§13. Manifolds

§13.1. Basics

Defn. A subset \( M \subset \mathbb{R}^n \) is called an \( n \)-dimensional (embedded) manifold, if it locally looks like a graph: \( \forall x_0 \in M \exists \text{ open set } U_0 \subset \mathbb{R}^n \) s.t. \( x_0 \in U \)

Such that \( M \cap U_0 = \{ x'' = F(x') \mid x' \in V_0 \} \)

Where we took some splitting \( \mathbb{R}^n = \mathbb{R}^n_x \times \mathbb{R}^n_{x''} \)

( might need to permute vector entries

e.g. \( x = (x_1, x_2, x_3, x_4) \)
\( x' = (x_2, x_3), \ x'' = (x_1, x_4) \),

\( V_0 \subset \mathbb{R}^n \) is some open set,

and \( F : V_0 \to \mathbb{R}^{n-n} \) is a \( C^\infty \) map.
Fundamental example:

\[ M = \{ x \in U \mid G(x) = 0 \} \]

where \( U \subset \mathbb{R}^n \) is some open set and \( G : U \to \mathbb{R}^{N-n} \) is a \( C^\infty \) map.

If \( dG \) is onto at each point of \( M \), then \( M \) is an \( n \)-dimensional manifold.

The proof uses Inverse Mapping Thm: fix \( x_0 \in M \) & choose a splitting \( x = (x', x'') \) such that \( \partial_{x''} G(x_0) \) is invertible (possible since \( dG(x_0) \) is onto).

Then the map \( \Phi : x \mapsto (x', G(x', x'')) \) has \( d\Phi(x_0) = (I \partial_{x'} G(x_0)) \) invertible
So by the Inverse Mapping Thm \( \Phi \) is a diffeomorphism when restricted to some neighborhood \( U_0 \) of \( x_0 \).

If its inverse is

\[
\Phi^{-1}(x) = (x', A(x))
\]

where \( A : \mathbb{R}^n \to \mathbb{R}^{n-n} \) is a \( C^\infty \) map,

\( W_0 := \Phi(U_0) \) open, then

\[
M \cap U_0 = \Phi^{-1}(W_0 \cap \{x'' = 0\})
\]

\[
= \{ (x', A(x', 0)) \}
\]

\[
= \{ x'' = F(x') \}
\]

where \( F(x') := A(x', 0) \).

(In effect we are reproving the Implicit Function Theorem...)
Some examples:

- Any open \( U \subseteq \mathbb{R}^n \) is an \( n \)-dim \textit{manifold}.
- \( S^n \subseteq \mathbb{R}^{n+1} \) is an \( n \)-dim \textit{compact} manifold.

where \( S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \)

(Here \( G(x) = |x|^2 - 1 \) say \( S^2 \) and \( \text{d}G(x) = 2x \neq 0 \) on \( S^n \)).

Coordinates & Parametrizations:

If \( M \cap V_0 = \{ x'' = F(x') : x' \in V_0 \} \) then \( x' \in M \cap V_0 \) \( \rightarrow \) \( x' \) is a (local) \textit{coordinate system} on \( M \) \( x' \in V_0 \) \( \rightarrow (x', F(x')) \) \( \in M \) is a \textit{parametrization}.

Transitions: if \( x : M \cap V_0 \rightarrow V_0 \) and \( \tilde{x} : M \cap V_0 \rightarrow \tilde{V}_0 \) are 2 coordinates then \( \exists \) a \( C^\infty \) \textit{diffeomorphism} \( \Phi : V_0 \rightarrow \tilde{V}_0 \) s.t. the diagram

\[
\begin{array}{ccc}
M \cap V_0 & \xrightarrow{x} & V_0 \\
\downarrow & & \downarrow \Phi \\
V_0 & \xrightarrow{\tilde{x}} & \tilde{V}_0
\end{array}
\]

is commutative.
We are generally interested in intrinsic objects on M (those depending only on M & the \( C^a \) structure given by local coordinates) rather than extrinsic ones (those depending on the way M is embedded into \( IR^n \)).

In fact, it would be better to use abstract manifolds:

"Defn." An \( n \)-dimensional (abstract) manifold is a metrizable topological space \( M \) together with a system of coordinate charts:

(open subset of \( M \)) \( \xrightarrow{\text{homeomorphism}} \) (open subset of \( IR^n \))

such that the transition maps are \( C^a \) diffeomorphisms.

For more details, see [18.101].
13.2. Basic objects on a manifold

Assume $M \subset \mathbb{R}^n$ is an n-dim. manifold.

- $C^\infty(M)$: consists of functions $f : M \to \mathbb{C}$ s.t.
  in each coordinate system $\varphi : M \ni u_0 \to V_0 (\subset \mathbb{R}^n \text{ open})$
  the map $f \circ \varphi^{-1} : V_0 \to \mathbb{C}$ is $C^\infty$.

- Can define $C^\infty$ maps & diffeomorphisms between manifolds. Intrinsic objects should transform naturally by diffeos.

Picture:

- Tangent space: if $x_0 \in M$ & $\varphi$ is a coordinate system then $T_{x_0}M$ (tangent space to $M$ at $x_0$) is the range of $d\varphi^{-1}(\varphi(x_0))$. It's an n-dimensional subspace of $\mathbb{R}^n$, since $\varphi^{-1} : V_0 \to \mathbb{R}^n$. 

$\varphi \circ T_{x_0}M$
If \( M = \{ x \in U : G(x) = 0 \} \) then \( T_{x_0} M = \{ v \in \mathbb{R}^n : dG(x_0)v = 0 \} \), i.e. the kernel of \( dG(x_0) \).

**Tangent bundle:**

\[
TM := \{ (x, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in M, \quad v \in T_x M \}
\]

is a \( 2n \)-dimensional manifold.

**Example:** \( M = \mathbb{S}^1 = \{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \} \).

Then \( TM = \{ (x, v) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \}
\]

\[
T_x \mathbb{S}^1 \quad \text{and} \quad \mathbb{S}^2
\]

**Smooth vector field:** a "\( C^\infty \) section" of the tangent bundle, i.e. \( X \) is a \( C^\infty \) vector field on \( M \) (write \( X \in \mathcal{C}^\infty(M; TM) \)) if it is a \( C^\infty \) map \( X : M \rightarrow \mathbb{R}^n \) such that \( \forall x \in M, \quad X(x) \in T_x M \).
Cotangent bundle:

\[ T^*M := \{ (x, \xi) \mid x \in M, \xi \in T_x^*M \} \]

Where \( T_x^*M \) is the dual space to \( T_xM \), i.e. \( T^*M = \{ \xi : T_xM \to \mathbb{R} \text{ linear} \} \)

\( T^*M \) is a 2n-dimensional manifold

Can identify \( T^*M \cong TM \) extrinsically (depending on the embedding MCIR^n)

by mapping \( \xi \in T_x^*M \) to \( \nu \in T_xM \)

such that \( \xi(\nu) = \nu \cdot w \) \( \forall w \in T_xM \)

Euclidean inner product

1-forms: \( C^\infty \) sections of \( T^*M \)

i.e. \( \omega : x \in M \mapsto \omega(x) \in T^*_xM \)

s.t. the map \( x \mapsto (x, \omega(x)) \) is \( C^\infty \)
Differential: if $f \in C^\infty(M)$ then $df \in C^\infty(M; T^*M)$ is a 1-form.

Defined as follows: if $x \in M$, $v \in T_xM$ then $df(x)v$ is the derivative of $f$ along $v$:

- Take any $C^\infty$ curve $\gamma: (-\varepsilon, \varepsilon) \to M$
  - $\gamma(0) = x$, $\gamma'(0) = v$

then $df(x)v = \left. \frac{d}{dt} \right|_{t=0} f(\gamma(t))$

(such $\gamma$ exist & $df(x)v$ is independent of $\gamma$)

$df$ is intrinsic: if $\Phi: M \to \tilde{M}$ is a diffeomorphism and $f \in C^\infty(\tilde{M})$ then $d(\Phi^*f) = \Phi^*df$ in the sense that $\forall x \in M$, $v \in T_xM$ we have

$$d(\Phi^*f)(x)v = df(\Phi(x))d\Phi(x)v$$

Here $d\Phi(x): T_xM \to T_{\Phi(x)}\tilde{M}$
A Riemannian metric on $M$:

- $g$ is a Riem. metric on $M$ if $\forall x \in M$, $g(x)$ is an inner product on $T_x M$.
- $g \in \mathcal{C}^\infty$ in the sense that $\forall$ vector fields $X, Y \in \mathcal{C}^\infty(M; TM)$ the function $x \mapsto g(x)(X(x), Y(x))$ is $\mathcal{C}^\infty$.

**Example**: (extrinsic!!) can put $g(x)(v, w) = v \cdot w$ Euclidean inner product $\forall x \in M$, $v, w \in T_x M \subset \mathbb{R}^n$.

- If $M = S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \}$ then this example gives the **round metric** on $S^n$. 


§13.3. Distributions on manifolds

We will fix a Riemannian metric $g$ on a manifold $M$.

(Major cheating... we did not need to do this at all. A better way would be to use the bundle of densities, see e.g. [Hörmander, §6.3])

Integration: for $f: M \to \mathbb{C}$ define $\int_M f(x) \, d\text{Vol}_g(x)$ as follows:
- if we have a coordinate system $x: U_0 \to V_0^{C^{r,n}}$ and $\text{supp} \ f \subset U_0$
  
  $\int_M f(x) \, d\text{Vol}_g(x) := \int_{V_0} f(x^{-1}(y)) J(y) \, dy$

  where $J(y) = \sqrt{\det g_{jk}(y)}$ and $g_{jk}(y) = g(x^{-1}(y))(dx^{-1}(y) e_j, dx^{-1}(y) e_k)$ are the coefficients of $g$ w.r.t. $x$.
Changing coordinates will not change the $\int_M f \text{dVol}_g$ defined above. This can be shown using Jacobi's formula (see Pset 9).

In general, split $f$ into functions supported in a single $U_0$ (partition of unity).

This gives the Lebesgue $\int \text{W.r.t.}$ the Riemannian volume measure on $(M, g)$ (which we kind of defined above:

$$\text{vol}_g(U) = \int_M 1_U \text{dVol}_g$$)

Can define spaces $L^p(M, g)$.

The spaces $L^p_{\text{loc}}(M)$, $L^p_c(M)$ (locally $L^p$) (compactly supported) are actually independent of the choice of $g$.

Define $C_c^\infty(M)$ similarly to $C_c^\infty(U)$, $U \subset \mathbb{R}^n$, can define convergence using coordinates.
More precisely, a sequence $\varphi_k \in C_0^\infty(M)$ converges to $0$ in $C_0^\infty(M)$ if:

1. Every compact $K \subset M$: $\forall k$, supp $\varphi_k \subset K$
2. For any coordinate system $\pi: U_0 \to V_0$, $M \to \mathbb{R}^n$,
   the functions $(\varphi_k \circ \pi^{-1}) \in C_0^\infty(V_0)$ converge to $0$ in $C_0^\infty(V_0)$.

**Defn.** A **distribution** on $M$ is a linear map $u: C_0^\infty(M) \to \mathbb{C}$ such that $\forall \varphi_k \to 0$ in $C_0^\infty(M)$, we have $(u, \varphi_k) \to 0$.

To embed functions into distributions, use the pairing

$$(f, \varphi) := \int_M f(x) \varphi(x) \, d\text{Vol}_g(x)$$

$\forall f \in L^1_{\text{loc}}(M)$, $\varphi \in C_0^\infty(M)$.

(depends on $d\text{Vol}_g$, not ideal... better to define distr. as dual to $C_0^\infty$ densities...)
• Denote by $D'(M)$ the space of distributions on $M$.
• $C_c^\infty(M)$ is still dense in $D'(M)$.
• Can define $u|_W$ for $u \in D'(M)$, $W$ open and can define $\text{supp } u \subset W$ closed.
• $\mathcal{E}'(M)$ distributions with compact support on $M$ (dual to $C_c^\infty(M)$).
• The sheaf property still holds on $D'(M)$.
• If $\varphi : U_0 \rightarrow V_0$ is a coordinate system, then can define $\varphi^* : \mathcal{E}'(V_0) \rightarrow \mathcal{E}'(U_0) \subset \mathcal{E}'(M)$.
• $\varphi^* : (\varphi^{-1})^* : D'(M) \rightarrow D'(U_0) \rightarrow D'(V_0)$, which lets us think of distributions on $M$ in local coordinates.
• If $M, \tilde{M}$ are manifolds and $\Phi : M \rightarrow \tilde{M}$ is a submersion (i.e. $d\Phi(x)$ is onto $T_x\tilde{M}$ at every $x \in M$), then can define $\Phi^* : D'(\tilde{M}) \rightarrow D'(M)$ (using local coordinates etc.).
Sobolev spaces

Let $s \in \mathbb{R}$. Define $H^s_{\text{loc}}(M) \subset D'(M)$ as follows:

$u \in D'(M)$ lies in $H^s_{\text{loc}}(M)$ if and only if a coordinate system $\tilde{x} : U_0 \to V_0$, the pullback

$\tilde{x}^* : \mathbb{R}^n \to M$

$\tilde{x}^* u \in D'(V_0)$ lies in $H^s_{\text{loc}}(V_0)$.

Define $H^s_c(M) = H^s_{\text{loc}}(M) \cap \mathcal{E}'(M)$.

Note: This is a reasonable definition because $H^s_{\text{loc}}$ is invariant under pullbacks by diffeomorphisms. (Pset 8, Exercise 7).

In particular, if $u \in H^s_c(V_0) \cap \mathcal{E}'(V_0)$ then $\tilde{x}^* u \in \mathcal{E}'(M)$ lies in $H^s_c(M)$.

Indeed, if $\tilde{x} : U_0 \to V_0$ is another coordinate system then $\tilde{x}^{-*} \tilde{x}^* u \in D'(V_0)$ is given by
\( \tilde{\mathcal{E}}^{-*} \mathcal{E}^* v = \Phi^* v \) where

\[
\Phi = \mathcal{E} \circ \tilde{\mathcal{E}}^{-1} : \tilde{V}_0 \wedge \mathcal{E}(U_0) \to \tilde{V}_0 \wedge \mathcal{E}(\tilde{U}_0)
\]

is a \( C^\alpha \) diffeomorphism:

\[
\begin{tikzcd}
\tilde{\mathcal{E}} & \mathcal{E} \\
\tilde{V}_0 \wedge \mathcal{E}(U_0) & V_0 \wedge \mathcal{E}(\tilde{U}_0)
\end{tikzcd}
\]

is commutative.

Since \( v \in H^s_c(V_0) \subset H^s_{loc}(V_0) \) we have

\[ \Phi^* v \in H^s_{loc}(\tilde{V}_0 \wedge \mathcal{E}(U_0)) \]

and \( \text{supp}(\Phi^* v) \) is the intersection of \( \tilde{V}_0 \) with a compact set =>

=> can extend \( \Phi^* v \) by 0 to \( D'(\tilde{V}_0) \)

and this will be in \( H^s_{loc}(\tilde{V}_0) \)...

**Multiplication**: if \( a \in C^\infty(M) \) then can define \( u \mapsto au \) as an operator \( D'(M) \to D'(M), \ H^s_{loc}(M) \to H^s_{loc}(M) \).