§16. More on Sobolev spaces

§16.1. Action by pseudodifferential operators

Here we show

Thm Assume $a \in S^\ell(U \times \mathbb{R}^n)$. Then

$$\mathrm{Op}(a) : C_c^\infty(U) \to C^\infty(U)$$

extends to a continuous operator

$$H_c^s(U) \to H_{loc}^{s-\ell}(U) \quad \forall s \in \mathbb{R}.$$

The proof will use

Lemma [Schur's bound]

Assume $B(\xi,\eta) \in C^0(\mathbb{R}^{2n})$ and define $Af(\xi) := \int_{\mathbb{R}^n} B(\xi,\eta) f(\eta)\, d\eta$, $A : C_c^0(\mathbb{R}^n) \to C^0(\mathbb{R}^n)$.

$$C_1 := \sup_\xi \int_{\mathbb{R}^n} |B(\xi,\eta)|\, d\eta \quad \text{and}$$

$$C_2 := \sup_\eta \int_{\mathbb{R}^n} |B(\xi,\eta)|\, d\xi \quad \text{are finite.}$$

Then A extends to a bdd operator on $L^2(\mathbb{R}^n)$ and $\|A\|_{L^2(\mathbb{R}^n)\circlearrowleft} \le \sqrt{C_1 C_2}$.

Proof Enough to show that

$\forall f \in C_c^0(\mathbb{R}^n)$,

$$\|Af\|_{L^2}^2 \leq C_1 C_2 \|f\|_{L^2}^2 .$$

We estimate $\forall \xi \in \mathbb{R}^n$

$$|Af(\xi)|^2 = \left| \int_{\mathbb{R}^n} B(\xi,\eta) f(\eta)\, d\eta \right|^2 \leq$$ ((Cauchy-Schwartz)

$$\leq \int_{\mathbb{R}^n} |B(\xi,\eta)|\, d\eta \cdot \int_{\mathbb{R}^n} |B(\xi,\eta)| |f(\eta)|^2 d\eta$$

$$\leq C_1 \int_{\mathbb{R}^n} |B(\xi,\eta)| \cdot |f(\eta)|^2 d\eta .$$

Integrating, we get

$$\int_{\mathbb{R}^n} |Af(\xi)|^2 d\xi \leq C_1 \int_{\mathbb{R}^{2n}} |B(\xi,\eta)| \cdot |f(\eta)|^2 d\eta\, d\xi$$

$$= C_1 \int_{\mathbb{R}^n} |f(\eta)|^2 \cdot \int_{\mathbb{R}^n} |B(\xi,\eta)| d\xi\, d\eta$$

$$\leq C_1 C_2 \int_{\mathbb{R}^n} |f(\eta)|^2 .$$ $\square$

We can now give

Proof of Thm

1. It suffices to show that $\forall \chi \in C_c^\infty(U)$,

$$\chi \operatorname{Op}(a) \chi : H^s(\mathbb{R}^n) \to H^{s-\ell}(\mathbb{R}^n)$$

For that it's enough to show:

$$\forall \chi \; \exists C \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

$$\|\chi \operatorname{Op}(a) \chi \varphi\|_{H^{s-\ell}} \leq C \|\varphi\|_{H^s}.$$

We have $\operatorname{Op}(a)\chi\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\eta} a(x,\eta) \widehat{\chi\varphi}(\eta)\, d\eta.$

Now, $\|\chi\varphi\|_{H^s} \leq C\|\varphi\|_{H^s}$, so we can write

$$\widehat{\chi\varphi}(\eta) = \langle \eta \rangle^{-s} v(\eta) \quad \text{where}$$

$v \in \mathcal{S}(\mathbb{R}^n)$ and $\|v\|_{L^2} \leq C\|\varphi\|_{H^s}$.

Now compute

$$\widehat{\chi \operatorname{Op}(a) \chi\varphi}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{ix\cdot(\eta-\xi)} \chi(x) a(x,\eta) \langle \eta \rangle^{-s} v(\eta)\, d\eta\, dx$$

So $\langle \xi \rangle^{s-\ell} \widehat{\chi \operatorname{Op}(a) \chi \varphi}(\xi) = \int_{\mathbb{R}^n} B(\xi,\eta) v(\eta)\, d\eta$

where

$$B(\xi,\eta) = (2\pi)^{-n} \langle\xi\rangle^{s-\ell} \langle\eta\rangle^{-s} \int_{\mathbb{R}^n} e^{ix\cdot(\eta-\xi)} \chi(x) a(x,\eta)\, dx$$

$$= (2\pi)^{-n} \langle\xi\rangle^{s-\ell} \langle\eta\rangle^{-s} \tilde{a}(\xi-\eta,\eta)$$

where $\tilde{a}(\zeta,\eta) = \int_{\mathbb{R}^n} e^{-ix\cdot\zeta} \chi(x) a(x,\eta)\, dx$

is the Fourier transform of χa in $x \to \zeta$ variable

2 We need to show that for

$$Av(\xi) = \int_{\mathbb{R}^n} B(\xi,\eta) v(\eta)\, d\eta, \quad \exists C \ \forall v$$

$$\|Av\|_{L^2(\mathbb{R}^n)} \leq C \|v\|_{L^2(\mathbb{R}^n)}.$$

By Schur's bound enough to show

$$\sup_{\xi} \int_{\mathbb{R}^n} |B(\xi,\eta)|\, d\eta < \infty \qquad (1) \quad \&$$

$$\sup_{\eta} \int_{\mathbb{R}^n} |B(\xi,\eta)|\, d\xi < \infty \qquad (2).$$

Integrating by parts in x and using that $a \in S^\ell$ we get $\forall N \ \exists C_N$:

$$|\tilde{a}(\zeta,\eta)| \leq C_N \langle\zeta\rangle^{-N} \langle\eta\rangle^{\ell}$$

So $\forall N\ \exists C_N$

$$|B(\xi,\eta)| \le C_N \left(\frac{\langle\xi\rangle}{\langle\eta\rangle}\right)^{s-\ell} \langle \xi-\eta\rangle^{-N}$$

Recall from §12.1 that

$$\left(\frac{\langle\xi\rangle}{\langle\eta\rangle}\right)^{s-\ell} \le C_{s,\ell} \langle \xi-\eta\rangle^{|s-\ell|}.$$

So $\forall N\ \exists \tilde{C}_N$:

$$|B(\xi,\eta)| \le \tilde{C}_N \langle \xi-\eta\rangle^{-N}.$$

Now (1) and (2) follow. $\square$

<u>Note</u>: if $a \in S^\ell(U\times\mathbb{R}^n)$ then the transpose $\mathrm{Op}(a)^t$ maps

$$H^s_c(U) \to H^{s-\ell}_{loc}(U) \quad \forall s$$

(follows because H^s_c, $H^{s-\ell}_{loc}$ are dual to each other; see Pset 10).

This shows that in Elliptic Regularity III, $Pu \in H^{s-\ell}_{loc}(M) \Rightarrow u \in H^s_{loc}(M)$.

In fact, we can get an estimate out of this:

Thm (Elliptic Estimate)

Assume that M is a manifold and $P \in \mathrm{Diff}^m(M)$ is elliptic. Fix $\psi, \chi \in C_c^\infty(M)$ such that $\chi = 1$ near $\operatorname{supp} \psi$. Also fix $s, N \in \mathbb{R}$. Then $\exists C$ such that $\forall u \in \mathcal{D}'(M)$,

$$\|\psi u\|_{H^s(M)} \le C \|\chi P u\|_{H^{s-m}(M)} + C \|\chi u\|_{H^{-N}(M)}$$

This is understood as follows: if $\chi P u \in H_c^{s-m}(M)$ then $\psi u \in H^s(M)$ & the estimate holds.

Here $\|\psi u\|_{H^s(M)}$ etc. are well-defined (up to equivalence) because ψ, χ are compactly supported.

Important special case:

if M is a compact manifold then can take $\psi = \chi = 1$ and get

$$\|u\|_{H^s(M)} \leq C\|Pu\|_{H^{s-m}(M)} + C\|u\|_{H^{-N}(M)}$$

Proof (1) Can reduce to the case when M is replaced by an open subset $U \subset \mathbb{R}^n$.

Indeed, use a partition of unity to write $\psi = \sum_{j=1}^{r} \psi_j$ where $\psi_j \in C_c^\infty(M)$ and each ψ_j is supported in U_j the domain of some coordinate system.

Bound $\|\psi u\|_{H^s} \leq \sum_{j=1}^{r} \|\psi_j u\|_{H^s}$

Now take $\chi_j \in C_c^\infty(U_j)$, $\chi_j = 1$ near $\operatorname{supp} \psi_j$. Then $\chi\chi_j = 1$ near $\operatorname{supp} \psi_j$ as well.
(can make $\operatorname{supp} \psi_j \subset \operatorname{supp} \psi$)

It suffices to show that $\forall j$

$$\|\psi_j u\|_{H^s} \le C\|\chi_j \chi P u\|_{H^{s-m}} + C\|\chi_j \chi u\|_{H^{-N}}$$

Since $\psi_j, \chi_j \chi$ are supported inside U_j, can pull this back to an open subset of $\mathbb{R}^n$ using the coordinate system.

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2 Now $M = U \subset \mathbb{R}^n$ open.

Recall the proof of Elliptic Regularity III in §15.4: we used

$$\tilde{Q} : \mathcal{E}'(U) \to \mathcal{D}'(U),\ C_c^\infty(U) \to C^\infty(U)$$

pseudolocal & such that $\tilde{Q}P - I$ is smoothing.

Moreover, $\tilde{Q} = \mathrm{Op}(q)^t$ for some

$$q \in S^{-m}(U \times \mathbb{R}^n), \text{ so}$$

$$\tilde{Q} : H_c^{s-m}(U) \to H^s_{loc}(U) \text{ is continuous.}$$

Write $I = \tilde{Q}P + R$, R smoothing

Fix $\tilde{\chi} \in C_c^\infty(U)$, $\chi = 1$ near supp $\tilde{\chi}$
$\tilde{\chi} = 1$ near supp ψ

Then $\forall u \in \mathcal{D}'(\overline{U})$, we have

$$\tilde{\chi} u = \tilde{Q} P \tilde{\chi} u + R \tilde{\chi} u, \text{ so}$$

$$\psi u = \psi \tilde{Q} P \tilde{\chi} u + \psi R \tilde{\chi} u.$$

Since R is smoothing and $\psi, \tilde{\chi} \in C_c^\infty(U)$

we have $\forall s, N$

$$\|\psi R \tilde{\chi} u\|_{H^s(\mathbb{R}^n)} \leq C_{s,N} \|\chi u\|_{H^{-N}(\mathbb{R}^n)}.$$

(if $\chi u_j \to 0$ in $H^{-N}(\mathbb{R}^n)$ then

$\tilde{\chi} u_j \to 0$ in $\mathcal{E}'(\overline{U})$, so

$R \tilde{\chi} u_j \to 0$ in $C^\infty(\overline{U})$, so

$\psi R \tilde{\chi} u_j \to 0$ in $C_c^\infty(U) \subset H^s(\mathbb{R}^n)$)

Next, $\psi \tilde{Q} P \tilde{\chi} u = \psi \tilde{Q} \tilde{\chi} P u + \psi \tilde{Q} [P, \tilde{\chi}] u.$

Since $\tilde{\chi} = 1$ near supp ψ, the coefficients of $[P, \tilde{\chi}]$ are supported away from supp ψ.

Since $\tilde{Q}$ is pseudolocal, $\psi \tilde{Q} [P, \tilde{\chi}]$ is smoothing.

So $\|\psi \tilde{Q} [P, \tilde{\chi}] u\|_{H^s} \leq C_{s,N} \|\chi u\|_{H^{-N}(\mathbb{R}^n)}$

(here $[P, \tilde{\chi}] = [P, \tilde{\chi}] \chi$) as well.

Finally, since $\tilde{Q} : H_c^{s-m}(U) \to H^s_{loc}(U)$ we get $\|\psi \tilde{Q} \tilde{\chi} P u\|_{H^s} \leq C \|\chi P u\|_{H^{s-m}}$. $\square$

§16.2 Compactness

Here we show that

H_c^s embeds compactly into H^t_{loc} when $s > t$:

Thm Assume that $s > t$ and $u_k \in H_c^s(\mathbb{R}^n)$ is a sequence such that $\exists\, C, R\ \forall k$

1. $\|u_k\|_{H^s} \leq C$
2. $\operatorname{supp} u_k \subset B(0, R)$.

Then u_k has a subsequence which converges in $H^t(\mathbb{R}^n)$.

Remark In fact one can relax 1 + 2 to $\exists\, \delta > 0$: $\|\langle x\rangle^\delta u_k\|_{H^s} \leq C$.

Basically, improved regularity + improved decay $\Rightarrow$ compactness

Proof

1 Since u_k is compactly supported, its Fourier transform $\hat{u}_k$ is in C^∞:

$$\hat{u}_k(\xi) = (u_k(x), e^{ix\cdot\xi}), \quad \xi \in \mathbb{R}^n$$

We next estimate for $e_\xi(x) := e^{ix\cdot\xi}$

$\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi = 1$ near $B(0,R)$

$$|\partial_\xi^\alpha \hat{u}_k(\xi)| \leq \|u_k\|_{H^s(\mathbb{R}^n)} \cdot \|\chi \cdot x^\alpha \cdot e_\xi\|_{H^{-s}(\mathbb{R}^n)}$$

$$\leq C \|\chi \cdot x^\alpha \cdot e_\xi\|_{H^N(\mathbb{R}^n)} \quad \left(\begin{array}{l} N \in \mathbb{N} \\ N \geq -s \end{array}\right)$$

$$\leq C \max_{|\beta| \leq N} \sup_x |\partial_x^\beta e_\xi(x)|$$

$$\leq C \langle \xi \rangle^N,$$ where C is independent of k

Taking this with $|\alpha| \leq 1$, we see that $\forall T$, the sequence $\hat{u}_k(\xi)$ is uniformly bdd & uniformly equicontinuous on the ball $B(0,T)$. Indeed, equicontinuity follows from the bound $|\hat{u}_k(\xi) - \hat{u}_k(\eta)| \leq C T^N |\xi - \eta|$

$\forall k \ \forall \xi, \eta \in B(0,T)$)

By Arzelà–Ascoli Thm and a diagonal argument (taking further subsequences for $T = 1, 2, \dots$) there exists a subsequence $\{u_{k_j}\}$ such that $\hat{u}_{k_j}(\xi) \longrightarrow v(\xi)$ locally uniformly in ξ for some continuous $v \in C^0(\mathbb{R}^n)$.

Then $\langle \xi \rangle^s \hat{u}_{k_j}(\xi) \to \langle \xi \rangle^s v(\xi)$ $\forall \xi$.

So by Fatou's Lemma, $\langle \xi \rangle^s v(\xi) \in L^2$.

Thus $v(\xi) = \hat{u}(\xi)$ for some $u \in H^s(\mathbb{R}^n)$.

2 It is not in general true that

$$\langle \xi \rangle^s (\hat{u}_{k_j}(\xi) - v(\xi)) \longrightarrow 0 \text{ in } L^2(\mathbb{R}^n)$$

(Think of a running step: $s = 0$, $n = 1$,

$$u_k(x) = e^{ikx} \varphi(x), \quad \varphi \in C_c^\infty(\mathbb{R});$$

$$\hat{u}_k(\xi) = \hat{\varphi}(\xi - k),$$

$\hat{u}_k(\xi) \to 0$ pointwise in ξ but not in L^2 in ξ)

However, for $t<s$ we do have $\langle\xi\rangle^t(\hat{u}_{k_j}(\xi)-v(\xi))\to 0$ in $L^2(\mathbb{R}^n)$

and thus $u_{k_j}\to u$ in $H^t(\mathbb{R}^n)$ (where $\hat{u}=v$).

Indeed, take any $T>0$.

Then $\int_{\mathbb{R}^n}\langle\xi\rangle^{2t}|\hat{u}_{k_j}(\xi)-v(\xi)|^2\,d\xi \leq$

$$\leq \int_{|\xi|\leq T}(\ldots)+\int_{|\xi|\geq T}(\ldots)$$

$$\leq C_T\sup_{|\xi|\leq T}|\hat{u}_{k_j}(\xi)-\hat{v}(\xi)|^2 +$$

$$+2\sup_{|\xi|\geq T}\langle\xi\rangle^{2(t-s)}\int_{\mathbb{R}^n}\langle\xi\rangle^{2s}(|\hat{u}_{k_j}(\xi)|^2+|v(\xi)|^2)\,d\xi$$

$\leq a_j(T)+b(T)$ where

- $a_j(T)\xrightarrow[k\to\infty]{} 0$ $\forall T$
- $b(T)$ is T- indepdt & $b(T)\xrightarrow[T\to\infty]{} 0$

since $\|\langle\xi\rangle^s\hat{u}_k(\xi)\|_{L^2}\sim\|u_k\|_{H^s}\leq C$

So $\forall T$,

$$\limsup_{j\to\infty} \int_{\mathbb{R}^n} \langle\xi\rangle^{2t} |\hat{u}_{k_j}(\xi) - v(\xi)|^2 \, d\xi$$

$$\leq \limsup_{j\to\infty} a_j(T) + b(T) \leq b(T)$$

Taking $T \to \infty$, we see that this $\limsup$ is $= 0$.

So $\langle\xi\rangle^t |\hat{u}_{k_j}(\xi) - v(\xi)|^2 \to 0$ in L^2 and u_{k_j} converges in H^t as needed. $\square$

Defn Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $A: \mathcal{X} \to \mathcal{Y}$ a bounded linear operator.

We say A is *compact*, if $\forall$ bounded sequence $u_k \in \mathcal{X}$ the sequence Au_k has a subsequence converging in $\mathcal{Y}$.

The Thm we just proved has

Corollary 1 Assume $\chi \in C_c^\infty(\mathbb{R}^n)$. Then $\forall\, s>t$, the multiplication operator $\chi : H^s(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ is compact.

Corollary 2 Let M be a compact manifold. Then $\forall\, s>t$, the inclusion $I : H^s(M) \longrightarrow H^t(M)$ is compact.
That is, if $u_k \in H^s(M)$ is bounded then u_k has a subsequence converging in $H^t(M)$.

Proof Write a partition of unity
$I = \chi_1 + \cdots + \chi_\ell$, each χ_ℓ is supported in the domain of a coordinate system.
multiplication op's

Use coordinates to show: $\forall j$,
$\chi_j : H^s(M) \to H^t(M)$ compact. Then $I = \chi_1 + \cdots + \chi_\ell$ is compact too. $\square$

§16.3. Fredholm Theory

Defn. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces.

A bounded operator $A: \mathcal{X} \to \mathcal{Y}$ is called a **Fredholm operator**, if:

$\boxed{1}$ $\operatorname{Ker} A = \{u \in \mathcal{X} \mid Au = 0\}$ is finite dimensional

$\boxed{2}$ The range $\operatorname{Ran} A := A(\mathcal{X})$ is closed in $\mathcal{Y}$

$\boxed{3}$ $A(\mathcal{X})$ has finite codimension in $\mathcal{Y}$

Basic properties: (18.102?)

① $\boxed{1} + \boxed{3} \Rightarrow \boxed{2}$ above (so $\boxed{2}$ not needed)

② If A is Fredholm, its index is $\operatorname{ind}(A) = \dim \operatorname{Ker} A - \operatorname{Codim}_{\mathcal{Y}} \operatorname{Ran} A$

③ If $\mathcal{X}, \mathcal{Y}$ are finite dimensional: $\dim \mathcal{X} = m$, $\dim \mathcal{Y} = n$, then any $A: \mathcal{X} \to \mathcal{Y}$ is Fredholm and

$$\operatorname{ind} A = m - n$$

(Rank / Nullity Thm)

"Fredholm operators are like matrices"

4. A is invertible $\Rightarrow$ A is Fredholm of index 0

5. A is Fredholm, $K : \mathcal{X} \to \mathcal{Y}$ is compact $\Rightarrow$ $A+K$ is Fredholm of same index as A

6. A is Fredholm $\Rightarrow$ $\exists\, \varepsilon > 0$ s.t. $\forall\, B : \mathcal{X} \to \mathcal{Y}$ with $\|B\|_{\mathcal{X}\to\mathcal{Y}} < \varepsilon$, $A+B$ is Fredholm of same index as A

7. $\mathcal{X} \xrightarrow{A} \mathcal{Y} \xrightarrow{B} \mathcal{Z}$, A, B Fredholm $\Rightarrow$ BA Fredholm and $\operatorname{ind}(BA) = \operatorname{ind} A + \operatorname{ind} B$.

<u>Thm</u> Assume M is a compact manifold and $P \in \mathrm{Diff}^m(M)$ is an elliptic differential operator. Then $\forall s \in \mathbb{R}$,

$$P_s := P : H^s(M) \to H^{s-m}(M)$$

is a Fredholm operator.

Proof 1. We first show that

$\operatorname{Ker} P_s = \{u \in H^s(M) \mid Pu = 0\}$

is finite dimensional.

We use the Elliptic Estimate: $\forall N \exists C$ $\forall u \in H^s(M)$

$$\|u\|_{H^s} \leq C\|Pu\|_{H^{s-m}} + C\|u\|_{H^{-N}} \qquad (*)$$

We see that $\forall u \in \operatorname{Ker} P_s$,

$$\|u\|_{H^s} \leq C\|u\|_{H^{-N}} \qquad (**)$$

where C is independent of u.

Take N s.t. $-N \leq s$.

Then since $H^s \hookrightarrow H^{-N}$ is a compact embedding,

we see that

$\forall$ sequence $u_k \in \operatorname{Ker} P_s$ s.t.

$\|u_k\|_{H^s} \leq 1$, there exists

a subsequence u_{k_j} converging in H^{-N}

and thus in H^s (as $(**)$ shows that

u_k Cauchy in $H^{-N} \Rightarrow u_k$ Cauchy in H^s)

Now, if $\dim \operatorname{Ker} P_s = \infty$ then take an orthonormal system $u_1, u_2, \ldots$ in H^s (w.r.t. the H^s inner product).

This cannot have a subsequence converging in H^s (as $\|u_k - u_\ell\|_{H^s} = \sqrt{2}$ $\forall k \neq \ell$ cannot be Cauchy), giving a contradiction.

So $\dim \operatorname{Ker} P_s < \infty$.

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(2) We next show that the range

$$\operatorname{Ran}(P_s) := \{ Pu \mid u \in H^s(M) \}$$

is a <u>closed</u> subspace of $H^{s-m}(M)$.

Assume that we have a sequence $u_k \in H^s(M)$ and $Pu_k \to v$ in $H^{s-m}(M)$.

We need to show that $v = Pu$ for some $u \in H^s(M)$.

We can add to u_k some element of $\operatorname{Ker} P_s$ to make sure that $u_k \perp \operatorname{Ker} P_s$ w.r.t. $\langle \cdot, \cdot \rangle_{H^s}$.

We first show that $\|u_k\|_{H^s}$ is bounded.

Assume not.

WLOG $\|u_k\|_{H^s} \to \infty$. Put $\tilde{u}_k := \dfrac{u_k}{\|u_k\|_{H^s}}$.

Then $\|\tilde{u}_k\|_{H^s} = 1$, $\tilde{u}_k \perp \operatorname{Ker} P_s$ wrt $\langle \cdot, \cdot \rangle_{H^s}$,

and $P\tilde{u}_k = \dfrac{Pu_k}{\|u_k\|_{H^s}} \to 0$ in H^{s-m}.

Passing to a subsequence, can assume that $\tilde{u}_k$ converges in H^{-N}.

Now use $(*)$:

$$\|\tilde{u}_k - \tilde{u}_\ell\|_{H^s} \le C\|P\tilde{u}_k - P\tilde{u}_\ell\|_{H^{s-m}} + C\|\tilde{u}_k - \tilde{u}_\ell\|_{H^{-N}}.$$

We see that $\tilde{u}_k$ is a Cauchy sequence in H^s. So $\tilde{u}_k \to$ some $\tilde{u}$ in H^s.

We have $P\tilde{u} = \lim_{k\to\infty} P\tilde{u}_k = 0 \Rightarrow \tilde{u} \in \operatorname{Ker} P_s$

and $\|\tilde{u}\|_{H^s} = 1$, $\tilde{u} \perp \operatorname{Ker} P_s$,

a <u>contradiction</u> (as $\tilde{u} \perp \tilde{u}$).

So $\|u_k\|_{H^s}$ is bounded.

Now, if u_k is bounded, then again pass to a subsequence to make u_k converge in H^{-N} & write again

$$\|u_k - u_\ell\|_{H^s} \le C\|Pu_k - Pu_\ell\|_{H^{s-m}} + C\|u_k - u_\ell\|_{H^{-N}}.$$

Then u_k is a Cauchy sequence in H^s $\Rightarrow$ $u_k \to$ some u in H^s.

$\Rightarrow Pu_k \to Pu$ in H^{s-m}

$\Rightarrow v = Pu$ as needed.

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3. Consider now the adjoint operator $P^* \in \mathrm{Diff}^m(M)$ such that

$$\langle P\varphi, \psi\rangle_{L^2} = \langle \varphi, P^*\psi\rangle_{L^2} \quad \forall \varphi, \psi \in C^\infty(M).$$

Here $\langle \varphi, \psi\rangle_{L^2} = \int_M \varphi\bar{\psi}\, d\mathrm{Vol}_g$ (fixed some Riem. metric g)

We can define $\langle u, v \rangle_{L^2} \in \mathbb{C}$
for $u \in H^s(M)$, $v \in H^{-s}(M)$,
any s; $|\langle u, v\rangle_{L^2}| \leq C \|u\|_{H^s} \cdot \|v\|_{H^{-s}}$.

and we still have

$$\langle P_s u, v \rangle_{L^2} = \langle u, P^*_{m-s} v \rangle_{L^2} \qquad (\star)$$

$\forall u \in H^s(M)$, $v \in H^{m-s}(M)$.

Indeed, take $\varphi_k \to u$ in H^s
$\psi_k \to v$ in H^{m-s}
$\varphi_k, \psi_k \in C^\infty(M)$. Then

$\langle P\varphi_k, \psi_k \rangle_{L^2} = \langle \varphi_k, P^* \psi_k \rangle_{L^2}$ and

$P\varphi_k \to Pu$ in H^{s-m}
$P^*\psi_k \to P^* v$ in H^{-s}.

Passing to the limit we get $(\star)$.

Using $(\maltese)$ we get the following characterization of the range of P_s:

$$\operatorname{Ran}(P_s) = \{ w \in H^{s-m}(M) : \forall v \in \operatorname{Ker} P^*_{m-s}, \text{ we have } \langle w, v \rangle_{L^2} = 0 \}.$$

Indeed,

$\underline{\subset}$: if $w \in \operatorname{Ran}(P_s)$ then $w = P_s u$ for some $u \in H^s(M)$.

Then $\forall v \in \operatorname{Ker} P^*_{m-s}$ we have

$$\langle w, v \rangle_{L^2} = \langle P_s u, v \rangle_{L^2} \overset{(\maltese)}{=} \langle u, P^*_{m-s} v \rangle_{L^2} = 0.$$

$\underline{\supset}$: Assume that $w \in H^{s-m}(M)$ but $w \notin \operatorname{Ran}(P_s)$.

Since $\operatorname{Ran}(P_s) \subset H^{s-m}(M)$ is closed, $\exists$ a bounded linear functional

$$F : H^{s-m}(M) \to \mathbb{C}, \quad F|_{\operatorname{Ran}(P_s)} = 0,$$

$F(w) = 1$.

But bounded linear functionals on $H^{s-m}(M)$ are $\langle \cdot, \cdot \rangle_{L^2}$ pairings with elements of $H^{m-s}(M)$

(Pset 8, Problem 1...
can make it work for manifolds...)

So $\exists\, v \in H^{m-s}(M)$ such that $\forall f \in H^{s-m}(M)$ we have

$$F(f) = \langle f, v \rangle_{L^2}.$$

Now, $\langle w, v \rangle_{L^2} = F(w) = 1$

and $\forall u \in H^s(M)$, we have $Pu \in \mathrm{Ran}(P_s) \Rightarrow$

$$\Rightarrow 0 = F(Pu) = \langle Pu, v \rangle_{L^2} \overset{(*)}{=} \langle u, P^* v \rangle_{L^2}.$$

This holds $\forall u \in C^\infty(M)$ in particular,

So $\langle u, P^* v \rangle_{L^2} = 0 \quad \forall u \in C^\infty(M)$

$\Downarrow$

$P^* v = 0$ (since $P^* v$ is a distribution)

So $\exists\, v \in \mathrm{Ker}\, P^*_{m-s} : \langle w, v \rangle_{L^2} \neq 0$

which gives $\underline{\supseteq}$.

4 We showed in 3 that

$$\mathrm{Ran}\,(P_s) = \{ w \in H^{s-m}(M) : \forall v \in \mathrm{Ker}\, P^*_{m-s},\ \langle w, v \rangle_{L^2} = 0 \}.$$

Since P^* is elliptic, we see that $\mathrm{Ker}\, P^*_{m-s}$ is finite dimensional.

So $\mathrm{Ran}\,(P_s)$ has finite codimension.

In fact, $\mathrm{Codim}_{H^{s-m}} \mathrm{Ran}\,(P_s) = \dim \mathrm{Ker}\, P^*_{m-s}$.

Since $v \longmapsto \langle \cdot, v \rangle : H^{s-m} \to \mathbb{C}$

is an isomorphism from $H^{m-s}(M)$ to the dual space to $H^{s-m}(M)$. $\square$

Rmk By Elliptic Reg. III,

$\mathrm{Ker}\, P_s = \mathrm{Ker}\, P = \{ u \in C^\infty(M) : Pu = 0 \}$

$\mathrm{Ker}\, P_s^* = \mathrm{Ker}\, P^* = \{ v \in C^\infty(M) : P^* v = 0 \}$

are independent of s

and $\mathrm{ind}\, P_s = \dim \mathrm{Ker}\, P - \dim \mathrm{Ker}\, P^*$.

In particular,

$$\operatorname{ind}(P_s^*) = -\operatorname{ind} P_s$$

and if P is self-adjoint (i.e. $P^* = P$) then $\operatorname{ind} P_s = 0 \quad \forall s.$

An important example of a self-adjoint operator is $P = -\Delta_g$ on a compact Riemannian manifold (M, g).

Here $P = P^*$ since $\forall \varphi, \psi \in C^\infty(M)$

$$\langle P\varphi, \psi \rangle_{L^2} = -\int_M (\Delta_g \varphi)\overline{\psi}\, d\mathrm{Vol}_g$$

$$= -\int_M \langle \nabla_g \varphi, \overline{\nabla_g \psi} \rangle\, d\mathrm{Vol}_g$$

$$= \langle \varphi, P\psi \rangle_{L^2}$$