[ = (2\pi)^{-n} \langle \xi \rangle^{-\ell} \langle \eta \rangle^{-\ell} \tilde{a}(\xi - \eta, \eta) \]

where \( \tilde{a}(\zeta, \eta) = \int_{\mathbb{R}^n} e^{-ix \cdot \zeta} X(x) a(x, \eta) \, dx \)

is the Fourier transform of \( Xa \) in \( x \to \zeta \) variable.

(2) We need to show that for

\[ A_v(\xi) = \int_{\mathbb{R}^n} B(\xi, \eta) v(\eta) \, d\eta, \quad \exists C \forall v \]

\[ \| A_v \|_{L^2(\mathbb{R}^n)} \leq C \| v \|_{L^2(\mathbb{R}^n)}. \]

By Schur's bound enough to show

\[ \sup_{\xi} \int_{\mathbb{R}^n} |B(\xi, \eta)| \, d\eta < \infty \quad (1) \]

\[ \sup_{\eta} \int_{\mathbb{R}^n} |B(\xi, \eta)| \, d\xi < \infty \quad (2). \]

Integrating by parts in \( x \) we get \( \forall N \exists C_N \)

\[ |\tilde{a}(\zeta, \eta)| \leq C_N \langle \zeta \rangle^{-N} \langle \eta \rangle^{\ell} \]
So \( \forall N \in \mathbb{C}_N \)
\[
|B(\xi, \eta)| \leq C_N \left( \frac{\langle \xi \rangle^{5-l}}{\langle \eta \rangle} \right) \langle \xi - \eta \rangle^{-N}
\]

Recall from \( \S 12.1 \) that
\[
\left( \frac{\langle \xi \rangle^{5-l}}{\langle \eta \rangle} \right) \leq C_{\xi, \eta} \langle \xi - \eta \rangle^{15 - l_1}.
\]

So \( \forall N \in \mathbb{C}_N \):
\[
|B(\xi, \eta)| \leq \tilde{C}_N \langle \xi - \eta \rangle^{-N}.
\]

Now (1) and (2) follow. \( \square \)

\underline{Note:} if \( a \in \mathcal{S}^l(U \times I^{n}) \) then
the transpose \( O(a)^t \) maps
\[
H^s_c(U) \rightarrow H^{s-l}_{loc}(U) \quad \forall s
\]
(follows because \( H^s_c, H^{s-l}_{loc} \) are dual to each other; see Pset 10).
This shows that in Elliptic Regularity III,
\[
P u \in H^{s-l}_{loc}(M) \Rightarrow u \in H^s_{loc}(M).
\]
In fact, we can get an estimate out of this:

**Thus (Elliptic Estimate)**

Assume that $M$ is a manifold and $P \in \text{Diff}^m(M)$ is elliptic. Fix $\eta, \chi \in C^\infty_c(M)$ such that

$\eta = 1$ near $\text{supp } \eta$. Also fix $s, N \in \mathbb{R}$.

Then $\exists C$ such that $\forall u \in \mathcal{D}'(M)$,

$$
\| \eta u \|_{H^s(M)} \leq C \| \chi P u \|_{H^{s-m}(M)} + C \| \chi \eta u \|_{H^{-N}(M)}
$$

This is understood as follows:

if $\chi P u \in H^{s-m}_c(M)$ then

$\eta u \in H^s(M)$ & the estimate holds.

Here $\| \eta u \|_{H^s(M)}$ etc. are well-defined (up to equivalence) because $\eta, \chi$ are compactly supported.
Important special case:

if $M$ is a compact manifold then can take $\gamma = \chi = 1$ and get

$$\|u\|_{H^s(M)} \leq C \|\text{Pull}_{H^{s-m}(M)} + C \|u\|_{H^{-N}(M)}$$

Proof: (1) Can reduce to the case when $M$ is replaced by an open subset $U \subset \mathbb{R}^n$.

Indeed, use a partition of unity to write $\gamma = \sum_{j=1}^{r} \gamma_j$ where $\gamma_j \in C_c^\infty(M)$ and each $\gamma_j$ is supported in $U_j$ the domain of some coordinate system.

Bound $\|\gamma u\|_{H^s} \leq \sum_{j=1}^{r} \|\gamma_j u\|_{H^s}$

Now take $\chi_j \in C_c^\infty(U_j)$,

$\chi_j = 1$ near $\text{supp} \gamma_j$. Then

$\chi \chi_j = 1$ near $\text{supp} \gamma_j$ as well.

(can make $\text{supp} \gamma_i \subset \text{supp} \gamma_j$)
It suffices to show that \( V_j \)
\[ \| \varphi_j \|_{H^s} < C \| X_j X \text{Pull}_{H^{s-m}} C \| \| X_j X \|_{H^{-n}} \]

Since \( \varphi_j, X_j X \) are supported inside \( U_j \), can pull this back to an open subset of \( \mathbb{R}^n \)
using the coordinate system.

2) Now \( M = U \subset \mathbb{R}^n \) open.
Recall the proof of Elliptic Regularity in \( \S 15.4 \) we used
\( Q : \mathcal{E}'(U) \to \mathcal{D}'(U), C^\infty_0(U) \to C^\infty(U) \)
pseudolocal & such that \( \tilde{Q} P - I \) is smoothing.
Moreover, \( \tilde{Q} = Op(q)^t \) for some
\( q \in \mathcal{E}^{-m}(U \times \mathbb{R}^n) \), so
\( \tilde{Q} : H^s_\mathcal{C} (U) \to H^s_{\text{loc}} (U) \) is continuous.

Write \( I = \tilde{Q} P + R, R \) smoothing
Fix \( X \in C^\infty_0(U), x = 1 \) near supp \( \varphi_j \)
Then \( \forall u \in D'(U) \), we have 
\[
\tilde{\chi}_u = \tilde{\mathcal{Q}} P \tilde{\chi}_u + R \tilde{\chi}_u , \quad \text{so} \quad 4u = 4 \tilde{\mathcal{Q}} P \tilde{\chi}_u + 4R \tilde{\chi}_u .
\]
Since \( R \) is smoothing and \( 4, \tilde{\chi}_u \in C^\omega(U) \), we have \( \forall s, N \)
\[
\| 4R \tilde{\chi}_u \|_{H^s(\mathbb{R}^n)} \leq C_{s, N} \| \tilde{\chi}_u \|_{H^{-N}(\mathbb{R}^n)} .
\]
If \( \chi_{u_j} \to 0 \) in \( H^{-N}(\mathbb{R}^n) \) then 
\[
\tilde{\chi}_{u_j} \to 0 \text{ in } \mathcal{E}'(U) , \quad \text{so} \quad R \tilde{\chi}_{u_j} \to 0 \text{ in } C^\omega(U) , \quad \text{so} \quad 4R \tilde{\chi}_{u_j} \to 0 \text{ in } C^\omega_c(U) \subset H^s(\mathbb{R}^n) .
\]
Next, 
\[
4 \tilde{\mathcal{Q}} P \tilde{\chi}_u = 4 \tilde{\mathcal{Q}} X P u + 4 \tilde{\mathcal{Q}} [P, X] u .
\]
Since \( \tilde{\chi} = 1 \) near \( \text{supp } 4 \), the coefficients 
of \([P, \tilde{\chi}]\) are supported away from \( \text{supp } 4 \).
Since \( \tilde{\mathcal{Q}} \) is pseudolocal, \( 4 \tilde{\mathcal{Q}} [P, X] \) is smoothing
so 
\[
\| 4 \tilde{\mathcal{Q}} [P, X] u \|_{H^s} \leq C_{s, N} \| \tilde{\chi}_u \|_{H^{-N}(\mathbb{R}^n)} .
\]
(here \([P, \tilde{\chi}] = [P, \tilde{\chi} X] \) as well.
Finally, since $\mathcal{Q} : H^{s-m}_c(U) \to H^s_{\text{loc}}(U)$ we get $\|\mathcal{Q}X\|_{H^s} \leq C \|X\|_{H^{s-m}}$.

§6.2. Compactness

Here we show that $H^s_c$ embeds compactly into $H^t_{\text{loc}}$ when $s > t$.

**Thm:** Assume that $s > t$ and $u_k \in H^s_c(\mathbb{R}^n)$ is a sequence such that $\exists C, R, \forall k$

1. $\|u_k\|_{H^s} \leq C$
2. $\text{supp } u_k \subset B(0, R)$.

Then $u_k$ has a subsequence which converges in $H^t(\mathbb{R}^n)$.

**Remark:** In fact one can relax 1 + 2 to $\exists \delta > 0$: $\|\langle x \rangle^{\delta} u_k\|_{H^s} \leq C$.

Basically, improved regularity + improved decay $\implies$ compactness.
Proof

Since $u_k$ is compactly supported, its Fourier transform $\hat{u}_k$ is in $C^\alpha$: $\hat{u}_k(\xi) = (u_k(x), e^{i\xi \cdot x})$, $\xi \in \mathbb{R}^n$

We next estimate $e^\xi(x) = e^{i\xi \cdot x}$, $x \in C_c^\infty(\mathbb{R}^n)$, $x = 1$ near $B(0, R)$

$|\partial_\xi \hat{u}_k(\xi)| \leq \|u_k\|_{H^s(\mathbb{R}^n)} \cdot \|x \cdot x^\alpha \cdot e^\xi\|_{H^{-s}(\mathbb{R}^n)}$

$\leq C \|x \cdot x^\alpha \cdot e^\xi\|_{H^N(\mathbb{R}^n)}$ ($N \in \mathbb{N}$, $N \geq -s$)

$\leq C \max_{|\beta| \leq N} \sup_x |\partial^\beta_x e^\xi(x)|$

$\leq C <\xi>^N$, where $C$ is independent of $k$

Taking this with $|\xi| \leq 1$, we see that for all $T$, the sequence $\hat{u}_k(\xi)$ is uniformly bounded and uniformly equicontinuous on the ball $B(0, T)$. Indeed, equicontinuity follows from the bound $|\hat{u}_k(\xi) - \hat{u}_k(\eta)| \leq C T^N |\xi - \eta|$ for all $k$, $\forall \xi, \eta \in B(0, T)$.
By Arzelà-Ascoli Thm and a diagonal argument (taking further subsequences for \( T = 1, 2, \ldots \)) there exists a subsequence \( \{ u_{k,j} \} \) such that \( \hat{u}_{k,j}(\xi) \to \hat{v}(\xi) \) locally uniformly in \( \xi \) for some continuous \( v \in C^0(\mathbb{R}^n) \). Then \( \langle \xi \rangle \hat{u}_{k,j}(\xi) \to \langle \xi \rangle \hat{v}(\xi) \) \( \forall \xi \).

So by Fatou's Lemma, \( \langle \xi \rangle \hat{v}(\xi) \in L^2 \).

Thus \( v(\xi) = \hat{u}(\xi) \) for some \( u \in H^s(\mathbb{R}^n) \).

\( \text{(2) It is not in general true that} \)

\[ \langle \xi \rangle \hat{u}_{k,j}(\xi) - v(\xi) \to 0 \text{ in } L^2(\mathbb{R}^n) \]

(Think of a running step: \( s = 0, h = 1, \))

\[ u_k(x) = e^{ikx} \varphi(x), \quad \varphi \in C_c^\infty(\mathbb{R}); \]

\[ \hat{u}_k(\xi) = \hat{\varphi}(\xi - k), \]

\( \hat{u}_k(\xi) \to 0 \) pointwise in \( \xi \) but not in \( L^2 \) in \( \xi \).
However, for $t < s$ we do have $\langle \xi^t (\hat{u}_{kj} (\xi) - v(\xi)) \rangle \to 0$ in $L^2 (\mathbb{R}^n)$.

and thus $u_{kj} \to u$ in $H^+ (\mathbb{R}^n)$ (where $\hat{u} = v$).

Indeed, take any $T > 0$.

Then $\int \langle \xi^2 t \hat{u}_{kj} (\xi) - v(\xi) \rangle^2 d\xi \leq \int_{\mathbb{R}^n} \leq \int_{|\xi| \leq T} + \int_{|\xi| > T}

\leq C_T \sup_{|\xi| \leq T} |\hat{u}_{kj} (\xi) - v(\xi)|^2 + 

+ 2 \sup_{|\xi| > T} \int_{|\xi| > T} \int_{\mathbb{R}^n} \leq a_j (T) + b(T)$ where

- $a_j (T) \to 0$ for all $T$,
- $b(T)$ is $T$-independent and $b(T) \to 0$ as $T \to \infty$.

Since $\|\langle \xi^t \hat{u}_{kj} (\xi) \rangle_{L^2} \| \leq \|u_{kj}\|_{H^s} \leq C$.
So \( \forall T, \)
\[
\limsup_{j \to \infty} \int_{\mathbb{R}^n} \left< \xi >^T \left| \hat{u}_{kj}(\xi) - v(\xi) \right|^2 d\xi \\
= \limsup_{j \to \infty} a_j(T) + b(T) \leq b(T)
\]

Taking \( T \to \infty, \) we see that this \( \limsup \) is \( = 0. \)

So \( \left< \xi >^T \left| \hat{u}_{kj}(\xi) - v(\xi) \right|^2 \to 0 \) in \( L^2 \)

and \( u_{kj} \) converges in \( H^t \) as needed. \( \square \)

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**Defn** Let \( X, Y \) be Banach spaces and \( A: X \to Y \) a bounded linear operator.

We say \( A \) is **compact**, if

A bounded sequence \( u_k \in X \)

the sequence \( Au_k \) has a subsequence converging in \( Y. \)
The Thm we just proved has

**Corollary 1** Assume \( X \in C_c^\infty(\mathbb{R}^n) \).
Then \( \forall s > t \), the multiplication operator \( X : H^s(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \)
is compact.

**Corollary 2** Let \( M \) be a compact manifold. Then \( \forall s > t \), the inclusion \( I : H^s(M) \to H^t(M) \) is compact.
That is, if \( u_k \in H^s(M) \) is bounded then \( u_k \) has a subsequence converging in \( H^t(M) \).

**Proof** Write a partition of unity \( I = X_1 + \cdots + X_e \), each \( X_e \) multiplication op's is supported in the domain of a coordinate system.

Use coordinates to show: \( X_j : H^s(M) \to H^t(M) \) compact. Then \( I = X_1 + \cdots + X_e \) is compact too. \( \square \)
§ 16.3. Fredholm Theory

Defn. Let $X, Y$ be Banach spaces. A bounded operator $A : X \to Y$ is called a Fredholm operator, if:

1. $\text{Ker } A = \{ u \in X \mid Au = 0 \}$ is finite dimensional
2. The range $\text{Ran } A := A(X)$ is closed in $Y$
3. $A(X)$ has finite codimension in $Y$

Basic properties: ($18.102$ ?)

2. If $A$ is Fredholm, its index is $\text{ind}(A) = \dim \text{Ker } A - \text{Codiag} \text{Ran } A$
3. If $X, Y$ are finite dimensional:
   \[ \dim X = m, \ \dim Y = n, \ \text{then} \]
   any $A : X \to Y$ is Fredholm and
   $\text{ind } A = m - n$
   (Rank/Nullity Thm)

"Fredholm operators are like matrices"
4. \( A \) is invertible \( \Rightarrow \)
   \( A \) is Fredholm of index 0

5. \( A \) is Fredholm, \( K: X \to Y \) is compact
   \( \Rightarrow A + K \) is Fredholm of same index as \( A \)

6. \( A \) is Fredholm \( \Rightarrow \exists \varepsilon > 0 \) s.t.
   \( \forall B: X \to Y \) with \( \|B\| < \varepsilon \),
   \( A + B \) is Fredholm of same index as \( A \)

7. \( X \xrightarrow{A} Y \xrightarrow{B} \mathbb{Z} \),
   \( A, B \) Fredholm \( \Rightarrow BA \) Fredholm
   and \( \text{ind} (BA) = \text{ind} A + \text{ind} B \).

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**Thm** Assume \( M \) is a compact
manifold and \( P \in \text{Diff}^m(M) \)
is an elliptic differential operator.
Then \( \forall s \in \mathbb{R}_+ \)
\( P_s := P: H^s(M) \to H^{s-m}(M) \)
is a Fredholm operator.
Proof: We first show that \( \text{Ker } P_s = \{ u \in H^s(M) \mid Pu = 0 \} \) is finite dimensional. 

We use the Elliptic Estimate: \( \forall u \in H^s(M) \)

\[ \| u \|_{H^s} \leq C \| Pu \|_{H^{s-m}} + C \| u \|_{H^{-N}} \quad (\star) \]

We see that \( \forall u \in \text{Ker } P_s, \)

\[ \| u \|_{H^s} \leq C \| u \|_{H^{-N}} \quad (\star \star) \]

Where \( C \) is independent of \( u \).

Take \( N \) s.t. \( -N \leq s \).

Then since \( H^s \hookrightarrow H^{-N} \) is a compact embedding,

we see that

a sequence \( u_k \in \text{Ker } P_s \) s.t.

\[ \| u_k \|_{H^s} \leq 1, \text{ there exists a subsequence } u_{k_j} \text{ converging in } H^{-N} \]

and thus in \( H^s \) (as \( (\star \star) \) shows that \( u_k \) Cauchy in \( H^{-N} \) implies \( u_k \) Cauchy in \( H^s \)).
Now, if \( \dim \ker P_5 = \infty \) then take an orthonormal system \( \{u_1, u_2, \ldots\} \) in \( H^s \) (w.r.t. the \( H^s \) inner product).

This cannot have a subsequence converging in \( H^s \) (as \( \|u_k - u\|_{H^s} = \sqrt{2} \) for all \( k \neq 0 \) cannot be Cauchy), giving a contradiction.

So \( \dim \ker P_5 < \infty \).

2. We next show that the range \( \text{Ran}(P_5) := \{ Pu \mid u \in H^s(M) \} \) is a closed subspace of \( H^{s-m}(M) \).

Assume that we have a sequence \( u_k \in H^s(M) \) and \( Pu_k \rightarrow v \) in \( H^{s-m}(M) \).

We need to show that \( v = Pu \) for some \( u \in H^s(M) \).

We can add to \( u_k \) some element of \( \ker P_5 \) to make sure that \( u_k \perp \ker P_5 \) w.r.t. \( < \cdot, \cdot >_{H^s} \).
We first show that $\|u_k\|_{H^s}$ is bounded.

WLOG $\|u_k\|_{H^s} \to \infty$. Put $\tilde{u}_k := \frac{u_k}{\|u_k\|_{H^s}}$.

Then $\|\tilde{u}_k\|_{H^s} = 1$, $\tilde{u}_k \perp \text{Ker } P_s$ wrt $\langle \cdot , \cdot \rangle_{H^s}$, and $P\tilde{u}_k = \frac{P u_k}{\|u_k\|_{H^s}} \to 0$ in $H^{s-m}$.

Passing to a subsequence, we can assume that $\tilde{u}_k$ converges in $H^{-N}$.

Now use (\star):

$\|\tilde{u}_k - \tilde{u}\|_{H^s} \leq C \|P\tilde{u}_k - P\tilde{u}\|_{H^{s-m}} + C \|\tilde{u}_k - \tilde{u}\|_{H^{-N}}$.

We see that $\tilde{u}_k$ is a Cauchy sequence in $H^s$. So $\tilde{u}_k \to$ some $\tilde{u}$ in $H^s$.

We have $P\tilde{u} = \lim_{k \to \infty} P\tilde{u}_k = 0 \Rightarrow \tilde{u} \in \text{Ker } P_s$ and $\|\tilde{u}\|_{H^s} = 1$, $\tilde{u} \perp \text{Ker } P_s$,

a contradiction (as $\tilde{u} \perp \tilde{u}$).

So $\|u_k\|_{H^s}$ is bounded.
New, if \( u_k \) is bounded, then again pass to a subsequence to make \( u_k \) converge in \( H^{-N} \) & write again
\[
\| u_k - u \|_{H^s} \leq C \| P_{u_k} - P_u \|_{H^{s-m}} + C \| u_k - u \|_{H^{-N}}.
\]
Then \( u_k \) is a Cauchy sequence in \( H^s \) \( \Rightarrow u_k \to \text{some } u \text{ in } H^s \)
\( \Rightarrow P_{u_k} \to P_u \text{ in } H^{s-m} \)
\( \Rightarrow v = P_u \text{ as needed.} \)

3. Consider now the adjoint operator
\[ P^* \in \text{Diff}^m(M) \text{ such that } \]
\[
\langle P\psi, \varphi \rangle_{L^2} = \langle \varphi, P^* \psi \rangle_{L^2} \forall \varphi, \psi \in C^0(M).
\]
Here \( \langle \varphi, \psi \rangle_{L^2} = \int_M \varphi \overline{\psi} \, dV_{\text{vol}}(g) \text{ (fixed some Riem. metric } g) \)
We can define \( \langle u, v \rangle_{L^2} \in C \) for \( u \in H^s(M), \ v \in H^{-s}(M) \), any \( s \); \( |\langle u, v \rangle_{L^2}| \leq C \|u\|_{H^s} \cdot \|v\|_{H^{-s}} \).

and we still have

\[
\langle Pu, v \rangle_{L^2} = \langle u, P^*_m v \rangle_{L^2}
\]

\( \forall u \in H^s(M), \ v \in H^{m-s}(M) \).

Indeed, take \( \varphi_k \to u \) in \( H^s \)
\( \psi_k \to v \) in \( H^{m-s} \)
\( \varphi_k, \psi_k \in C^\infty(M) \). Then

\[
\langle P\varphi_k, \psi_k \rangle_{L^2} = \langle \varphi_k, P^* \psi_k \rangle_{L^2}
\]

\( P\varphi_k \to Pu \) in \( H^{s-m} \)
\( P^* \psi_k \to P^* v \) in \( H^{-s} \).

Passing to the limit we get (\( * \)).
Using (4) we get the following characterization of the range of $P_5$:

$$\text{Ran} \ (P_5) = \{ w \in H^{s-m}(M) : \forall v \in \ker P_{m-5}^* \text{ we have } \langle w, v \rangle_{L^2} = 0 \}.$$ 

Indeed,

$\subseteq$: if $w \in \text{Ran} \ (P_5)$ then $w = P_5 u$ for some $u \in H^s(M)$.

Then $\forall v \in \ker P_{m-5}^*$ we have

$$\langle w, v \rangle_{L^2} = \langle P_5 u, v \rangle_{L^2} = \langle u, P_{m-5}^* v \rangle_{L^2} = 0.$$ 

$\supseteq$: Assume that $w \in H^{s-m}(M)$ but $w \notin \text{Ran} \ (P_5)$.

Since $\text{Ran} \ (P_5) \subset H^{s-m}(M)$ is closed, there exists a bounded linear functional $F : H^{s-m}(M) \to \mathbb{C}$, $F |_{\text{Ran} \ (P_5)} = 0$, $F(w) = 1$. 


But bounded linear functionals on $H^{s-m}(M)$ are $<\cdot, \cdot>_{L^2}$ pairings with elements of $H^{m-s}(M)$ (Pset 8, Problem 1... can make it work for manifolds...).

So, $\exists \, v \in H^{m-s}(M)$ such that

$\forall f \in H^{s-m}(M)$ we have

$F(f) = <f, v>_{L^2}$.

Now, $<w, v>_{L^2} = F(w) = 1$ and $\forall u \in H^s(M)$, we have $Pu \in \text{Ran}(P_s) \Rightarrow O = F(Pu) = <Pu, v>_{L^2} = <u, P^*v>_{L^2}$.

This holds $\forall u \in C^\infty(M)$ in particular, so $<u, P^*v>_{L^2} = 0 \ \forall u \in C^\infty(M)$.

$P^*v = 0$ (since $P^*v$ is a distribution.)

So $\exists \, v \in \text{Ker} \, P^*_{m-s} : <w, v>_{L^2} \neq 0$

which gives $\supset$. 
We showed in (3) that
\[ \text{Ran } (P_s) = \{ w \in H^{s-m}(M) : \forall \nu \in \text{Ker } P_{m-s}^* \} \]
\[ \langle w, \nu \rangle_{L^2} = 0 \]
Since \( P^* \) is elliptic, we see that \( \text{Ker } P_{m-s}^* \) is finite dimensional.
So \( \text{Ran } (P_s) \) has finite co-dimension.
In fact, \( \text{Codim}_{H^{s-m}} \text{Ran } (P_s) = \dim \text{Ker } P_{m-s}^* \).
Since \( \nu \mapsto \langle \cdot, \nu \rangle : H^{s-m} \to \mathbb{C} \)
is an isomorphism from \( H^{m-s}(M) \) to the dual space to \( H^{s-m}(M) \). \( \square \)

**Remark by Elliptic Reg. III**, \( \text{Ker } P_s = \text{Ker } P = \{ u \in C^\infty(M) : P_u = 0 \} \)
\( \text{Ker } P_s^* = \text{Ker } P^* = \{ \nu \in C^\infty(M) : P^*_\nu = 0 \} \)
are independent of \( s \)
and \( \text{ind } P_s = \dim \text{Ker } P - \dim \text{Ker } P^* \).
In particular,
\[ \text{ind} (P^*_s) = - \text{ind} P_s \]
and if \( P \) is self-adjoint (i.e. \( P^* = P \))
then \( \text{ind} P_s = 0 \) \( \forall s \).

An important example of a self-adjoint operator is
\[ P = -\Delta_g \text{ on a compact Riemannian manifold } (M, g). \]

Here \( P = P^* \) since \( \forall \psi, \psi \in C^\infty (M) \)
\[ \langle P\psi, \psi \rangle_{L^2} = - \int_M (\Delta_g \psi) \bar{\psi} \, dV_g \]
\[ = - \int_M \langle \nabla_g \psi, \nabla_g \bar{\psi} \rangle \, dV_g \]
\[ = \langle \psi, P \psi \rangle_{L^2} \]